Notes on Functional Analysis

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ABSTRACT. The present set of notes are written to support our students at the mathematics 4 and 5 levels.

Contents

CHAPTER 1

Introduction

Functional Analysis is a vast area within mathematics. Briefly phrased, it concerns a number of features common to the many vector spaces met in various branches of mathematics, not least in analysis. For this reason it is perhaps appropriate that the title of the topic contains the word “analysis”.

Even though the theory is concerned with vector spaces, it is not at all the same as linear algebra; it goes much beyond it. This has a very simple explanation, departing from the fact that mainly the infinite dimensional vector spaces are in focus. So, if V denotes a vector space of infinite dimension, then one could try to carry over the succesful notion from linear algebra of a basis to the infinite dimensional case. That is, we could look for families $(v_j)_{j\in J}$ in V such that an arbitrary vector $v \in V$ would be a sum

$$v = \sum \lambda_j v_j, \tag{1.0.1}$$

for some uniquely determined scalars λ_j. However, although one may add two or any finite number of vectors in V, we would need to make sense of the above sum, where the number of summands would be infinite in general. Consequently the discussion of existence and uniqueness of such decompositions of v would have to wait until such sums have been *defined*.

More specifically, this indicates that we need to define convergence of infinite series; and so it seems inevitable that we need to have a metric d on V. (One can actually make do with a topology, but this is another story to be taken up later.) But given a metric d, it is natural to let (1.0.1) mean that $v = \lim_{j\to\infty}(\lambda_1 v_1 + \dots + \lambda_j v_j)$ with respect to d.

Another lesson from linear algebra could be that we should study maps $T\colon V \to V$ that are *linear*. However, if T is such a linear map, and if there is a metric d on V so that series like (1.0.1) make sense, then T should also be linear with respect to infinite sums, that is

$$T\big(\sum \lambda_j v_j\big) = \sum \lambda_j T v_j. \tag{1.0.2}$$

This is just in order that the properties of V and T play well together. But it is a consequence, however, that (1.0.2) holds if T is merely assumed to be a *continuous*, linear map $T\colon V \to V$.

This indicates in a clear way that, for vector spaces V of infinite dimension, various objects that a priori only have an algebraic content (such as bases or linear maps) are intimately connected with topological properties (such as convergence or continuity). This link is far more important for the infinite dimensional case than, say bases and matrices are — the study of the mere connection constitutes the theory. [1]

In addition to the remarks above, it has been known (at least) since the milestone work of Stephan Banach [**Ban32**] that the continuous linear maps $V \to \mathbb{F}$ from a metric vector space V to its scalar field $\mathbb{F}$ (say $\mathbb{R}$ or $\mathbb{C}$) furnish a tremendous tool. Such maps are called *functionals* on V, and they are particularly useful in establishing the abovementioned link between algebraic and topological properties. When infinite dimensional vector spaces and their operators are studied from this angle, one speaks of *functional analysis* — not to hint at what functionals are (there isn't much to add), but rather because one analyses *by means of* functionals.

[1] When applying functional analysis to problems in, say mathematical analysis, it is often these 'connections' one needs. However, this is perhaps best illustrated with words from Lars Hörmander's lecture notes on the subject [**Hör89**]: "functional analysis alone rarely solves an analytical problem; its role is to clarify what is essential in it".

CHAPTER 2

Topological and metric spaces

As the most fundamental objects in functional analysis, the *topological* and *metric* spaces are introduced in this chapter. However, emphasis will almost immediately be on the metric spaces, so the topological ones are mentioned for reference purposes.

2.1. Rudimentary Topology

A topological space T is a set T considered with some collection τ of subsets of T, such that τ fulfils

$$T \in \tau, \qquad \emptyset \in \tau \tag{2.1.1}$$

$$\bigcap_{j=1}^{k} S_j \in \tau \text{ for } S_1, \dots, S_k \in \tau \tag{2.1.2}$$

$$\bigcup_{i \in I} S_i \in \tau \text{ for } S_i \in \tau \text{ for } i \in I; \tag{2.1.3}$$

hereby I is an arbitrary index set (possibly infinite). Such a family τ is called a *topology* on T; the topological space T is rather the pair (T, τ). A trivial example is to take $\tau = \mathscr{P}(T)$, the set of all subsets of T, which clearly satisfies the above requirements.

When τ is fixed, a subset $S \subset T$ is called an *open* set of T if $S \in \tau$; and it is said to be *closed* if the complement of S is open. The *closure*, or closed hull, of S is the smallest closed subset containing S, written $\overline{S}$. The *interior* of S, denoted S°, is the largest open set $O \subset S$. A set U is called a *neighbourhood* of a point $x \in T$ if there is some open set O such that $x \in U \subset O$. On any subset $A \subset T$ there is an induced topology $\alpha = \{ A \cap O \mid O \in \tau \}$.

In this setting, a subset $K \subset T$ is *compact*, if every open covering of K contains a finite subcovering; that is, whenever $K \subset \bigcup_{i \in I} O_i$ where $O_i \in \tau$ for every $i \in I$, then there exist some $i_1, \dots, i_N$ such that also $K \subset O_{i_1} \cup \dots \cup O_{i_N}$. In addition, T itself is a *connected* space, if it is not a disjoint union of two non-trivial open sets, that is if $T = O_1 \cup O_2$ for some O_1, $O_2 \in \tau$ implies that either $O_1 = \emptyset$ or $O_2 = \emptyset$.

The space T is called a *Hausdorff* space, and τ a Hausdorff topology, if to different points x and y in T there exists disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$ (then τ is also said to separate the points in T).

Using open sets, *continuity* of a map can also be introduced:

DEFINITION 2.1.1. For topological spaces (S,σ) and (T,τ), a map $f\colon S \to T$ is said to be continuous if $f^{-1}(O) \in \sigma$ for every $O \in \tau$.

Here $f^{-1}(O)$ denotes the preimage of O, ie $f^{-1}(O) = \{\, p \in S \mid f(p) \in O \,\}$. When f is a bijection, so that the inverse map $f^{-1}\colon T \to S$ is defined, then f is called a *homeomorphism* if both f and f^{-1} are continous with respect to the topologies σ, τ.

REMARK 2.1.2. The notion of continuity depends heavily on the considered topologies. Indeed, if $\sigma = \mathscr{P}(S)$ then every map $f\colon S \to T$ is clearly continuous; the same conclusion is valid if $\tau = \{\emptyset, T\}$.

2.2. Metric and topological concepts

Most topological spaces met in practice have more structure than just a topology; indeed, it is usually possible to measure distances between points by means of metrics:

A non-empty set M is called a metric space when it is endowed with a map $d\colon M \times M \to \mathbb{R}$ fulfilling, for every x, y and $z \in M$,

$$\text{(d1)} \quad d(x,y) \geq 0, \text{ and } d(x,y) = 0 \text{ only for } x = y, \tag{2.2.1}$$

$$\text{(d2)} \quad d(x,y) = d(y,x), \tag{2.2.2}$$

$$\text{(d3)} \quad d(x,z) \leq d(x,y) + d(y,z). \tag{2.2.3}$$

Such a map d is said to be a *metric* on M; stricly speaking the metric space is the pair (M,d). The inequality (d3) is called the triangle inequality.

The main case of interest is when the set is a vector space V on which there is a norm $\|\cdot\|$, so that d is the induced metric $d(x,y) = \|x-y\|$. However, for clarity, this chapter will review some necessary prerequisites from the theory of abstract metric spaces (but this will not in general be studied *per se*).

In a metric space (M,d) the *open ball* centered at $x \in M$, with radius $r > 0$ is the set

$$B(x,r) = \{\, y \in M \mid d(x,y) < r \,\}. \tag{2.2.4}$$

A subset $A \subset M$ is said to be open, if to every $x \in A$ there is some $r > 0$ such that $B(x,r) \subset A$ (such x are called interior points of A). It straightforward to check that every open ball is an open set and that, moreover, the collection τ of open sets is a topology as defined in Section 2.1.

By referring to the general definitions in Section 2.1, notions such as closed and compact sets in M now also have a meaning. Whether a subset $A \subset M$ is, say closed or not, this is a *topological property* of A in the sense that it may be settled as soon as one knows the open sets, that is knows the topology τ.

However, some properties are not topological, but rather *metric* in the sense that they depend on which metric M is endowed with. For example, $A \subset M$ is called *bounded* if there is some open ball $B(x,r)$ such that $A \subset$

$B(x,r)$. But it is not difficult to see that

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \tag{2.2.5}$$

in any case defines a metric on M and that the two metrics d and d' give rise to the same topology on M; but since $d'(x,y) < 1$ for all x, y it is clear that every $A \subset M$ is bounded in (M,d'). (In particular $\mathbb{R}$ is unbounded with respect to $d(x,y) = |x-y|$, while bounded with respect to $d'(x,y) = \frac{|x-y|}{1+|x-y|}$.)

Another example of a topological property is convergence of a sequence:

DEFINITION 2.2.1. A sequence (x_n) in a metric space M is *convergent* if there is some $x \in M$ for which $d(x_n,x) \to 0$ for $n \to \infty$. In this case x is called the *limit point* of (x_n), and one writes $x_n \to x$ or $x = \lim_{n\to\infty} x_n$.

Since the requirement for convergence is whether, for every $\varepsilon > 0$, it holds eventually that $x_n \in B(x,\varepsilon)$, convergence is a topological property.

Basic exercises show that the limit point of a sequence is unique (if it exists), and that every convergent sequence is a so-called fundamental sequence, or Cauchy sequence:

DEFINITION 2.2.2. In a metric space (M,d) a sequence is a Cauchy sequence if to every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d(x_n,x_m) < \varepsilon$ for all n, $m > N$.

The space (M,d) itself is said to be *complete* if every Cauchy sequence is convergent in M.

Completeness is another metric property. Eg $\mathbb{R}$ is incomplete if one uses the metric $d(x,y) = |\arctan x - \arctan y|$, which moreover gives the usual topology on $\mathbb{R}$.

For many reasons it is rather inconvenient that not all metric spaces are complete. Although this cannot be changed, there is a remedy in the fact that every metric space has a completion, as we shall see below.

