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Extensions of certain classical integrals of Erdélyi for Gauss hypergeometric functions

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Abstract

It is shown how series manipulation technique and certain classical summation theorems for hypergeometric series can be used to prove Erdélyi's integral representations for ${}_2F_1(z)$, originally proved using fractional calculus. The method not only leads to generalizations but also leads to new integrals of Erdélyi type for certain ${}_{q+1}F_q(z)$ and corresponding Pochhammer contour integrals. The technique outlined here, compared to the method of fractional calculus, seems to be more effective as it not only provides transparent elementary proofs of Erdélyi's integrals but even leads to various generalizations.

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1. Introduction

Although the integrals involving and representing hypergeometric functions have numerous applications in pure and applied mathematics (see, for example, [6]), not all such integrals have been collected in tables or are readily available in the mathematical literature. In this context, we shall consider the integrals of Erdélyi type.

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Let $z \neq 1$ and $|\arg(1-z)| < \pi$. In 1939, Erdélyi [4] showed that Euler's integral

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}\, z\right] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,\mathrm{d}t, \tag{1.1}$$

where $Re(\gamma) > R(\beta) > 0$, and Bateman's [1] extension of (1.1)

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}\, z\right] = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)}\int_0^1 t^{\mu-1}(1-t)^{\gamma-\mu-1}{}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \mu;\end{matrix}\, zt\right]\mathrm{d}t, \tag{1.2}$$

where $R(\gamma) > Re(\mu) > 0$, have extensions of the forms

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}\, z\right] = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)}\int_0^1 t^{\mu-1}(1-t)^{\gamma-\mu-1}(1-tz)^{\lambda-\alpha-\beta}$$
$$ \times {}_2F_1\left[\begin{matrix}\lambda-\alpha,\lambda-\beta; \\ \mu;\end{matrix}\, tz\right]{}_2F_1\left[\begin{matrix}\alpha+\beta-\lambda,\lambda-\mu; \\ \gamma-\mu;\end{matrix}\, \frac{(1-t)z}{1-tz}\right]\mathrm{d}t, \tag{1.3}$$

where $Re(\gamma) > Re(\mu) > 0$,

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}\, z\right] = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)}\int_0^1 t^{\mu-1}(1-t)^{\gamma-\mu-1}(1-tz)^{-\alpha'}$$
$$ \times {}_2F_1\left[\begin{matrix}\alpha-\alpha',\beta; \\ \mu;\end{matrix}\, tz\right]{}_2F_1\left[\begin{matrix}\alpha',\beta-\mu; \\ \gamma-\mu;\end{matrix}\, \frac{(1-t)z}{1-tz}\right]\mathrm{d}t, \tag{1.4}$$

where $Re(\gamma) > Re(\mu) > 0$ and

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}\, z\right] = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(v)\Gamma(\gamma+\mu-\lambda-v)}\int_0^1 t^{v-1}(1-t)^{\gamma+\mu-\lambda-v-1}$$
$$ \times {}_2F_1\left[\begin{matrix}\mu-\lambda,\gamma-\lambda; \\ \gamma+\mu-\lambda-v;\end{matrix}\, 1-t\right]{}_3F_2\left[\begin{matrix}\alpha,\beta,\mu; \\ \lambda,v;\end{matrix}\, tz\right]\mathrm{d}t, \tag{1.5}$$

where $Re(\lambda, v, \gamma+\mu-\lambda-v) > 0$. It may be noted that, when $\mu=\beta$, (1.2) becomes (1.1) and when $\lambda=\alpha+\beta, \alpha'=0$ and $\lambda=\mu$, then (1.3), (1.4) and (1.5), respectively, get converted into (1.2). Erdélyi also considered special cases and confluent limit cases of his formulas and some formulas obtained by applying transformation formulas to the hypergeometric functions in the integrand.

Gasper [7] pointed out some important applications of Erdélyi's fractional integral (1.3), such as to derive Dirichlet–Mehler-type integral representations for Jacobi polynomials and for generalized Legendre functions, and to prove the positivity of certain sums of generalized Legendre functions and derived the discrete analogue of (1.3). Later in [8], this work was extended for (1.4) and (1.5) to q-analogues, along with the derivation of a generalization of a q-Kampé de Fériet sum, which was conjectured in the work [14] on the evaluation of the $9-j$ recoupling coefficients appearing in the quantum theory of angular momentum.

On the other hand, as recorded in [12, pp. 287–288], Erdélyi [3] gave a general form for (1.2) as

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix} z\right]\\
&=\prod_{j=1}^{m}\left\{\frac{\Gamma(\alpha_j)}{\Gamma(a_j)\Gamma(\alpha_j-a_j)}\right\}\prod_{j=1}^{n}\left\{\frac{\Gamma(b_j)}{\Gamma(\beta_j)\Gamma(b_j-\beta_j)}\right\}\\
&\quad\times\int_0^1\cdots\int_0^1\prod_{j=1}^{m}\{u_j^{a_j-1}(1-u_j)^{\alpha_j-a_j-1}\}\prod_{j=1}^{n}\{v_j^{b_j-1}(1-v_j)^{b_j-\beta_j-1}\}\\
&\quad\times{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_m,a_{m+1},\ldots,a_p;\\ \beta_1,\ldots,\beta_n,b_{n+1},\ldots,b_q;\end{matrix} zu_1\ldots u_m v_1\ldots v_n\right]\mathrm{d}u_1\ldots\mathrm{d}u_m\,\mathrm{d}v_1\ldots\mathrm{d}v_n,
\end{aligned}
\tag{1.6}
$$