In a metric space M, one subset A is said to be everywhere dense, or just *dense*, in another subset B if to every point $b \in B$ one can find points of A arbitrarily close to b; that is if every ball $B(b,\delta)$ has a non-empty intersection with A. Rephrasing this one has

DEFINITION 2.2.3. For subsets A and B of M, one calls A dense in B if $B \subset \overline{A}$.

As an example, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in one another; notice that these sets are disjoint and that the definition actually allows this.

By abuse of language, a sequence (x_n) in M is called dense if its range $\{x_n \mid n \in \mathbb{N}\}$ is dense in M.

The notion of denseness is very important: for example it allows

DEFINITION 2.2.4. A completion of a metric space (M,d) is a pair (M',T) consisting of a complete metric space (M',d') and an isometry T of M onto a dense subspace $T(M)$ of M'.

In view of the next theorem, one speaks about *the* completion of M.

THEOREM 2.2.5. *To any metric space (M,d) there exists a completion, and it is uniquely determined up to isometry.*

PROOF. Let C be the vector space of continuous bounded maps $M \to \mathbb{R}$. This is complete with the metric $\sup_M |f-g|$.

To get a map $M \to C$, one can set $F_x(y) = d(x,y) - d(m,y)$ for a fixed $m \in M$. Indeed, continuity on M of F_x follows from that of the metric, while boundedness results from the triangle inequality,

$$|F_x(y)| = |d(x,y) - d(m,y)| \le d(x,m). \tag{2.2.6}$$

Similarly any x, y, $z \in M$ gives

$$|F_x(y) - F_z(y)| = |d(x,y) - d(z,y)| \le d(x,z), \tag{2.2.7}$$

so $\sup |F_x - F_z| = d(x,z)$; equality follows for $y = z$. Therefore $\Phi(x) = F_x$ is isometric, and if M' is defined to be the closure of $\{ F_x \mid x \in M \}$, it is clear that $\Phi(M)$ is dense in M', so that this is a completion. By composing with Φ^{-1}, any completion is isometric to M'. □

A metric space M is called *separable* if there is a dense sequence of points $x_n \in M$. This is a rather useful property enjoyed by most spaces met in applications, hence the spaces will be assumed separable whenever convenient in the following.

2.3. An example of density: uniform approximation by polynomials

If $\mathscr{P}$ denotes the set of polynomials on the real line, consider then the question whether a given continuous function $f\colon [a,b] \to \mathbb{C}$, on a compact interval $[a,b]$, can be uniformly approximated on $[a,b]$ by polynomials; ie, does there to every $\varepsilon > 0$ exist a $p \in \mathscr{P}$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a,b]$.

Using the sup-norm on $C([a,b])$, this classic question amounts to whether $\mathscr{P}$ (or rather the restrictions to $[a,b]$) is dense in the Banach space $C([a,b])$.

THEOREM 2.3.1. *The polynomials are dense in $C([a,b])$.*

This is the Weierstrass approximation theorem.

PROOF. By means of an affine transformation, $y = a + x(b-a)$, it suffices to treat the case $[a,b] = [0,1]$. So let f be given in $C([0,1])$, $\varepsilon > 0$.

Consider then the so-called Bernstein polynomials associated with f,

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k}. \tag{2.3.1}$$

Since $1 = (x+(1-x))^n$ the binomial formula gives

$$f(x) - p_n(x) = \sum_{k=0}^{n} (f(x) - f(k/n)) \binom{n}{k} x^k (1-x)^{n-k}. \tag{2.3.2}$$

By uniform continuity there is some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$ for $0 \le x, y \le 1$. Taking out the terms of the above sum for which k is such that $|x - \frac{k}{n}| < \delta$,

$$|f(x) - p_n(x)| \le \varepsilon + \sum_{|x-\frac{k}{n}| \ge \delta} |f(x) - f(k/n)| \binom{n}{k} x^k (1-x)^{n-k}. \tag{2.3.3}$$

With $M = \sup |f|$, insertion of $1 \le \frac{|x-\frac{k}{n}|^2}{\delta^2}$ in the sum yields

$$|f(x) - p_n(x)| \le \varepsilon + \tfrac{2M}{n^2\delta^2} \sum_{k=0}^{n} |k - xn|^2 \binom{n}{k} x^k (1-x)^{n-k}. \tag{2.3.4}$$

Because the variance of the binomial distribution is $nx(1-x)$, this entails

$$\sup |f - p| \le \varepsilon + \tfrac{2M}{4n\delta^2}. \tag{2.3.5}$$

So by taking $n > M/(\varepsilon\delta^2)$, the conclusion $\|f - p_n\| < 2\varepsilon$ follows. $\square$

CHAPTER 3

Banach spaces

Recall that a family $(v_j)_{j\in J}$ of vectors in a space V is said to be linearly independent, if every finite subfamily $(u_{j_1},\dots,u_{j_n})$ has the familiar property that the equation

$$0=\lambda_1 u_{j_1}+\dots+\lambda_n u_{j_n} \tag{3.0.6}$$

only has the trivial solution $0=\lambda_1=\dots=\lambda_n$.

The vector space V itself is said to have *infinite dimension*, if for every $n\in\mathbb{N}$ there exists n linearly independent vectors in V. In this case one writes $\dim V=\infty$ (regardless of how 'many' linearly independent vectors there are); the study of such 'wild' spaces is a key topic in functional analysis.

As an example, $\dim C(\mathbb{R})=\infty$, for the family of 'tent functions' is uncountable and linearly independent: the functions $f_k(x)$ that, with $k\in\mathbb{R}$ as a parameter, grow linearly from 0 to 1 on $[k-\frac{1}{3},k]$ and decrease linearly to 0 on $[k,k+\frac{1}{3}]$, with the value 0 outside of $[k-\frac{1}{3},k+\frac{1}{3}]$, are linearly independent because only $\lambda_1=\dots=\lambda_n=0$ have the property that

$$\lambda_1 f_{k_1}(x)+\dots+\lambda_n f_{k_n}(x)=0 \quad\text{for all}\quad x\in\mathbb{R}. \tag{3.0.7}$$

For $k_1,\dots k_n\in\mathbb{Z}$ this is clear since $f_{k_1},\dots,f_{k_n}$ have disjoint supports then. Generally, when the k_m are real, the claim follows by considering suitable values of x (supply the details!).

Notice that $\dim V=\infty$ means precisely that the below set has no majorants in $\mathbb{R}$:

$$\mathscr{N}=\{\,n\in\mathbb{R}\mid V\text{ contains a linearly independent }n\text{-tupel.}\,\}. \tag{3.0.8}$$

By definition V is *finite-dimensional* (or has finite dimension) if the above set $\mathscr{N}$ is upwards bounded. In any case, *the* dimension of V is defined as

$$\dim V=\sup\mathscr{N}. \tag{3.0.9}$$

Recall from linear algebra that finite-dimensional spaces V and W over the same field are isomorphic if and only if $\dim V=\dim W$. The proof of this non-trivial result relies on suitable choises of bases.

The general concept of bases brings us back to the questions mentioned in the introduction, so it is natural to let V be normed now.

DEFINITION 3.0.2. In a normed vector space V, a sequence (u_n), which may be finite, is a *basis* if for every $x\in V$ there is a unique sequence (λ_n) of scalars in $\mathbb{F}$ such that $x=\sum\lambda_n u_n$.

In the definition of a basis U, uniqueness of the expansions clearly implies that U is a linearly independent family. So if $\dim V < \infty$, every basis is finite, and the expansions $x = \sum \lambda_n u_n$ are consequently finite sums; hence the notion of a basis is just the usual one for finite dimensional spaces. For the infinite dimensional case the term Schauder basis is also used.

A subset $W \subset V$ is called *total* if $\overline{\operatorname{span}}\, W = V$. Clearly any basis U is a total set.

A normed space V is said to be *separable* if there is a sequence (v_n) with dense range, ie $V \subset \overline{\{ v_n \mid n \in \mathbb{N} \}}$. It is straightforward to see that V is separable, if V has a basis (use density of $\mathbb{Q}$ in $\mathbb{R}$). For simplicity we shall stick to separable spaces in the sequel (whenever convenient).

EXAMPLE 3.0.3. For every p in $[1,\infty[$ the sequence space ℓ^p has the canonical basis (e_n) with

$$e_n = (0,\dots,\underbrace{0,1,0}_{n\text{th entry}},\dots). \tag{3.0.10}$$

This is evident from the definition of basis. ℓ^∞ does not have a basis because it is unseparable.

REMARK 3.0.4. It is not clear whether a total sequence $U = (u_n)$ will imply the existence of expansions as in the definition of a basis: given $x \in V$, there is some (s_n) in $\operatorname{span} U$ converging to x, hence $x = \sum_{n=1}^\infty y_n$ with $y_n = s_n - s_{n-1}$ (if $s_0 = 0$); here $y_n \in \operatorname{span} U$, so $y_j = \alpha_{j,1}u_{j,1} + \dots + \alpha_{j,n_j}u_{j,n_j}$ with $u_{j,m} \in U$ for every $j \in \mathbb{N}$ and $m = 1,\dots,n_j$. By renumeration one is lead to consideration of the series $\sum_{j=1}^\infty \alpha_n u_n$, from which $\sum y_n$ is obtained by introduction of parentheses; the convergence of $\sum \alpha_n u_n$ is therefore unclear.

However, it would be nice if denseness (viz. $V = \overline{\operatorname{span}}\, U$) would be the natural replacement for the requirement, in the finite dimensional case, that $\operatorname{span} U = V$.

REMARK 3.0.5 (Hamel basis). There is an alternative notion of a basis of an arbitrary vector space V over $\mathbb{F}$: a family $(v_i)_{i\in I}$ is a Hamel basis if every vector has a unique representation as a *finite* linear combination of the v_i, that is, if every $v \in V$ has a unique expansion

$$v = \sum_{i\in I} \lambda_i v_i, \quad \text{with } \lambda_i \neq 0 \text{ for only finitely many } i. \tag{3.0.11}$$

While (also) this coincides with the basis concept for finite-dimensional spaces, it is in general rather difficult to show that a vector space has a Hamel basis. In fact the difficulties lie at the heart of the foundations of mathematics; phrased briefly, one has to use transfinite induction (eg Zorn's lemma) to prove the existence.