$m\leqslant p$; $n\leqslant q$; $Re(\alpha_j)>Re(a_j)>0$, $j=1,\ldots,m$; $Re(\beta_j)>Re(b_j)>0$, $j=1,\ldots,n$; $p\leqslant q$ and $|z|<\infty$ (or $p=q+1, z\neq 1$ and $|\arg(1-z)|<\pi$), which, for $m=0$ and $n=1$, yields the elegant result

$$
{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix} z\right]=\frac{\Gamma(b_1)}{\Gamma(\lambda)\Gamma(b_1-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{b_1-\lambda-1}\,{}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ \lambda,b_2,\ldots,b_q;\end{matrix} zt\right]\mathrm{d}t,
\tag{1.7}
$$

$p\leqslant q$ and $|z|<\infty$ ($p=q+1, z\neq 1$ and $|\arg(1-z)|<\pi$); $Re(b_1)>Re(\lambda)>0$, whose special case, when $p=2$ and $q=1$, corresponds to (1.2). This work on (1.2) has been further extended by many authors (see [12, p. 288]) for Appell, Kampé de Fériet, Lauricella and other multiple series by using fractional calculus.

Erdélyi used fractional integration by parts and transformation formulas for ${}_2F_1$ hypergeometric functions to derive his integrals (1.3)–(1.5).

In this paper, the series manipulation technique and the classical summation theorems are used to give an alternative way of proof for Erdélyi integrals (1.3)–(1.5). Motivated, from the above way of proof, in Section 3, new integrals of Erdélyi type for certain ${}_{q+1}F_q(z)$ are conjectured and proved. A generalization and unification of (1.3), (1.4) and (3.1) is obtained in Section 4(i). Multidimensional cases of integrals (1.3), (1.4), (3.1), (4.1) and of integrals obtained in Section 3, are contained in Section 4(ii). A generalization of (1.5), in terms of a most general power series is found, whose particular case in terms of ${}_pF_q(z)$, includes (1.7) as a particular case, is obtained in Section 4(iii). Furthermore, in Section 4(iv), we are able to give an interesting multidimensional case of the generalization of (1.5), whose particular case in terms of ${}_pF_q(z)$, includes the integral (1.6) in its fold. In Section 5, it is shown that how the method outlined in this paper can be used to prove the corresponding Pochhammer contour analogues of all the extensions of Erdélyi's integrals developed here.

2. Alternative proofs for the Erdélyi's integrals (1.3)–(1.5)

(i) Let us denote the right-hand side of (1.3) by I, then replacing the ${}_2F_1$'s of the integrand by their series form and interchanging the order of summations and integration, which is valid when $|z| < 1$, we get

$$I = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} \sum_{m,n=0}^{\infty} \frac{(\lambda-\alpha)_m(\lambda-\beta)_m(\lambda-\mu)_n(\alpha+\beta-\lambda)_n}{(\mu)_m(\gamma-\mu)_n} \frac{z^m}{m!}\frac{z^n}{n!}$$

$$\times \int_0^1 t^{\mu+m-1}(1-t)^{\gamma-\mu+n-1}(1-tz)^{\lambda-\alpha-\beta-n}\,\mathrm{d}t. \tag{2.1}$$

By using (1.1), with the prescribed conditions, (2.1) becomes

$$I = \sum_{m,n=0}^{\infty} \frac{(\lambda-\alpha)_m(\lambda-\beta)_m(\lambda-\mu)_n(\alpha+\beta-\lambda)_n(\mu)_m(\gamma-\mu)_n}{(\mu)_m(\gamma-\mu)_n(\gamma)_{m+n}} \frac{z^m}{m!}\frac{z^n}{n!}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha+\beta-\lambda+n, \mu+m; \\ \gamma+m+n;\end{matrix}\; z\right]$$

or

$$I = \sum_{k,m,n=0}^{\infty} \frac{(\alpha+\beta-\lambda)_{k+n}(\mu)_{k+m}(\lambda-\alpha)_m(\lambda-\beta)_m(\lambda-\mu)_n}{(\gamma)_{k+m+n}(\mu)_m} \frac{z^k}{k!}\frac{z^m}{m!}\frac{z^n}{n!}. \tag{2.2}$$

Now applying the series manipulation technique [10, p. 56, Lemma 10] or [13, p. 100, (1)] twice or more explicitly, using the following triple series manipulation:

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(k,m,n) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{k-m} A(k-m-n,m,n)$$

on the triple series of (2.2), we can write

$$I = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{(\alpha+\beta-\lambda)_{k-m}(\mu)_k(\lambda-\alpha)_m(\lambda-\beta)_m(-k)_m(-1)^m}{(\gamma)_k(\mu)_m m!} \frac{z^k}{k!} {}_2F_1\left[\begin{matrix}\lambda-\mu, m-k; \\ 1-\mu-k;\end{matrix}\; 1\right]. \tag{2.3}$$

After summing the inner ${}_2F_1$ of (2.3) by the Vandermonde theorem, [11, p. 243, (III. 4)] we can write

$$I = \sum_{k=0}^{\infty} \frac{(\lambda)_k(\alpha+\beta-\lambda)_k}{(\gamma)_k} \frac{z^k}{k!} {}_3F_2\left[\begin{matrix}\lambda-\alpha, \lambda-\beta, -k; \\ 1+\lambda-\alpha-\beta-k, \lambda;\end{matrix}\; 1\right].$$

The above inner ${}_3F_2(1)$ can be summed by the Saalschütz theorem [11, p. 243, (III. 2)] to give

$$I = {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma;\end{matrix}\; z\right].$$

Thus the proof of (1.3) is completed.

(ii) Similarly, in the right-hand side of (1.4), replacing the ${}_2F_1$'s of the integrand by their series form and using (1.1), we obtain a triple series. Applying the series manipulation technique, on this triple series, twice, we can take the first inner series as a ${}_2F_1$ and sum it by the Vandermonde theorem to have a double series, from which again, we can take the inner series as a ${}_2F_1$ and sum it by the Vandermonde theorem and get the ${}_2F_1$, which on the left of (1.4) and thus, we complete the proof of (1.4).

(iii) The proof of (1.5) is slightly different and simple. Let the right-hand side of (1.5) be I. Replacing ${}_2F_1$ and ${}_3F_2$ of the integrand by their series form and using Euler's Beta integral, we can have

$$I = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\gamma+\mu-\lambda)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\mu)_n}{(\gamma+\mu-\lambda)_n(\lambda)_n} \frac{z^n}{n!} {}_2F_1\left[\begin{matrix} \mu-\lambda, \gamma-\lambda; \\ \gamma+\mu-\lambda+n; \end{matrix} 1\right].$$

Summing the above inner ${}_2F_1$ by the Gauss theorem [11, p. 243 (III.3)], we get that

$$I = {}_2F_1\left[\begin{matrix} \alpha, \beta; \\ \gamma; \end{matrix} z\right]$$

and thus the proof of (1.5) is completed.

3. Integrals of Erdélyi type for certain ${}_{q+1}F_q(z)$

Motivated from the above way of proof of Erdélyi's integrals which uses the Gauss, Vandermonde and Saalschütz theorems, similar integrals are conjectured below, whose proof uses other summation theorems as well. We call these "Erdélyi-type integrals" because they follow the similar method of proof and are integrals of the combination of powers and hypergeometric functions, like (1.3)–(1.5). For example,

$$\begin{aligned} {}_3F_2\left[\begin{matrix} \alpha, \beta, v+\alpha'; \\ \gamma, v+\alpha; \end{matrix} z\right] &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}(1-tz)^{-\alpha'} \\ &\quad \times {}_3F_2\left[\begin{matrix} \alpha-\alpha', \beta, v; \\ \lambda, v+\alpha; \end{matrix} zt\right] {}_2F_1\left[\begin{matrix} \alpha', \beta-\lambda; \\ \gamma-\lambda; \end{matrix} \frac{z(1-t)}{1-tz}\right] \mathrm{d}t, \end{aligned} \tag{3.1}$$

where $|\arg(1-z)| < \pi$, $Re(\gamma) > Re(\lambda) > 0$ and $z \neq 1$. As $v \to \infty$ (3.1) becomes (1.4) and can be proved by following the method explained in Section 2(i) and the proof will use the Vandermonde and Saalschütz theorems. The other integrals are

$${}_3F_2\left[\begin{matrix} \alpha, \dfrac{\gamma-\beta}{2}, \dfrac{1+\gamma-\beta}{2}; \\ \dfrac{\gamma}{2}, \dfrac{1+\gamma}{2}; \end{matrix} z\right]$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma-\alpha-\beta)}\int_0^1 t^{\gamma-\alpha-\beta-1}(1-t)^{\alpha+\beta-1}(1+tz)^{-\beta}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha-\beta,\gamma-\beta;\\ \gamma-\alpha-\beta;\end{matrix} zt\right] {}_3F_2\left[\begin{matrix}\alpha,\frac{\beta}{2},\frac{1+\beta}{2};\\ \frac{\alpha+\beta}{2},\frac{1+\alpha+\beta}{2};\end{matrix} \frac{-z(1-t)^2}{(1+tz)^2}\right]\mathrm{d}t, \tag{3.2}$$

where $|\arg(1-z)|<\pi$, $Re(\gamma)>Re(\alpha+\beta)>0$ and $z\neq 1$. It can be proved by using the method of Section 2(i) and the proof shall use the Saalschütz theorem twice.

$$ {}_3F_2\left[\begin{matrix}\alpha,\beta,1+\gamma-\mu-\lambda;\\ 1+\gamma-\mu,1+\gamma-\lambda;\end{matrix} z\right]$$

$$=\frac{\Gamma(1+\gamma)}{\Gamma(\beta)\Gamma(1+\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta}(1-tz)^{-\alpha}$$

$$\times {}_5F_4\left[\begin{matrix}\gamma,1+\frac{\gamma}{2},\mu,\lambda,\alpha;\\ \frac{\gamma}{2},1+\gamma-\mu,1+\gamma-\lambda,1+\gamma-\beta;\end{matrix} \frac{-t(1-t)}{1-tz}\right]\mathrm{d}t, \tag{3.3}$$

where, $|\arg(1-z)|<\pi$, $Re(1+\gamma)>Re(\beta)>0$, $z\neq 1$. For proving (3.3), in the right-hand side of it, replacing the ${}_5F_4$ of the integrand by the series form and using (1.1), we get a double series. Applying series manipulation, once, on this double series and summing the resulting inner series by the Dougall's theorem [11, p. 243, (III.14)] we get a ${}_3F_2$ which is on the left of (3.3) and, thus, we complete the proof of (3.3).