It is well known that the existence of a Hamel basis has startling consequences. One such is when $\mathbb{R}$ is considered as a vector space over the field of rational numbers, $\mathbb{Q}$. Clearly $v = 1$ is then not a basis, for $\sqrt{2} = \lambda v$ does not hold for any $\lambda \in \mathbb{Q}$; but there exists a Hamel basis $(v_i)_{i\in I}$ in $\mathbb{R}$, whence

(3.0.11) holds for every $v \in \mathbb{R}$ with rational scalars λ_i. By the uniqueness, there are $\mathbb{Q}$-linear maps $p_i \colon \mathbb{R} \to \mathbb{Q}$ given by $p_i(v) = \lambda_i$. In particular they solve the *functional equation*

$$f(\lambda x) = \lambda f(x) \quad \text{for all} \quad x \in \mathbb{R}, \lambda \in \mathbb{Q}, \tag{3.0.12}$$

and every p_i is a *discontinuous* function $\mathbb{R} \to \mathbb{R}$, since a continuous function on $\mathbb{R}$ has an interval as its image.

Moreover, with $a = f(1)$, clearly any solution fulfils $f(\lambda) = a\lambda$ for $\lambda \in \mathbb{Q}$. Since $\mathbb{R} = \overline{\mathbb{Q}}$, every continuous solution to the functional equation is a scaling $x \mapsto ax$; these are not just continuous but actually C^∞. So it is rather striking how transfinite induction gives rise to an abundance of other solutions, that are effectively outside of the class of continuous functions $C(\mathbb{R}, \mathbb{R})$. However, it should be emphasised that no-one is able to write down expressions for these more general solutions.

Notes. An exposition on Schauder bases may be found in [**You01**]. Schauder's definition of a basis was made in 1927, and in 1932 Banach raised the question whether every Banach space has a basis. This was, however, first settled in 1973 by Per Enflo, who gave an example of a separable Banach space without any basis.

CHAPTER 4

Hilbert spaces

The familiar spaces $\mathbb{C}^n$ may be seen as subspaces of the infinite dimensional space $\ell^2(\mathbb{N})$ of square-summable sequences (by letting the sequences consist only of zeroes from index $n+1$ onwards). The space $\ell^2(\mathbb{N})$ has many geometric properties in common with $\mathbb{C}^n$ because it is equipped with the inner product $\sum x_n \bar{y}_n$. It is therefore natural to study infinite dimensional vector spaces with inner products; this is the theory of *Hilbert spaces* which is developed in this chapter. However, a separable Hilbert space is always isomorphic to ℓ^2 or $\mathbb{C}^n$, as we shall see. In addition a firm basis for manipulation of coordinates is given, including Bessel's inequality and Parseval's identity. We shall also later in Chapter 5 verify that all this applies to Fourier series of functions in $L^2(-\pi,\pi)$.

4.1. Inner product spaces

Before Hilbert spaces can be defined, it is necessary to introduce the concept of an *inner product*. This is a generalisation of the scalar product from linear algebra.

DEFINITION 4.1.1. An inner product on a vector space V is a map $(\cdot\,|\,\cdot)$ from $V\times V$ to $\mathbb{C}$ which for all x, y, $z\in V$, all λ, $\mu\in\mathbb{F}$ fulfills

(i) $(\lambda x+\mu y\,|\,z)=\lambda(x\,|\,z)+\mu(y\,|\,z)$;
(ii) $(x\,|\,y)=\overline{(y\,|\,x)}$;
(iii) $(x\,|\,x)\geq 0$, with $(x\,|\,x)=0$ if and only if $x=0$.

The pair $(V,(\cdot\,|\,\cdot))$ is called an *inner product space*.

Notice that $(x\,|\,\lambda y+\mu z)=\overline{\lambda}(x\,|\,z)+\overline{\mu}(x\,|\,y)$ is a consequence of the first two conditions.

Thus an inner product is an example of a *sesqui-linear* form on V: this is an arbitrary map $V\times V\to\mathbb{F}$, which is linear in the first and conjugate linear in the second variable.

For a sesqui-linear form $s(\cdot,\cdot)$ one has the *polarisation identities*, that express that s is determined by its values on the diagonal:

$$s(x,y)=\frac{1}{4}\sum_{k=0,\dots,3}\mathrm{i}^k s(x+\mathrm{i}^k y,x+\mathrm{i}^k y)\qquad\text{for }\mathbb{F}=\mathbb{C}\tag{4.1.1}$$

$$s(x,y)=\frac{1}{4}s(x+y,x+y)-\frac{1}{4}s(x-y,x-y)\qquad\text{for }\mathbb{F}=\mathbb{R}.\tag{4.1.2}$$

These are verified by using sesqui-linearity on the right hand sides.

Moreover, (i),(ii) and (iii) imply that for $x,\ y \in V$ and $\lambda \in \mathbb{F}$,

$$\begin{aligned} 0 &\le (\,x+\lambda y\,|\,x+\lambda y\,) \\ &= (\,x\,|\,x\,) + \lambda(\,y\,|\,x\,) + \overline{\lambda}(\,x\,|\,y\,) + \lambda\overline{\lambda}(\,y\,|\,y\,). \end{aligned} \tag{4.1.3}$$

Notice that for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{F}^n$ there is the familiar expression,

$$(\,x\,|\,y\,) = x_1\overline{y_1} + \cdots + x_n\overline{y_n}. \tag{4.1.4}$$

Usually this will define the inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$.

EXAMPLE 4.1.2. As a less obvious inner product space, one may consider the space $C_0(\mathbb{R}^n)$ is endowed with

$$(\,f\,|\,g\,) = \int_{\mathbb{R}^n} f(x)\overline{g(x)}m(x)\,dx. \tag{4.1.5}$$

4.1.1. Identities and inequalities for inner products. A fundamental fact about inner products is that they give rise to a norm on the vector space. This is made precise in

DEFINITION 4.1.3. On a vector space V with inner product $(\,\cdot\,|\,\cdot\,)$ the *induced* norm on V is given as

$$\|x\| = \sqrt{(\,x\,|\,x\,)}. \tag{4.1.6}$$

The definition is permissible, for the square root makes sense because of condition (iii); whence (4.1.6) may be used as a short-hand. Moreover, the two first conditions for a norm are trivial to verify (do it!), but the triangle inequality could deserve an explanation.

However, it turns out that there are *three* fundamental facts about inner products which are based on (4.1.3). Indeed, both the triangle inequality, the Cauchy–Schwarz inequality and a vector version of how to "square the sum of two terms" result from this:

PROPOSITION 4.1.4. *For a vector space V with inner product $(\,\cdot\,|\,\cdot\,)$, the following relations hold for arbitrary $x,\ y \in V$:*

$$\text{(i)} \quad \|x+y\| \le \|x\| + \|y\| \tag{4.1.7}$$

$$\text{(ii)} \quad |(\,x\,|\,y\,)| \le \|x\|\,\|y\| \tag{4.1.8}$$

$$\text{(iii)} \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\,x\,|\,y\,). \tag{4.1.9}$$

PROOF. Clearly $\lambda = 1$ in (4.1.3) yields (iii). For (ii) we may assume $y \neq 0$; then $\lambda = -\frac{(\,x\,|\,y\,)}{(\,y\,|\,y\,)}$ in (4.1.3) yields $|(\,x\,|\,y\,)|^2 \le (\,x\,|\,x\,)(\,y\,|\,y\,)$, ie (ii).

Now (ii) gives $\mathrm{Re}(\,x\,|\,y\,) \le |(\,x\,|\,y\,)| \le \|x\|\,\|y\|$, so (iii) entails $\|x+y\|^2 \le (\|x\| + \|y\|)^2$, whence (i) holds. □

In view of (i) in this proposition, Definition 4.1.3 has now been justified. For simplicity, given an inner product space V, the symbols $(\,\cdot\,|\,\cdot\,)$ and $\|\cdot\|$ will often be used, without further notification, to denote the inner product and the induced norm on V, respectively. Eg the *polarisation identity* takes the form, in case $\mathbb{F}=\mathbb{C}$,

$$(\,x\,|\,y\,)=\tfrac{1}{4}\sum_{k=0}^{3}\mathrm{i}^k\,\|x+\mathrm{i}^k y\|^2. \tag{4.1.10}$$

Replacing y by $-y$ in (4.1.9) and adding the resulting formula to (4.1.9) itself, one obtains the *parallellogram law*:

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2). \tag{4.1.11}$$

Conversely, if a norm $\|\cdot\|$ on a vector space V fulfils this, one can show that $(\,x\,|\,y\,)$ *defined* by the expression on the right hand side of (4.1.10) actually is an inner product on V, that moreover induces the given norm $\|\cdot\|$.

REMARK 4.1.5. From the use of (4.1.3) in the proof, it is evident that equality in Cauchy–Schwarz' inequality holds if and only if x and y are proportional. In the triangle inequality (i), equality $\|x+y\|=\|x\|+\|y\|$ implies $\operatorname{Re}(\,x\,|\,y\,)=\|x\|\,\|y\|\geq|(\,x\,|\,y\,)|$, so therefore either $x=\lambda y$ or $y=\lambda x$ and $(\,x\,|\,y\,)=|(\,x\,|\,y\,)|$; then $\lambda=|\lambda|\geq 0$, ie the factor λ is *positive*.

EXAMPLE 4.1.6. Using the above, it is now easy to show the classical fact from geometry, that if a map $T\colon\mathbb{R}^n\to\mathbb{R}^n$ is an *isometry*, ie

$$\|T(x)-T(y)\|=\|x-y\|\quad\text{for all}\quad x,y\in\mathbb{R}^n, \tag{4.1.12}$$

then T is *affine*; ie $T(x)=Ax+b$ for some orthogonal matrix A, $b\in\mathbb{R}^n$.