$$ {}_4F_3\left[\begin{matrix}\alpha,\beta,\frac{\gamma-\mu}{2},\frac{1+\gamma-\mu}{2};\\ \gamma-\mu,\frac{\gamma}{2},\frac{1+\gamma}{2};\end{matrix} z\right]$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha,\mu;\\ \gamma-\beta;\end{matrix} \frac{-t(1-t)z}{1-tz}\right]\mathrm{d}t, \tag{3.4}$$

where $|\arg(1-z)|<\pi$, $Re(\gamma)>Re(\beta)>0$ and $z\neq 1$.

To prove (3.4), in the right-hand side of it we replace $(1-tz)^{-\alpha}$ by $(1-tz)^{-\mu}(1-tz)^{\mu-\alpha}$ and write the ${}_2F_1$ and $(1-tz)^{\mu-\alpha}$ in series, followed by the use of Euler's integral (1.1) to obtain a triple series. On which, applying series manipulation twice, followed by the use of the Vandermonde and Saalschütz theorems, we get the left-hand side of (3.4) and we thus, complete the proof of (3.4).

$$
{}_4F_3\left[\begin{matrix}\alpha,\beta,\gamma,1+\alpha+\mu-\beta-\gamma;\\ 1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma-\mu;\end{matrix}\ z\right]
$$

$$
=\frac{\Gamma(1+\alpha-\mu)}{\Gamma(\lambda)\Gamma(1+\alpha-\lambda-\mu)}\int_0^1 t^{\lambda-1}(1-t)^{\alpha-\mu-\lambda}(1-tz)^{-\mu}{}_2F_1\left[\begin{matrix}\mu,\alpha-\lambda;\\ 1+\alpha-\mu-\lambda;\end{matrix}\ \frac{(1-t)z}{1-tz}\right]
$$

$$
\times{}_7F_6\left[\begin{matrix}\alpha-\mu,1+\dfrac{\alpha-\mu}{2},\beta-\mu,\gamma-\mu,1+\alpha-\beta-\gamma,\dfrac{\alpha}{2},\dfrac{1+\alpha}{2};\\ \dfrac{\alpha-\mu}{2},1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma-\mu,\dfrac{\lambda}{2},\dfrac{1+\lambda}{2};\end{matrix}\ zt^2\right]\mathrm{d}t, \tag{3.5}
$$

where $|\arg(1-z)|<\pi$, $Re(1+\alpha-\mu)>Re(\lambda)>0$ and $z\neq 1$. Eq. (3.5) can be proved by using the method of Section 2(i) and the proof will use the Vandermonde and Dougall theorems.

$$
{}_5F_4\left[\begin{matrix}\beta,\gamma,1+\alpha-\mu-\lambda,\dfrac{1+\alpha}{2},\dfrac{2+\alpha}{2};\\ 1+\alpha-\mu,1+\alpha-\lambda,\dfrac{\beta+\lambda}{2},\dfrac{1+\beta+\lambda}{2};\end{matrix}\ z\right]
$$

$$
=\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{\beta-1}(1-tz)^{\gamma-\alpha-1}(1-t^2z)^{1+\alpha-\beta-\gamma}
$$

$$
\times{}_4F_3\left[\begin{matrix}\alpha,1+\dfrac{\alpha}{2},\mu,\lambda;\\ \dfrac{\alpha}{2},1+\alpha-\mu,1+\alpha-\lambda;\end{matrix}\ \frac{t(1-t)z}{1-tz}\right]\mathrm{d}t, \tag{3.6}
$$

where $|\arg(1-z)|<\pi$, $Re(\gamma)>0$ and $z\neq 1$. Eq. (3.6) can be proved by the method explained in the Section 2(i) and the proof will use the Saalchütz and Dougall's ${}_5F_4(1)$ [11, p. 243, (III.13)] theorems.

$$
{}_6F_5\left[\begin{matrix}\alpha,\beta,\gamma,1+2\alpha-\beta-\gamma-\mu,\dfrac{1+\mu}{2},\dfrac{2+\mu}{2};\\ 1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma+\mu-\alpha,\dfrac{\lambda}{2},\dfrac{1+\lambda}{2};\end{matrix}\ z\right]
$$

$$
=\frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\lambda-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\lambda-\alpha-1}(1-tz)^{\mu-\alpha}
$$

$$
\times{}_5F_4\left[\begin{matrix}\mu,1+\dfrac{\mu}{2},\beta+\mu-\alpha,\gamma+\mu-\alpha,1+\alpha-\beta-\gamma;\\ \dfrac{\mu}{2},1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma+\mu-\alpha;\end{matrix}\ zt^2\right]
$$

$$
\times{}_2F_1\left[\begin{matrix}\alpha-\mu,\lambda-\mu-1;\\ \lambda-\alpha;\end{matrix}\ \frac{-zt(1-t)}{1-tz}\right]\mathrm{d}t, \tag{3.7}
$$

where $|\arg(1-z)| < \pi$, $Re(\lambda) > Re(\alpha) > 0$ and $z \neq 1$. Eq. (3.7) can be proved by the method explained in Section 2(i) and the proof will use the Vandermonde and Dougall theorems.

The $\beta = 0$ case of (3.2) gives an interesting integral representation for binomial function ${}_1F_0(z)$, involving ${}_2F_1(z)$. Integrals (3.3) and(3.4) provide new extensions of well-known Euler's integral (1.1), which follows from (3.3), when $\mu = 0$ or $\lambda = 0$ and from (3.4), when $\mu = 0$.

It will be of interest to mention here that following the argument of Rainville [10, pp. 48–49], we can put $z = 1$ in the above mentioned integrals provided the convergence condition of the series ${}_{q+1}F_q(z)$, on the LHS, for $|z| = 1$, is being satisfied. Taking $z = 1$ in (3.3) and using Euler's Beta integral, we get a well-known transformation of a well-poised ${}_6F_5(-1)$ into a ${}_3F_2(1)$. Taking $z = 1$ in (3.4) and using Euler's Beta integral, we get a known transformation of a ${}_4F_3(1)$ into a ${}_3F_2(-1)$ due to Whipple. Taking $z = 1$ in (3.5) and using the Gauss theorem and Euler's Beta integral, we get a new transformation of a well-poised ${}_4F_3(1)$ into a well-poised ${}_7F_6(1)$. Taking $z = 1$, in (3.6) and (3.7) and using Euler's Beta integral, we get two new transformations of special double Srivastava–Daoust series [12] into ${}_5F_4(1)$ and ${}_6F_5(1)$, respectively. Hence, the integrals of this section generalize certain hypergeometric transformations.

4. Generalizations of Erdélyi's integrals

(i) Here, we give an interesting unification and generalization of both the Erdélyi's integrals (1.3) and (1.4) along with (3.1).

The integral established is

$$ {}_3F_2\left[\begin{matrix} \nu, \xi, \lambda; \\ \gamma, \delta; \end{matrix} z\right] = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} \int_0^1 t^{\mu-1}(1-t)^{\gamma-\mu-1}(1-tz)^{\delta-\nu-\xi} \times {}_3F_2\left[\begin{matrix} \delta-\xi, \delta-\nu, \lambda; \\ \mu, \delta; \end{matrix} zt\right] {}_2F_1\left[\begin{matrix} \lambda-\mu, \nu+\xi-\delta; \\ \gamma-\mu; \end{matrix} \frac{(1-t)z}{1-tz}\right] \mathrm{d}t, \tag{4.1} $$

where $|\arg(1-z)| < \pi$, $Re(\gamma) > Re(\mu)$ and $z \neq 1$. The proof follows on the lines of the method explained in Section 2(i) and the proof will use the Vandermonde and Saalchütz theorems. It may be noted that in (4.1), when $\nu = \alpha, \xi = \beta, \delta = \lambda$, integral (1.3) follows; when $\xi = \delta + \alpha' - \alpha, \nu = \alpha, \lambda = \beta$ and then as $\delta \to \infty$ integral (1.4) follows and when $\xi = \alpha, \lambda = \beta, \nu = \nu + \alpha', \delta = \nu + \alpha$, integral (3.1) follows. Again, using the argument of Rainville [10, pp. 48–49], we can put $z = 1$ in this integral. Taking $z = 1$ and using the Gauss theorem and Euler's Beta integral, (4.1) converts into the well-known Kummer–Thomae–Whipple ${}_3F_2(1)$ transformation. Thus (4.1) is a generalization of Kummer–Thomae–Whipple ${}_3F_2(1)$ transformation.

(ii) A multidimensional case of Euler's integral (1.1) follows in the form

$$ {}_{q+1}F_q\left[\begin{matrix} a_1, \ldots, a_{q+1}; \\ b_1, \ldots, b_q; \end{matrix} z\right] = \prod_{j=1}^{q} \left\{ \frac{\Gamma(b_j)}{\Gamma(a_j)\Gamma(b_j-a_j)} \right\} \int_0^1 \cdots \int_0^1 \prod_{j=1}^{q} \{t_j^{a_j-1}(1-t_j)^{b_j-a_j-1}\} \times (1-t_1\ldots t_q z)^{-a_{q+1}} \,\mathrm{d}t_1 \ldots \mathrm{d}t_q, \tag{4.2} $$

where $|\arg(1-z)| < \pi$, $Re(b_j) > Re(a_j) > 0$, $j = 1, \ldots, q$.

From the repeated application of the functional equation [10, p. 85, Theorem 28]

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix} a_2,\ldots,a_p; \\ b_2,\ldots,b_q; \end{matrix} zt\right] \mathrm{d}t \quad (4.3) $$

and then using the binomial theorem.