Indeed, as $T(x)-T(0)$ is isometric too, $b:=T(0)=0$ can be assumed. Then $y=0$ yields that T is norm preserving, ie $\|T(x)\|=\|x\|$ for all x. T is also inner product preserving, for a calculation of both sides of (4.1.12) by means of (4.1.9) gives

$$(\,T(x)\,|\,T(y)\,)=(\,x\,|\,y\,)\quad\text{for all}\quad x,y\in\mathbb{R}^n. \tag{4.1.13}$$

So for the natural basis $(e_1,\dots,e_n)$ one has $(\,T(e_j)\,|\,T(e_k)\,)=\delta_{jk}$, whence $(T(e_1),\dots,T(e_n))$ is another orthonormal basis. Writing $T(x)=\sum\lambda_jT(e_j)$ it follows by taking inner products with $T(e_k)$ that $\lambda_k=(\,T(x)\,|\,T(e_k)\,)$, so

$$T(x)=\sum_{j=1,\dots,n}(\,T(x)\,|\,T(e_j)\,)T(e_j)=\sum_{j=1,\dots,n}(\,x\,|\,e_j\,)T(e_j). \tag{4.1.14}$$

The last expression is linear in x, so that $T(x)=Ax$ for an $n\times n$-matrix A. Here (4.1.13) gives $A^{\mathrm{t}}A=I$, hence $A^{\mathrm{t}}=A^{-1}$ as desired.

4.1.2. Continuity of inner products. In analogy with the fact that a norm always is continuous, it holds for every inner product that it is jointly continuous in both variables, that is, continuous as a map

$$(\,\cdot\,|\,\cdot\,)\colon\ \begin{matrix}V\\ \times\\ V\end{matrix}\longrightarrow\mathbb{C}. \tag{4.1.15}$$

Hereby $V\times V$ is normed by $\|(x,y)\|=(\|x\|^2+\|y\|^2)^{1/2}$.

PROPOSITION 4.1.7. *In an inner product space V it holds for any pair of sequences converging with respect to the induced norm, say $x_n \to x$ and $y_n \to y$ in V, that*

$$(x_n \,|\, y_n) \to (x \,|\, y) \quad \textit{for} \quad n \to \infty. \tag{4.1.16}$$

PROOF. By the triangle and Cauchy–Schwarz inequalities,

$$\begin{aligned} &|(x_n \,|\, y_n) - (x \,|\, y)| \\ &\qquad \le |(x_n - x \,|\, y_n - y)| + |(x \,|\, y_n - y)| + |(x_n - x \,|\, y)| \\ &\qquad \le \|x_n - x\| \, \|y_n - y\| + \|x\| \, \|y_n - y\| + \|x_n - x\| \, \|y\|. \end{aligned} \tag{4.1.17}$$

Here all terms on the right hand side goes to 0, so (4.1.16) follows. □

It is clear that the Cauchy–Schwarz inequality is crucial for the above result. In addition it will be seen in Proposition 4.3.1 below that also the parallellogram law has rather striking consequences.

4.1.3. Orthogonality. In an inner product space V, the vectors x and y are called *orthogonal* if $(x \,|\, y) = 0$; this is symbolically written $x \perp y$.

From (4.1.9) one can now read off Pythagoras' theorem (indeed, a generalisation to the infinite dimensional case) :

PROPOSITION 4.1.8. *If $x \perp y$ for two vectors x, y in an inner product space V, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

For subsets M and N of an inner product space V, one says that M, N are orthogonal, written $M \perp N$ if $(x \,|\, y) = 0$ for every $x \in M$, $y \in N$. In addition the *orthogonal complement* of such M is defined as

$$M^\perp = \{\, y \in V \mid \forall x \in M \colon \, (x \,|\, y) = 0 \,\}. \tag{4.1.18}$$

Clearly $M^\perp$ is a subspace of V, and *closed* by Proposition 4.1.7: if $y_n \to y$ and $y_n \in M^\perp$, then any $x \in M$ yields $(y \,|\, x) = \lim (y_n \,|\, x) = 0$, so $y \perp M$. Analogously it is seen that $\overline{M}^\perp = M^\perp$.

Clearly $M \subset M^{\perp\perp}$, so that $M^{\perp\perp}$ is a closed subspace containing M. In fact, $M^{\perp\perp}$ is the smallest such set, the so-called closed linear hull of M. As first step towards this, note that $M_1 \subset M_2$ implies $M_1^\perp \supset M_2^\perp$. Then, if $M \subset X$ for some closed subspace $X \subset H$, one has $M \subset M^{\perp\perp} \subset X^{\perp\perp} = X$. Here the last identity follows from the fundamental Projection Theorem proved further below.

As another main example, note that

$$V^\perp = \{\, 0 \,\}. \tag{4.1.19}$$

Indeed, $0 \in V^\perp$, and if $z \in V^\perp$ then $(z \,|\, z) = 0$, whence $z = 0$. This fact is used repeatedly (as a theme in proofs) in the following.

Finally, a family $(u_j)_{j \in J}$ in an inner product space V is called an *orthogonal family* provided

$$(u_j \,|\, u_k) = 0 \text{ for } j \ne k \text{ and } u_j \ne 0 \text{ for every } j \in J. \tag{4.1.20}$$

The same terminology applies to a sequence, for this is the case with $J = \mathbb{N}$.

4.2. Hilbert spaces and orthonormal bases

To get a useful generalisation of the Euclidean spaces $\mathbb{R}^n$, that are complete with respect to the metric induced by the inner product, Hilbert spaces are defined as follows:

DEFINITION 4.2.1. A vector space H with inner product is called a *Hilbert space* if it complete with respect to the induced norm.

In particular all Hilbert spaces are Banach spaces. As an example one has $H = \ell^2$ endowed with the inner product

$$(\,(x_n)\,|\,(y_n)\,) = \sum_{n=1}^{\infty} x_n\overline{y_n} \quad \text{for} \quad (x_n),(y_n) \in \ell^2. \tag{4.2.1}$$

Notice that the series converges because $|x_n\overline{y_n}| \le \frac{1}{2}(|x_n|^2 + |y_n|^2)$. The completeness may be verified directly (try it!).

In the rest of this chapter focus will be on Hilbert spaces, for the completion of an inner product space may be shown to have the structure of a Hilbert space, because the inner product extends to the completion in a unique way.

For convenience, it will also often be assumed that the Hilbert spaces are separable. This will later have the nice consequence, that an orthonormal basis will be at most countable.

DEFINITION 4.2.2. An *orthonormal basis* of a Hilbert space is a basis $(e_j)_{j\in J}$ which is also an orthonormal set, that is, which also satisfies $(\,e_j\,|\,e_k\,) = \delta_{jk}$ for all $j,\,k \in J$.

For a an orthonormal family $(e_j)_{j\in J}$ to be a basis it is by Definition 3.0.2 required that every $x \in H$ can be written as a convergent series $x = \sum_{j\in J}\lambda_j e_j$ for uniquely determined scalars λ_j. It will follow later that it suffices, in addition to the orthogonality, that the family $(e_j)_{j\in J}$ is total.

In case $J = \mathbb{N}$ the uniqueness is a consequence of (4.2.4):

PROPOSITION 4.2.3. *Let (e_n) be an orthonormal sequence in a Hilbert space H, and let (λ_n) be a sequence in $\mathbb{F}$. Then*

$$\sum_{n=1}^{\infty} \lambda_n e_n \quad \textit{converges in } H \iff \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty. \tag{4.2.2}$$

In the affirmative case, with $x := \sum \lambda_n e_n$,

$$\|x\| = (\sum_{n=1}^{\infty} |\lambda_n|^2)^{1/2} \tag{4.2.3}$$

$$\lambda_n = (\,x\,|\,e_n\,) \quad \textit{for every } n \in \mathbb{N}. \tag{4.2.4}$$

PROOF. Setting $s_n = \sum_{j=1}^{n}\lambda_j e_j$, Pythagoras' theorem implies

$$\|s_{n+p} - s_n\|^2 = \sum_{j=n+1}^{n+p} \|\lambda_j e_j\|^2 = \sum_{j=n+1}^{n+p} |\lambda_j|^2. \tag{4.2.5}$$

Therefore (s_n) is fundamental precisely when $\sum|\lambda_n|^2$ is a convergent series. And in this case, continuity of the norm and the inner product yields

$$\|x\| = \lim_{n\to\infty}\|s_n\| = \lim_{n\to\infty}(\sum_{j=1}^{n}|\lambda_j|^2)^{1/2} = (\sum_{n=1}^{\infty}|\lambda_n|^2)^{1/2} \tag{4.2.6}$$

$$(\,x\,|\,e_n\,) = \lim_{k\to\infty}(\,\sum_{j=1}^{k}\lambda_j e_j\,|\,e_n\,) = \lambda_n. \tag{4.2.7}$$

□

Frequently, the proposition is also useful for cases with only finitely many scalars λ_n; the trick is then to add infinitely many zeroes to obtain a sequence (λ_n). This observation is convenient for the proof of

PROPOSITION 4.2.4 (Bessel's inequality). *Let (e_n) be an orthonormal sequence in a Hilbert space H. For all $x\in H$ and $n\in\mathbb{N}$,*

$$\|x-\sum_{j=1}^{n}(\,x\,|\,e_j\,)e_j\|^2 = \|x\|^2-\sum_{j=1}^{n}|(\,x\,|\,e_j\,)|^2 \tag{4.2.8}$$

$$\sum_{j=1}^{\infty}|(\,x\,|\,e_j\,)|^2 \le \|x\|^2, \tag{4.2.9}$$

and the series $\sum_{j=1}^{\infty}(\,x\,|\,e_j\,)e_j$ converges in H.