The multidimensional case of (4.1) admits the form

$$ {}_{q+2}F_{q+1}\left[\begin{matrix} a_1,\ldots,a_{q-1},v,\xi,\lambda; \\ b_1,\ldots,b_{q-1},\gamma,\delta; \end{matrix} z\right] $$

$$ = \prod_{j=1}^{q-1}\left\{\frac{\Gamma(b_j)}{\Gamma(a_j)\Gamma(b_j-a_j)}\right\}\frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)} $$

$$ \times \int_0^1\cdots\int_0^1 \prod_{j=1}^{q-1}\{(t_j)^{a_j-1}(1-t_j)^{b_j-a_j-1}\} $$

$$ \times\, t_q^{\mu-1}(1-t_q)^{\gamma-\mu-1}(1-t_1\ldots t_q z)^{\delta-\xi-v} $$

$$ \times\, {}_3F_2\left[\begin{matrix} \delta-\xi,\delta-v,\lambda; \\ \mu,\delta; \end{matrix} t_1\ldots t_q z\right] $$

$$ \times\, {}_2F_1\left[\begin{matrix} \lambda-\mu, v+\xi-\delta; \\ \gamma-\mu; \end{matrix} \frac{t_1\ldots t_{q-1}(1-t_q)z}{(1-t_1\ldots t_q z)}\right] \mathrm{d}t_1\ldots \mathrm{d}t_q, \quad (4.4) $$

where $|\arg(1-z)|<\pi$, $Re(\gamma)>Re(\mu)>0$, $z\neq 1$ and $Re(b_j)>Re(a_j)>0$, $j=1,\ldots,q-1$. For $q=1$, (4.4) becomes (4.1). The proof of (4.4) follows along the lines of the method, explained in Section 2(i), but uses (4.2) in place of (1.1) and the used summation theorems will be of the Vandermonde and Saalschütz. Obviously, the multidimensional cases of Erdélyi's integrals (1.3) and (1.4) will follow from (4.4), since (4.1) is a generalization and unification of (1.3) and (1.4). Similarly, multidimensional cases of the integrals discussed in Section 3 can also be developed.

(iii) Motivated from the alternative proof of Erdélyi's integral (1.5), given in Section 2(iii), it is possible to suggest a general setting for the integral in the form

$$ \sum_{n=0}^{\infty} c_n \frac{(\lambda)_n(v)_n}{(\gamma)_n(\mu)_n} z^n = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(v)\Gamma(\gamma+\mu-\lambda-v)} $$

$$ \times \int_0^1 t^{v-1}(1-t)^{\gamma+\mu-\lambda-v-1}\,{}_2F_1\left[\begin{matrix} \mu-\lambda,\gamma-\lambda; \\ \gamma+\mu-\lambda-v; \end{matrix} 1-t,\right] \phi(zt)\,\mathrm{d}t, \quad (4.5) $$

where $Re(\lambda, v, \gamma + \mu - \lambda - v) > 0$ and $\phi(z)$ is assumed to be a convergent series defined by

$$\phi(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{4.6}$$

where c_n is any bounded sequence of complex numbers. Following the method of proof for (1.5) given in Section 2(iii), (4.5) can also be proved.

If we take

$$c_n = \frac{(\alpha)_n(\beta)_n(\mu)_n}{(\gamma)_n(v)_n n!},$$

then (4.5) leads us to (1.5).

If we choose $\gamma = b_1$ and

$$c_n = \frac{(a_1)_n \ldots (a_p)_n(\mu)_n}{(b_2)_n \ldots (b_q)_n(\lambda)_n(v)_n n!},$$

then (4.5) gives us

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(v)\Gamma(b_1 + \mu - \lambda - v)} \int_0^1 t^{v-1}(1-t)^{b_1+\mu-\lambda-v-1} \times {}_2F_1\left[\begin{matrix} \mu - \lambda, b_1 - \lambda; \\ b_1 + \mu - \lambda - v; \end{matrix} 1 - t\right] {}_{p+1}F_{q+1}\left[\begin{matrix} a_1, \ldots, a_p, \mu; \\ b_2, \ldots, b_q, \lambda, v; \end{matrix} zt\right] \mathrm{d}t, \tag{4.7}$$

where $p \leqslant q$ and $|z| < \infty$ (or $p = q + 1, z \neq 1$, and $|\arg(1-z)| < \pi$); $Re(\lambda, v, b_1 + \mu - \lambda - v) > 0$. Eq. (4.7) generalizes (1.7) since $\mu = v$ case of (4.7) is (1.7).

(iv) At this stage, one can observe that the method of proof used for (4.5) above can be applied any number of times to prove the following multidimensional case of the integral given by (4.5):

$$\sum_{k=0}^{\infty} c_k \prod_{i=1}^{m}\left\{\frac{(a_i)_k(\lambda_i)_k}{(\alpha_i)_k(\mu_i)_k}\right\} \prod_{j=1}^{n}\left\{\frac{(\beta_j)_k(\xi_j)_k}{(b_j)_k(v_j)_k}\right\} z^k$$
$$= \prod_{i=1}^{m}\left\{\frac{\Gamma(\alpha_i)\Gamma(\mu_i)}{\Gamma(a_i)\Gamma(\lambda_i)\Gamma(\alpha_i + \mu_i - a_i - \lambda_i)}\right\} \times \prod_{j=1}^{n}\left\{\frac{\Gamma(b_j)\Gamma(v_j)}{\Gamma(\xi_j)\Gamma(\beta_j)\Gamma(b_j + v_j - \xi_j - \beta_j)}\right\}$$
$$\times \int_0^1 \cdots \int_0^1 \prod_{i=1}^{m}\left\{u_i^{\lambda_i - 1}(1-u_i)^{\alpha_i+\mu_i-a_i-\lambda_i-1} {}_2F_1\left[\begin{matrix} \mu_i - a_i, \alpha_i - a_i; \\ \alpha_i + \mu_i - a_i - \lambda_i; \end{matrix} (1-u_i)\right]\right\}$$
$$\times \prod_{j=1}^{n}\left\{v_j^{\beta_j - 1}(1-v_j)^{b_j+v_j-\xi_j-\beta_j-1} {}_2F_1\left[\begin{matrix} v_j - \xi_j, b_j - \xi_j; \\ b_j + v_j - \xi_j - \beta_j; \end{matrix} (1-v_i)\right]\right\}$$
$$\times \phi(zu_1 \ldots u_m \; v_1 \ldots v_n)\, \mathrm{d}u_1 \ldots \mathrm{d}u_m\, \mathrm{d}v_1 \ldots \mathrm{d}v_n, \tag{4.8}$$

where

$$\phi(z)=\sum_{k=0}^{\infty}c_k z^k$$

is assumed to be a convergent series and c_k is any bounded sequence of complex numbers and $Re(a_i,\lambda_i,\alpha_i+\mu_i-a_i-\lambda_i,\xi_j,\beta_j,b_j+v_j-\xi_j-\beta_j)>0,\ i=1,\dots,m,\ j=1,\dots,n.$

Now in (4.8), setting

$$c_k=\frac{\prod_{i=1}^{m}\{(\alpha_i)_k(\mu_i)_k\}\prod_{r=1}^{p}(a_{m+r})_k\prod_{j=1}^{n}(v_j)_k}{\prod_{j=1}^{n}\{(\beta_i)_k(\xi_i)_k\}\prod_{l=1}^{q}(b_{n+l})_k\prod_{i=1}^{m}(\lambda_i)_k k!},\quad m\leqslant p,\ n\leqslant q,$$

we get

$$\begin{aligned}&{}_pF_q\left[\begin{matrix}a_1,\dots,a_p;\\ b_1,\dots,b_q;\end{matrix}z\right]\\ &=\prod_{i=1}^{m}\left\{\frac{\Gamma(\alpha_i)\Gamma(\mu_i)}{\Gamma(a_i)\Gamma(\lambda_i)\Gamma(\alpha_i+\mu_i-a_i-\lambda_i)}\right\}\cdot\prod_{j=1}^{n}\left\{\frac{\Gamma(b_j)\Gamma(v_j)}{\Gamma(\xi_j)\Gamma(\beta_j)\Gamma(b_j+v_j-\xi_j-\beta_j)}\right\}\\ &\cdot\int_0^1\cdots\int_0^1\prod_{i=1}^{m}\left\{u_i^{\lambda_i-1}(1-u_i)^{\alpha_i+\mu_i-a_i-\lambda_i-1}\cdot{}_2F_1\left[\begin{matrix}\mu_i-a_i,\alpha_i-a_i;\\ \alpha_i+\mu_i-a_i-\lambda_i;\end{matrix}(1-u_i)\right]\right\}\\ &\cdot\prod_{j=1}^{n}\left\{v_j^{\beta_j-1}(1-v_j)^{b_j+v_j-\xi_j-\beta_j-1}\cdot{}_2F_1\left[\begin{matrix}v_j-\xi_j,b_j-\xi_j;\\ b_j+v_j-\xi_j-\beta_j;\end{matrix}(1-v_j)\right]\right\}{}_{p+m+n}F_{q+m+n}\\ &\cdot\left[\begin{matrix}\alpha_1,\dots,\alpha_m,a_{m+1},\dots,a_p,\mu_1,\dots\mu_m,v_1,\dots v_n;\\ \beta_1,\dots,\beta_n,b_{n+1},\dots,b_q,\lambda_1,\dots\lambda_m,\xi_1,\dots\xi_n;\end{matrix}zu_1\dots u_m v_1\dots v_n\right]\mathrm{d}u_1\dots\mathrm{d}u_m\,\mathrm{d}v_1\dots\mathrm{d}v_n,\end{aligned}\tag{4.9}$$

where, $m\leqslant p;\ n\leqslant q;\ Re(a_i,\lambda_i,\alpha_i+\mu_i-a_i-\lambda_i)>0,\ i=1,\dots,m;\ Re(\xi_j,\beta_j,b_j+v_j-\xi_j-\beta_j)>0,\ j=1,\dots,n;\ p\leqslant q$ and $|z|<\infty$ (or $p=q+1, z\neq 1$, and $|\arg(1-z)|<\pi$), which for $\mu_i=\lambda_i$ and $v_j=\xi_j,\ i=1,\dots,m,\ j=1,\dots,n$ gives the Erdélyi's result (1.6). It may be noted that as (1.6) is a multidimensional extension of (1.2), similarly (4.9) provides a multidimensional case for Erdélyi's extension (1.5).