PROOF. Using (4.1.9), the first claim is a direct consequence of (4.2.3), for if $x_n = \sum_{j=1}^{n}(\,x\,|\,e_j\,)e_j$,

$$\|x-x_n\|^2 = \|x\|^2+\|x_n\|^2-2\,\mathrm{Re}(\,x\,|\,x_n\,) = \|x\|^2-\sum_{j=1}^{n}|(\,x\,|\,e_j\,)|^2. \tag{4.2.10}$$

Since the left hand side is non-negative, $\sum_{j=1}^{n}|(\,x\,|\,e_j\,)|^2\le\|x\|^2$ for each n, whence (4.2.9). The convergence of the series is then a consequence of Proposition 4.2.3. □

Whether equality holds in Bessel's inequality (4.2.9) for all vectors x in H or not, this depends on whether the given orthonormal sequence (e_n) contains enough vectors to be a basis or not; cf (iii) in

THEOREM 4.2.5. *For an orthonormal sequence (e_n) in a Hilbert space H the following properties are equivalent:*

(i) *(e_n) is an orthonormal basis for H.*

(ii) $(\,x\,|\,y\,) = \sum_{n=1}^{\infty}(\,x\,|\,e_n\,)(\,e_n\,|\,y\,)$ *for all x, y in H.*

(iii) $\|x\|^2 = \sum_{n=1}^{\infty}|(\,x\,|\,e_n\,)|^2$ *for all x in H.*

(iv) *If x in H is such that $(\,x\,|\,e_n\,)=0$ for all $n\in\mathbb{N}$, then $x=0$.*

In the affirmative case $x=\sum_{n=1}^{\infty}(\,x\,|\,e_n\,)e_n$ holds for every $x\in H$.

PROOF. Notice that when (e_n) is an orthonormal basis, then the last statement is true because the basis property shows that $x = \sum \lambda_n e_n$ holds; then $\lambda_n = (x \,|\, e_n)$ by Proposition 4.2.3.

Now (i) implies (ii) by the continuity of $(\cdot \,|\, y)$. Moreover, (iii) is a special case of (ii), and (iv) is immediate from (iii), since $\|x\| = 0$ only holds for $x = 0$. Given that (iv) holds, one can for any x consider $y = \sum_{n=1}^{\infty} (x \,|\, e_n) e_n$, which converges by Proposition 4.2.3 and Bessel's inequality. But then $(x - y \,|\, e_n) = 0$ is seen for every $n \in \mathbb{N}$ by substitution of y; hence $x = y$. Therefore every $x = \sum_{n=1}^{\infty} \lambda_n e_n$ with the coefficients uniquely determined by $\lambda_n = (x \,|\, e_n)$ according to Proposition 4.2.3, so (i) holds. □

The identity in (iii) is known as Parseval's equation (especially in connection with Fourier series). Notice that in the affirmative case, (ii) expresses that the inner product $(x \,|\, y)$ may be computed from the coordinates of x, y, for since $y = \sum y_n e_n$ with $y_n = (y \,|\, e_n)$ and similarly for x, the identity in (ii) amounts to

$$(x \,|\, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n. \tag{4.2.11}$$

It is not a coincidence that the right hand side equals the inner product in ℓ^2 of the coordinate sequences (x_n), (y_n) — these are clearly in ℓ^2 because of (iii). Indeed, this fact leads to the proof of the next result.

To formulate it, an operator $U\colon H_1 \to H_2$, where H_1, H_2 are Hilbert spaces, will be called *unitary* when U is a linear bijection fulfilling

$$(Ux \,|\, Uy) = (x \,|\, y) \quad \text{for all } x,\, y \in H_1. \tag{4.2.12}$$

THEOREM 4.2.6. *Every separable Hilbert space H has an orthonormal basis $(e_j)_{j \in J}$ with index set $J \subset \mathbb{N}$; and the corresponding map $x \mapsto ((x \,|\, e_j))_{j \in J}$ is a unitary operator from H onto $\ell^2(J)$.*

Observe that $\ell^2(J)$ is either ℓ^2 or $\mathbb{F}^n$; the latter possibility occurs if J is finite, for by a renumeration $J = \{1, \dots, n\}$ may be obtained.

PROOF. Let (v_n) be dense in H; then $V := \operatorname{span}(v_n)$ is dense in H. By extracting a subsequence, one obtains a family $(v_j)_{j \in J}$, with $J \subset \mathbb{N}$, such that $v_j \notin \operatorname{span}(v_1, \dots, v_{j-1})$ for every $j \in J$ and $V = \operatorname{span}(v_j)_{j \in J}$. Using Gram–Schmidt orthonormalisation, there is a family (e_j) with $V = \operatorname{span}(e_j)_{j \in J}$. It follows that $(e_j)_{j \in J}$ is an orthonormal basis for H, for if $x \in H$ is orthogonal to every e_j, then $x \in V^\perp = H^\perp = (0)$, so that (iv) in Theorem 4.2.5 is fulfilled. (The proof of (iv) $\Longrightarrow$ (i) applies *verbatim* when J is finite.)

The operator $U\colon H \to \ell^2(J)$ given by $Ux = ((x \,|\, e_j))_{j \in J}$ is linear and injective (for $J = \mathbb{N}$ this is because of (iii)). It is also surjective because any $(\alpha_j) \in \ell^2(J)$ gives rise to the vector $x = \sum_J \alpha_j e_j$ in H by Proposition 4.2.3;

by continuity of the inner product $Ux = (\alpha_j)$ clearly holds. Finally it follows from (ii) that for all x, $y \in H$,

$$(x\,|\,y) = \sum_J (x\,|\,e_j)(e_j\,|\,y) = (Ux\,|\,Uy). \tag{4.2.13}$$

Hence U is unitary as claimed. □

Note that (iii) expresses that $\|Ux\| = \|x\|$ holds for the map U in the above proof, ie that U is norm-preserving, so U is clearly a homeomorphism. As (4.2.13) shows, it also preserves inner products, so one cannot distinguish the Hilbert spaces H and $\ell^2(J)$ from one another (two vectors are orthogonal in H if and only if their images are so in $\ell^2(J)$ and so on).

Generalising from this, it is seen that the unitary operators constitute the natural class of isomorphisms on the set of Hilbert spaces; two Hilbert spaces H_1, H_2 are also called *unitarily equivalent* if there is an isomorphism (ie a unitary operator) from H_1 onto H_2. And Theorem 4.2.6 gives

COROLLARY 4.2.7. *Two separable Hilbert spaces H_1 and H_2 over $\mathbb{F}$ are unitarily equivalent if and only if they both have orthonormal bases indexed by $\mathbb{N}$ (or by $\{1,\dots,n\}$ for some $n \in \mathbb{N}$). In particular all orthonormal bases of a separable Hilbert space have the same index set.*

From the proof of Theorem 4.2.6 one can read off the next criterion.

COROLLARY 4.2.8. *If H is a Hilbert space and $(e_j)_{j\in J}$ is a countable orthonormal family, then $(e_j)_{j\in J}$ is an orthonormal basis if it is* total.

4.3. Minimisation of distances

It is a crucial geometric property of a Hilbert space H that for any subspace U of finite dimension and any $x \in H$ there exists a uniquely determined point $u_0 \in U$ with the least possible distance to x. Ie this u_0 fulfils

$$\|x - u_0\| = \inf\{\, \|x-u\| \mid u \in U\,\}. \tag{4.3.1}$$

Since the infimum exists and is ≥ 0, the crux is that it actually is *attained* at a certain point u_0 (hence is a minimum).

To see that u_0 exists, it suffices to take an orthonormal basis for U, say $(e_1,\dots,e_n)$ and verify that for arbitrary $\lambda_1,\dots,\lambda_n$ in $\mathbb{F}$,

$$\|x - \sum_{j=1}^{n} (x\,|\,e_j)e_j\| \leq \|x - \sum_{j=1}^{n} \lambda_j e_j\|. \tag{4.3.2}$$

Indeed, one can then let $u_0 = \sum_{j=1}^{n} (x\,|\,e_j)e_j$, for this belongs to U and clearly minimises the distance to x. It is straightforward to prove (4.3.2) in the same manner as in the proof of (4.2.8) above.

However, since a finite dimensional subspace always is closed (cf the below Lemma 6.2.1) and moreover convex, the above result also follows from the next result. Recall that $C \subset H$ is convex if $\theta x + (1-\theta)y \in C$ for every $\theta \in [0,1]$ and all x, $y \in C$.

PROPOSITION 4.3.1. *Let C be a closed, convex subset of a Hilbert space H. For each $x \in H$ there exists a uniquely determined point $y \in C$ such that*

$$\|x-y\| \leq \|x-v\| \quad \text{for all } v \in C. \tag{4.3.3}$$

PROOF. Let (y_n) be chosen in C so that $\|x-y_n\| \to \delta$ where $\delta = \inf\{\, \|x-v\| \mid v \in C\}$. Applying the parallelogram law and the convexity,

$$\begin{aligned} \|y_n-y_m\|^2 &= 2\|y_n-x\|^2+2\|x-y_m\|^2 - \|y_n-x-(x-y_m)\|^2 \\ &= 2\|y_n-x\|^2+2\|x-y_m\|^2 - 4\|\tfrac{1}{2}(y_n+y_m)-x)\|^2 \\ &\leq 2\|y_n-x\|^2+2\|x-y_m\|^2-4\delta^2. \end{aligned} \tag{4.3.4}$$

Since the last expression can be made arbitrarily small, (y_n) is a Cauchy sequence, hence converges to a limit point y. Since C is closed $y \in C$, and by the continuity of the norm $\|x-y\| = \delta$.

If also $\delta = \|x-z\|$ for some $z \in C$, one can substitute y_n and y_m by y and z, respectively, in the above inequality and derive that $\|y-z\|^2 \leq 2\delta^2+2\delta^2-4\delta^2 = 0$. Whence $z = y$. □

By the proposition, there is a map $P_C\colon H \to H$ given by $P_C x = y$, in the notation of (4.3.3). Clearly $(P_C)^2 = P_C$, so P_C is called the projection onto C.

Notice that when U is a subspace of dimension $n \in \mathbb{N}$, say with orthonormal basis $(e_1,\dots,e_n)$, then (4.3.2) ff. shows that

$$P_U x = \sum_{j=1}^{n} (x\,|\,e_j)e_j \quad \text{for } x \in H. \tag{4.3.5}$$

With this structure P_U is linear and bounded for every n-dimensional subspace $U \subset H$. Since $x - P_U x$ is orthogonal to every e_j and therefore to any vector in U, the operator P_U is called the *orthogonal* projection on U.

4.4. The Projection Theorem and self-duality

It was seen above that orthogonality was involved in the process of finding the minimal distance from a point to a subspace. There are also other geometric properties of Hilbert spaces that are linked to orthogonality, and a few of these are presented here.