5. Pochhammer contour analogues

(i) The convergence conditions for integral (1.1) have been relaxed by taking the integral round Pochhammer's double-loop contour and the result that follows is [5, p. 115(4)]; [11, p. 19, (1.6.18)];

[9, p. 58, (8)]

$$
{}_2F_1\left[\begin{matrix}\alpha,\beta;\\ \gamma;\end{matrix}\ z\right]=\frac{-\Gamma(\gamma)\mathrm{e}^{-\mathrm{i}\pi\gamma}}{4\Gamma(\beta)\Gamma(\gamma-\beta)\sin(\pi\beta)\sin\{\pi(\gamma-\beta)\}}
\times\int^{(1+,0+,1-,0-)} t^{\gamma-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,\mathrm{d}t \qquad (5.1)
$$

for all $\arg z$ and for β and $\gamma-\beta\neq 1,2,3,\ldots$ only.

In this context the method of proof outlined in the previous sections, suggest the corresponding Pochhammer contour analogues of the Erdélyi-type integrals, generalized Erdélyi's integrals and their multidimensional extensions.

For instance, the Pochhammer contour analogue of (3.7) assumes the form

$$
{}_6F_5\left[\begin{matrix}\alpha,\beta,\gamma,1+2\alpha-\beta-\gamma-\mu,\frac{1+\mu}{2},\frac{2+\mu}{2};\\ 1+\alpha-\beta,1+\alpha-\gamma,\beta+\gamma+\mu-\alpha,\frac{\lambda}{2},\frac{1+\lambda}{2};\end{matrix}\ z\right]
$$

$$
=\frac{-\Gamma(\gamma)\mathrm{e}^{-\mathrm{i}\pi\lambda}}{4\Gamma(\alpha)\Gamma(\lambda-\alpha)\sin\pi\alpha\sin\pi(\lambda-\alpha)}\int^{(1+,0+,1-,0-)} t^{\alpha-1}(1-t)^{\lambda-\alpha-1}(1-tz)^{\mu-\alpha}
$$

$$
\times\,{}_5F_4\left[\begin{matrix}\mu,1+\frac{\mu}{2},\beta+\mu-\alpha,\gamma+\mu-\alpha,1+\alpha-\beta-\gamma;\\ \frac{\mu}{2},1+\alpha-\beta,1+\alpha-\lambda,\beta+\gamma+\mu-\alpha;\end{matrix}\ zt^2\right]
$$

$$
\times\,{}_2F_1\left[\begin{matrix}\alpha-\mu,\lambda-\mu-1;\\ \lambda-\alpha;\end{matrix}\ \frac{-zt(1-t)}{1-tz}\right]\mathrm{d}t \qquad (5.2)
$$

for all $\arg z$ and for α and $\lambda-\alpha\neq 1,2,3,\ldots$ only.

This can be proved by the method explained in Section 2(i) but the proof will use (5.1) instead of (1.1) and the used summation theorems will be of the Vandermonde and Dougall. Similarly, Pochhammer contour analogues of other Erdélyi-type integrals and (4.1) can also be developed.

(ii) The Pochhammer contour analogue of Beta integral [9, p. 18]; [2, p. 214], is given by

$$
\int^{(1+,0+,1-,0-)} t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t=\frac{-4\sin(\pi\alpha)\sin(\pi\beta)}{\mathrm{e}^{-\mathrm{i}\pi(\alpha+\beta)}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (5.3)
$$

where $\alpha,\beta\neq 0,1,2,\ldots$ only.

As pointed out in [9, p. 61], the contour analogue of (4.3) can be obtained, by using (5.3). We can use the contour analogue of (4.3) to obtain the contour analogue of (4.2), which in turn can be used to develop the contour analogues of (4.4) and the multidimensional cases of Erdélyi-type integrals, on the line of Section 4(ii).

(iii) The contour analogue of (1.5) admits the form

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta;\\ \gamma;\end{matrix}\, z\right] = \frac{-\Gamma(\gamma)\Gamma(\mu)\mathrm{e}^{-\mathrm{i}\pi(\gamma+\mu-\lambda)}}{4\sin(\pi v)\sin\{\pi(\gamma+\mu-\lambda-v)\}\Gamma(\lambda)\Gamma(v)\Gamma(\gamma+\mu-\lambda-v)} \times \int^{(1+,0+,1-,0-)} t^{v-1}(1-t)^{\gamma+\mu-\lambda-v-1}\,{}_2F_1\left[\begin{matrix}\mu-\lambda,\gamma-\lambda;\\ \gamma+\mu-\lambda-v;\end{matrix}\, 1-t\right] \times {}_3F_2\left[\begin{matrix}\alpha,\beta,\mu;\\ \lambda,v;\end{matrix}\, tz\right]\mathrm{d}t \tag{5.4}$$

for all $\arg z$ and $Re(\lambda) > 0$; $v, \gamma-\lambda+\mu-v \neq 1, 2\ldots$, only.

The proof of (5.4) is on the lines of the method explained in Section 2(iii) but that will use (5.3) in place of beta integral.

Similarly, the generalization and multidimensional extensions of (5.4) can be developed on the lines of Sections 4(iii) and (iv).

6. Conclusion

In conclusion, this paper illustrate the superiority of series manipulation technique over the fractional calculus technique, particularly, in the derivations of the Erdélyi-type integrals and their various generalizations. The complete list of such integrals and details on their formation (i.e., the determination of the parameters and the combination of z and t in the integrals) are being prepared and will be communicated soon, in a separate paper. The multiple series analogues and q-series analogues of the results of this paper are also under preparation and will be announced soon.

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