4.4.1. On orthogonal projection. For orthogonal subspaces M and N, ie $M \perp N$, the orthogonal sum is defined as

$$M \oplus N = \{\, x+y \mid x \in M, \quad y \in N\,\}. \tag{4.4.1}$$

Hence any vector $z \in M \oplus N$ has a decomposition $z = x+y$ with $x \in M$ and $y \in N$. The orthogonality shows that this decompostion is unique (since $M \perp N \implies M \cap N = \{0\}$).

When both M and N are closed in H, then $M \oplus N$ is a *closed* subspace too, for if $z_n \in M \oplus N$ converges in H, Pythagoras' theorem applied to the decompositions $z_n = x_n + y_n$ gives Cauchy sequences (x_n), (y_n) in M and

N, and the sum of these converges to an element of $M \oplus N$ (since M, N are closed) as well as to $\lim z_n$.

Recall that for a closed subspace M of H, the orthogonal complement is denoted $M^\perp$; alternatively $H \ominus M$ may be used to make it clear that the orthogonal complement is calculated with respect to H. When $H = M \oplus N$ both M and N are called *direct summands* of H, but for a given M there is, by the orthogonality, only one possible choice of N: this is a consequence of the next result known as the *Projection Theorem*, which states that as the direct summand N one can take $N = H \ominus M$.

As a transparent example, take the familiar orthogonal sum $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, with $\mathbb{R}^k \simeq \{\, (x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{R}^n \mid x_1, \dots, x_k \in \mathbb{R} \,\}$, for $0 \le k \le n$, and a similar identification for $\mathbb{R}^{n-k}$. The Projection Theorem is a non-trivial generalisation to Hilbert spaces:

THEOREM 4.4.1. *Let M be a closed subspace of a Hilbert space H. Then there is an orthogonal sum*

$$H = M \oplus M^\perp. \tag{4.4.2}$$

Moreover, every x has a unique decomposition $x = y + z$ for y and z equal to the closest point of M and $M^\perp$, respectively, to x.

PROOF. Given $x \in H$ there is a $y \in M$ such that $\|x - y\| \le \|x - v\|$ for all $v \in M$, by Proposition 4.3.1. Letting $z = x - y$ it remains to be verified that $z \in M^\perp$; but for $\lambda \in \mathbb{F}$ and $v \in M$ with $\|v\| = 1$,

$$\|z\|^2 \le \|x - (y + \lambda v)\|^2 = \|z\|^2 + |\lambda|^2 - 2\operatorname{Re}\overline{\lambda}(z \,|\, v), \tag{4.4.3}$$

so $\lambda = (z \,|\, v)$ entails $|\lambda|^2 \le 0$; whence $(z \,|\, v) = 0$ for any $v \in M$. Thence $M \oplus M^\perp = H$.

The uniqueness of the decomposition was seen after (4.4.1); its existence implies that $(M^\perp)^\perp = M$, cf Corollary 4.4.2 below. Applying the construction to $M^\perp$ therefore gives $x = y' + z'$ where $z' \in M^\perp$, $y' \in M$ and z' equal to the point of $M^\perp$ closest to x. But by the uniqueness, $y = y'$ and $z = z'$. □

COROLLARY 4.4.2. *For every subset $M \subset H$ the bi-orthogonal complement $M^{\perp\perp}$ is the closed linear hull of M; in particular, $\overline{M} = M^{\perp\perp}$ if M is a subspace.*

PROOF. For a subspace M, any $x \in M^{\perp\perp}$ is decomposed $x = y + z$ for $y \in M$ and $z \in M^\perp$, by the Projection Theorem. Since $M \subset M^{\perp\perp}$, clearly $z = x - y \in M^\perp \cap M^{\perp\perp} = (0)$. Then $x = y \in M$. For an arbitrary subset, the question was previously reduced to the case of a subspace. □

By the uniqueness of the decomposition in the theorem, there is an map $P \colon x \mapsto y$, which moreover is a linear operator (why?). P is bounded, for by Pythagoras' theorem the splitting $x = y + z$ in $M \oplus M^\perp$ gives

$$\|Px\|^2 = \|y\|^2 \le \|y\|^2 + \|z\|^2 = \|x\|^2. \tag{4.4.4}$$

Hence $\|P\| \leq 1$.

If M has finite dimension, P equals the previously introduced orthogonal projection on M (cf the proof of Theorem 4.4.1), and Px is called the orthogonal projection of x onto M. A characterisation of the operators in $\mathbb{B}(H)$ that are ortogonal projections follows in Proposition 6.1.1 below.

4.4.2. On the self-duality. In a Hilbert space H it is immediate that every vector $y \in H$ gives rise to the linear functional $x \mapsto (x\,|\,y)$; by Cauchy–Schwarz' inequality this is bounded,

$$|(x\,|\,y)| \leq \|x\|\,\|y\|. \tag{4.4.5}$$

It is a very important fact that all elements in H^* arise in this way; cf the next theorem, known as Frechet–Riesz' theorem (or the Riesz Representation Theorem).

THEOREM 4.4.3. *For each $\varphi \in H^*$ there exists a vector $z \in H$ such that $\varphi(x) = (x\,|\,z)$ for all $x \in H$.*

PROOF. With $N = Z(\varphi)$, which is a closed subspace by the continuity of φ, the Projection Theorem gives $H = N \oplus N^\perp$. Clearly $\varphi \equiv 0$ if and only if $N = H$, in which case $z = 0$ will do. For $\varphi \neq 0$ there is some $y \in N^\perp$ with $\|y\| = 1$, and then $v = \varphi(x)y - \varphi(y)x$ belongs to N, regardless of $x \in H$. So it suffices to let $z = \overline{\varphi}(y)y$, for

$$0 = (v\,|\,y) = \varphi(x)(y\,|\,y) - \varphi(y)(x\,|\,y) = \varphi(x) - (x\,|\,z). \tag{4.4.6}$$

For the uniqueness, assume $\varphi = (\cdot\,|\,z) = (\cdot\,|\,w)$; then $(x\,|\,z-w) = 0$ for all x, yielding $z - w \in H^\perp$ and $z = w$. $\square$

Notice that eg $(\ell^p)^* \neq \ell^p$ for $p \in [1,\infty[$ with $p \neq 2$: the sequence with $x_n = n^{-1/r}$ is only in ℓ^q for $q > r$ so that there are *strict* inclusions

$$\ell^p \subsetneq \ell^q \quad \text{for} \quad 1 \leq p < q \leq \infty. \tag{4.4.7}$$

Hence Banach spaces do not identify with their duals in general.

CHAPTER 5

Examples of Hilbert spaces. Fourier series.

The basic non-trivial example of a Hilbert space is $L^2(A,\mathbb{A},\mu)$, consisting of (equivalence classes of) square-integrable functions on an arbitrary measure space $(A,\mathbb{A},\mu)$. (ℓ^2 is also covered by considering the counting measure on $\mathbb{N}$).

For an open set $\Omega\subset\mathbb{R}^n$ there is the standard Hilbert space $L^2(\Omega)$ with inner product $(f\,|\,g)=\int_\Omega f(x)\overline{g(x)}\,dx$. However, certain subsets of $L^2(\Omega)$ are Hilbert spaces in their own right.

EXAMPLE 5.0.4 (Sobolev spaces). Let the subset $H^1(\Omega)\subset L^2(\Omega)$ be defined by the requirement that to each $f\in H^1(\Omega)$ there exist other functions $f_1',\dots,f_n'$ in $L^2(\Omega)$ such that for every $\varphi\in C_0^\infty(\Omega)$ it holds that

$$\int_\Omega f(x)(-\tfrac{\partial}{\partial x_j}\varphi(x))\,dx=\int_\Omega f_j'(x)\varphi(x)\,dx \quad\text{for}\quad j=1,\dots,n. \tag{5.0.8}$$

Notice that for f in $C_0^1(\Omega)$ one can take $f_j'=\frac{\partial f}{\partial x_j}$; hence $C_0^1(\Omega)\subset H^1(\Omega)$.

For $f\in H^1(\Omega)$ the functions f_j' are called the (generalised) derivatives of f of the first order, and these are written in operator notation as

$$\partial_j f=\partial_{x_j}f=\tfrac{\partial f}{\partial x_j}=f_j',\quad\text{for}\quad j=1,\dots,n. \tag{5.0.9}$$

Here it was used that the derivatives f_j' are *determined* by f: if $\tilde f_1,\dots,\tilde f_n$ is another set of functions in $L^2(\Omega)$ fulfilling (5.0.8), then $f_1'-\tilde f_1\in C_0^\infty(\Omega)^\perp=L^2(\Omega)^\perp=(0)$; similarly $f_j'=\tilde f_j$ for all j. As a consequence these partial differential operators give well defined maps

$$\partial_j\colon H^1(\Omega)\to L_2(\Omega)\quad\text{for}\quad j=1,\dots,n. \tag{5.0.10}$$

(In $C^1(\Omega)\cap H^1(\Omega)$ these maps are given by limits of difference quotients. In general the f_j' equal the so-called distribution derivatives $\partial_j f$ of f.)

A topology on $H^1(\Omega)$ may be obtained eg as a metric subspace of $L^2(\Omega)$. But to have some control over $f_1',\dots,f_n'$, it is stronger to note that, by the uniqueness and linearity of the generalised derivatives, there is a well defined inner product on $H^1(\Omega)$ given by

$$(f\,|\,g)_{H^1}=(f\,|\,g)_{L^2}+(f_1'\,|\,g_1')_{L^2}+\dots+(f_n'\,|\,g_n')_{L^2}; \tag{5.0.11}$$

the norm induced is clearly given by

$$\|f\|_{H^1}=\Big(\int_\Omega(|f(x)|^2+\sum_{j=1}^n|\partial_j f(x)|^2)\,dx\Big)^{1/2}. \tag{5.0.12}$$

Actually $H^1(\Omega)$ is a Hilbert space, because it is complete with respect to this norm (verify this!). Notice that the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, because for every $f \in H^1(\Omega)$ one has $\|f\|_{L^2} \leq \|f\|_{H^1}$. Moreover, the expression for $\|\cdot\|_{H^1}$ implies directly that the differential operators $\partial_1, \dots, \partial_n$ in (5.0.10) above all are *continuous* maps $H^1 \to L^2$.

$H^1(\Omega)$ is called the *Sobolev* space of order 1 over Ω; this Hilbert space plays a very significant role in the theory of partial differential equations. It is also convenient to introduce the subspace $H_0^1(\Omega)$ by taking the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, ie

$$H_0^1(\Omega) = \big\{ f \in H^1(\Omega) \bigm| \exists \varphi_k \in C_0^\infty(\Omega) \colon \lim_{k\to\infty} \|f - \varphi_k\|_{H^1} = 0 \big\}. \tag{5.0.13}$$

Clearly $H_0^1(\Omega)$ is Hilbert space with the induced inner product from $H^1(\Omega)$.

EXAMPLE 5.0.5. The Sobolev spaces have generalisations to Hilbert spaces $H^m(\Omega)$ incorporating higher order derivatives up to some order $m \in \mathbb{N}$. For this it is useful to adopt the multiindex notation, say for $f \in C^\infty(\Omega)$:

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, which is said to have length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, one writes

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}. \tag{5.0.14}$$

Then the subspace $H^m(\Omega) \subset L^2(\Omega)$ is defined as the set of f to which there for every $|\alpha| \leq m$ exists some $f_\alpha \in L^2(\Omega)$ fulfilling the condition $(f \,|\, \partial^\alpha \varphi)_{L^2} = (-1)^{|\alpha|} (f_\alpha \,|\, \varphi)$ for all $\varphi \in C_0^\infty(\Omega)$.

Since the f_α are uniquely determined, there are maps $\partial^\alpha f := f_\alpha$ defined for $f \in H^m(\Omega)$. This gives rise to an inner product on $H^m(\Omega)$, namely

$$(f \,|\, g)_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha f(x) \overline{\partial^\alpha g(x)} \, dx. \tag{5.0.15}$$

The induced norm has the expression

$$\|f\|_{H^m} = \big(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^2}^2 \big)^{1/2}. \tag{5.0.16}$$

With this $H^m(\Omega)$ is a Hilbert space. The subspace $H_0^m(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$, that clearly is a Hilbert space. By inspection of the norms, there are *bounded*, hence continuous maps

$$\partial^\alpha \colon H^m(\Omega) \to H^{m-|\alpha|}(\Omega), \quad \text{for} \quad |\alpha| \leq m. \tag{5.0.17}$$

5.1. Examples of orthonormal bases.

Some of the elementary functions provide basic examples of orthonormal bases. Eg one has

PROPOSITION 5.1.1. *The Hilbert space $L^2(0, \pi)$ has an orthonormal basis $(e_n)_{n \in \mathbb{N}_0}$ consisting of*

$$e_0 \equiv \frac{1}{\sqrt{\pi}}, \quad e_n = \sqrt{\frac{2}{\pi}} \cos(nt), \quad \textit{for } n \in \mathbb{N}. \tag{5.1.1}$$

Indeed, orthonormality is easy to derive from the periodicity and Euler's identities (do it!). It remains to show that $(e_n)_{n\in\mathbb{N}_0}$ is total in $L_2(0,\pi)$, and for this it suffices by density to approximate an arbitrary $f \in C([0,\pi])$. But to $g(t) = f(\arccos t)$ and $\varepsilon > 0$, Weierstrass' approximation theorem (2.3.1) furnishes a polynomial $p = \sum_{j=0}^{N} a_j t^j$ such that $|g - p| < \varepsilon\pi^{-1/2}$ on $[-1,1]$; thence

$$|f(t) - \sum_{j=0}^{N} a_j (\cos t)^j| < \varepsilon\pi^{-1/2}, \quad \text{for} \quad t \in [0,\pi]. \tag{5.1.2}$$

Here Euler's identities yield that $(\cos t)^j = \sum_{k=-j}^{j} b_k e^{\mathrm{i}kt}$ for scalars satisfying $b_k = b_{-k}$, whence $(\cos t)^j$ is in $E_j = \operatorname{span}(e_0,\dots,e_j)$. Then $p\circ\cos$ is in E_N, and $\|f - p\circ\cos\| < \varepsilon$ in $L_2(0,\pi)$ as desired.

Similarly the sine function gives rise to an orthonormal basis.

PROPOSITION 5.1.2. *The Hilbert space $L_2(0,\pi)$ has an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ given by $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ for $n \in \mathbb{N}$.*

The orthonormality is verified as for the cosines; but that the sequence is total follows at once from the totality of the cosines: if $f \perp \overline{\operatorname{span}}(e_n)$, then $(f\,|\,e_n) = \int_0^\pi f(x)\sin(nx)\,dx = 0$ for all n; this yields

$$\begin{aligned}(f\sin\,|\,\cos(n\cdot)) &= \int_0^\pi f(x)\sin(x)\cos(nx)\,dx \\ &= (f\,|\,\tfrac12\sin((n+1)\cdot)) - (f\,|\,\tfrac12\sin((n-1)\cdot)) = 0\end{aligned} \tag{5.1.3}$$

Since $\{0\} = \operatorname{span}(\cos(n\cdot))^\perp$, this gives $f\sin = 0$, hence $f = 0$ a.e. Therefore (e_n) is total.

5.2. On Fourier series

It is known from elementary calculus that eg $f(x) = \cos^2(x)\sin(3x)$ may be resolved into a sum of oscillations with frequencies $\frac{1}{2\pi}$, $\frac{3}{2\pi}$ and $\frac{5}{2\pi}$, simply by use of Euler's identities:

$$f(x) = \cos^2(x)\sin(3x) = \tfrac14\sin(5x) + \tfrac12\sin(3x) + \tfrac14\sin x. \tag{5.2.1}$$

This way, a harmonic or *Fourier analysis* of f is obtained.

The classical claim of J. Fourier (made around 1820?!) is that any function f on the interval $[-\pi,\pi]$ may expressed as an infinite series of harmonic functions, namely

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n\cos(nx) + B_n\sin(nx)) \tag{5.2.2}$$

with the coefficients

$$A_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)\,dy \quad \text{for} \quad n = 0,1,2,\dots \tag{5.2.3}$$

$$B_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)\,dy \quad \text{for} \quad n = 1,2,\dots. \tag{5.2.4}$$

Later as the notion of functions was crystallised, it became increasingly important to clarify Fourier's claim.

It is quite remarkable that his assertion is true for functions as general as those in the class $L^2(-\pi,\pi)$ (and similarly in dimensions $n>1$).

5.2.1. The one-dimensional case. The results on orthonormal bases of sines and cosines on $[0,\pi]$ lead to the following main result. It is formulated for the Hilbert space $L^2(-\pi,\pi;\frac{1}{2\pi}m_1)$, where the one-dimensional Lebesgue measure m_1 is normalised for convenience. Hence $(f\,|\,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$ is the inner product of f, g.

THEOREM 5.2.1. *The functions $e_n(x) = e^{\mathrm{i}nx}$, with $n\in\mathbb{Z}$, constitute an orthonormal basis for $L^2(-\pi,\pi;\frac{1}{2\pi}m_1)$, and for every f in this space,*

$$f = \sum_{n=-\infty}^{\infty} c_n e_n \tag{5.2.5}$$

with coefficients $c_n = (f\,|\,e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-\mathrm{i}ny}\,dy$ for $n\in\mathbb{Z}$.

PROOF. It is straightforward to see that the sequence is orthonormal, for by the periodicity of $e^{\mathrm{i}(k-n)x}/(k-n)$ for $k\neq n$,

$$(e_k\,|\,e_n) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(k-n)y}\,dy = \delta_{kn}. \tag{5.2.6}$$

It therefore suffices to see that $\{\, e_n \mid n\in\mathbb{Z}\,\}$ is a total subset, which follows if every f in $L^2(-\pi,\pi)$ satisfies

$$f = \lim_{n\to\infty} \sum_{k=-n}^{n} (f\,|\,e_k)e_k. \tag{5.2.7}$$

(This is the meaning of (5.2.5).)

First the case of an even f is considered, ie $f(x) = f(-x)$. For such f it holds that $B_n = 0$ for every $n\in\mathbb{N}$, for the substitution $y=-x$ leads to

$$\int_{-\pi}^{0} f(y)\sin(ny)\,dy = -\int_{0}^{\pi} f(x)\sin(nx)\,dx. \tag{5.2.8}$$

Classically f is therefore assigned the Fourier series

$$f(x) = \tfrac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(nx). \tag{5.2.9}$$

However, for $x\in[0,\pi]$ this holds as an identity in $L^2(0,\pi;\frac{2}{\pi}m_1)$. Indeed, in view of Proposition 5.1.1 the functions $g_0(x) = \frac{1}{\sqrt{2}}$ and $g_n(x) = \cos(nx)$

with $n \in \mathbb{N}$ for an orthonormal basis, so since f is even,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (f \,|\, g_n) g_n(x) \\ &= \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(y)\, dy \frac{1}{2} + \sum_{n=1}^{\infty} \big(\tfrac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos(ny)\, dy \big) \cdot \cos(nx) \\ &= \tfrac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx). \end{aligned} \tag{5.2.10}$$

Actually (5.2.9) even holds in $L^2(-\pi,\pi;\frac{1}{2\pi}m_1)$, for if s_n denotes the n^{th} partial sum on the right hand side of (5.2.9),

$$\|f - s_n\|^2_{L^2(-\pi,\pi)} \le \|f - s_n\|^2_{L^2(0,\pi)} \searrow 0. \tag{5.2.11}$$

(For the inequality one may use that $|f - s_n|^2$ has the same integral on $[-\pi,0]$ and $[0,\pi]$, since f and the cosines are even.)

Using a completely analogous argument, and that $L^2(0,\pi;\frac{2}{\pi}m_1)$ has another orthonormal basis given by $(\sin(n\cdot))_{n\in\mathbb{N}}$, cf Proposition 5.1.2, it is seen that for odd functions, ie $f(x) = -f(-x)$, all the A_n vanish and

$$f = \sum_{n=0}^{\infty} B_n \sin(n\cdot) \quad \text{in} \quad L^2(-\pi,\pi;\tfrac{1}{2\pi}m_1). \tag{5.2.12}$$

Now any function f may be written $f = f_1 + f_2$ where $f_1(x) := (f(x) + f(-x))/2$ is even and $f_2(x) := (f(x) - f(-x))/2$ is odd, and the above analyses apply to these terms. Chosing new scalars

$$\begin{aligned} C_n &= \tfrac{1}{2}(A_n - \mathrm{i}B_n) \quad \text{for} \quad n \in \mathbb{N}_0, \\ C_{-n} &= \tfrac{1}{2}(A_n + \mathrm{i}B_n) \quad \text{for} \quad n \in \mathbb{N}, \end{aligned} \tag{5.2.13}$$

Euler's formula leads to

$$\begin{aligned} f &= f_1 + f_2 \\ &= \tfrac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(n\cdot) + B_n \sin(n\cdot)) \\ &= \tfrac{A_0}{2} + \sum_{n=1}^{\infty} (C_n e^{\mathrm{i}n\cdot} + C_{-n} e^{-\mathrm{i}n\cdot}) = \lim_{n\to\infty} \sum_{k=-n}^{n} C_k e^{\mathrm{i}k\cdot}. \end{aligned} \tag{5.2.14}$$

Since insertion of (5.2.3) and (5.2.4) into (5.2.13) shows that $C_n = c_n$ for every $n \in \mathbb{Z}$, (5.2.14) proves (5.2.7), hence the theorem. □

Observe that the classical Parseval's equation for Fourier series now is a *gratis* consequence of Theorem 4.2.5:

$$\tfrac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n(f)|^2. \tag{5.2.15}$$

Here $c_n(f) = (f \,|\, e_n)$, but the main change is that as the index set for the basis vectors, $\mathbb{Z}$ must now be used instead of $\mathbb{N}$. (Here $c_n(f) := (f \,|\, e_n)$.)

In addition, Proposition 4.2.3 shows that the sequence $(c_n(f))$ is in $\ell^2(\mathbb{Z})$ for every $f \in L^2(-\pi,\pi)$, and that conversely *any* (α_n) in $\ell^2(\mathbb{Z})$ equals the Fourier coefficients of some function $g \in L^2(-\pi,\pi)$; indeed, $g = \sum \alpha_n e_n$ by Proposition 4.2.3. (Actually this is just an example of the unitary equivalence mentioned in Theorem 4.2.6!)

5.2.2. Fourier series in higher dimensions. Using the above results it is now possible to deduce the corresponding facts in n dimensions. So consider the cube $Q =]-\pi,\pi]^n$ and the corresponding Hilbert space $L^2(Q)$ (the Lebegue measure m_n is now tacitly normalised by $(2\pi)^{-n}$).

It is easy to see that there is an orthonormal sequence of functions

$$e_k(x) = e^{\mathrm{i}k\cdot x} = e^{\mathrm{i}(k_1x_1+\dots+k_nx_n)} \tag{5.2.16}$$

with $x = (x_1,\dots,x_n) \in Q$ and a 'multi-integer' $k = (k_1,\dots,k_n) \in \mathbb{Z}^n$. In fact for k, $m \in \mathbb{Z}^n$,

$$(\,e_k\,|\,e_m\,) = \tfrac{1}{(2\pi)^n}\prod_{j=1}^{n}\int_{-\pi}^{\pi} e^{\mathrm{i}(k_j-m_j)x_j}\,dx_j = \delta_{k_1m_1}\dots\delta_{k_nm_n} = \delta_{km}. \tag{5.2.17}$$

This system is moreover an orthonormal basis. Indeed, assume that $f \in L^2(Q)$ is orthogonal to e_k for every $k \in \mathbb{Z}^n$. Since $L^2(Q) \subset L^1(Q)$, the below auxiliary function is well defined (a.e.) by Fubini's theorem,

$$g(x_n) = \int_{]-\pi,\pi]^{n-1}} \tfrac{f(x_1,\dots,x_{n-1},x_n)}{\exp(\mathrm{i}(k_1x_1+\dots+k_{n-1}x_{n-1}))(2\pi)^n}\,d(x_1,\dots,x_{n-1}). \tag{5.2.18}$$

Moreover, it is in $L^2(-\pi,\pi)$ because Hölder's inequality gives that

$$|g(x_n)| \le \Big(\int |f(x_1,\dots,x_{n-1},x_n)|^2\,d(x_1,\dots,x_{n-1})\Big)^{1/2}, \tag{5.2.19}$$

where the right hand side is quadratic integrable. However, by the identity above, $(\,g\,|\,e^{\mathrm{i}k_n\cdot}\,) = (\,f\,|\,e_k\,) = 0$. Then the results for dimension 1 give that $g \equiv 0$. Hence $f(\cdot,x_n)$ is orthogonal to all the exponentials in $n-1$ dimensions. Repeating this argument, it is seen that for fixed $x_2,\dots,x_n \in]-\pi,\pi]$, the function $f(\cdot,x_2,\dots,x_n)$ is 0 in $L^2(-\pi,\pi)$; by Fubini $\|f\| = 0$ in $L^2(Q)$, whence $f = 0$.

Thereby the following generalisation of Theorem 5.2.1 is an immediate consequence of the general Hilbert space theory:

THEOREM 5.2.2. *The functions $e_k(x) = e^{\mathrm{i}k\cdot x}$, with $k \in \mathbb{Z}^n$, constitute an orthonormal basis for $L^2(Q)$, where $Q =]-\pi,\pi]^n$, and for every f in this space,*

$$f = \sum_{k\in\mathbb{Z}^n} c_k e_k \tag{5.2.20}$$

with coefficients $c_k = (\,f\,|\,e_k\,) = (2\pi)^{-n}\int_Q f(y)e^{-\mathrm{i}k\cdot y}\,dy$ for $k \in \mathbb{Z}$. The sequence (c_k) is in $\ell^2(\mathbb{Z}^n)$, and Parseval's identity holds.

Conversely, to any $(c_k) \in \ell^2(\mathbb{Z}^n)$ there exists a unique function f in $L^2(Q)$ having (c_k) as its Fourier coefficients.

The convergence in (5.2.20) means that for any $\varepsilon > 0$ there exists a finite set $K \subset \mathbb{Z}^n$ such that

$$\|f - \sum_{k\in K} c_k e_k\|_{L^2(Q)} < \varepsilon. \tag{5.2.21}$$

EXAMPLE 5.2.3. For the subspace $H^1(Q)$ of $L^2(Q)$, introduced in Example 5.0.4 at least if we now take $Q =]-\pi,\pi[^n$, it is natural to ask for characterisations in terms of Fourier series.

However, this is easier to carry out for the subspace

$$H^1(\mathbb{T}) = \{\, f \in H^1(Q) \mid \forall j = 1,\dots,n\colon \\ x_j = 0 \implies f(x+\pi e_j) = f(x-\pi e_j)\,\}. \tag{5.2.22}$$

The reason is that any such f may be extended to a 2π-periodic function in all variables without loosing the H^1-property (whereas such extensions of functions in $H^1(Q)$ would have jump discontinuities at the boundary of Q). Observe, however, that there is an important technical remnant, namely to account for the fact that the elements of $H^1(\mathbb{T})$ are so regular that the values at $x \pm \pi e_j$ may be calculated in an unambiguous way.

We shall abstain from that here, and just mention the resulting characterisation instead. Indeed, defining $h^1(\mathbb{Z}^n) \subset \ell^2(\mathbb{Z}^n)$ to be the subspace of sequences (c_k) fulfilling

$$\Big(\sum_{k\in\mathbb{Z}^n} (1 + k_1^2 + \cdots + k_n^2)|c_k|^2\Big)^{1/2} < \infty, \tag{5.2.23}$$

then $u \in H^1(\mathbb{T})$ holds precisely when its Fourier coefficients (c_k) belong to $h^1(\mathbb{Z}^n)$. And $\|u\|_{H^1}$ equals the left hand side of the above inequality. Proof of this is given later.

On these grounds, Hilbert space theory is customarily deemed the natural framework for Fourier series.

CHAPTER 6

Operators on Hilbert spaces

6.1. The adjoint operator

As an application of the notion of adjoint operators, one can give the following characterisation of orthogonal projections.

PROPOSITION 6.1.1. *Let $P \in \mathbb{B}(H)$. Then P is an orthogonal projection onto a closed subspace M of H if and only if $P^* = P^2 = P$, that is if P is a self-adjoint idempotent.*

In the affirmative case $M = P(H) = Z(I-P) = \{\, x \in H \mid Px = x \,\}$, so $H = P(H) \oplus Z(P)$.

PROOF. That the orthogonal projection P onto a closed subspace M of H is a bounded, self-adjoint and idempotent operator is easy to see from the definition of P.

Conversely, if $P = P^* = P^2$ holds for some $P \in \mathbb{B}(H)$, then the identity $I = P + (I-P)$ shows that

$$\forall x \in H \colon\, x \in P(H) + (I-P)(H). \tag{6.1.1}$$

Now it is straightforward to verify that also $I-P$ is a self-adjoint idempotent. Using this, both subspaces $P(H)$, $(I-P)(H)$ are seen to be closed: if $x_n \to x$ in H for a sequence (x_n) in eg $P(H)$, then $x_n = Px_n \to Px$, so $x = Px$. They are ortogonal since $P^*(I-P) = P - P^2 \equiv 0$, so $H = P(H) \oplus (I-P)(H)$ in view of (6.1.1). Since $P = P^2$, it also follows from (6.1.1) that P is the orthogonal projection onto $P(H)$. The remaining facts are uncomplicated to verify. □

The folowing formula is sometimes useful.

LEMMA 6.1.2. *If $T \in \mathbb{B}(H)$ is self-adjoint, ie $T^* = T$, then*

$$\|T\| = \sup\bigl\{\, |(Tx\,|\,x)| \bigm| x \in H,\ \|x\| = 1 \,\bigr\}. \tag{6.1.2}$$

PROOF. If M_T denotes the supremum in (6.1.2), it follows from Cauchy-Schwarz' inequality that $M_T \le \|T\|$.

Whenever $Tx \ne 0$ it is clear that $\|y\| = \|x\|$ by setting $y = s^{-1}Tx$ for $s = \|Tx\|/\|x\|$. Then a polarisation and the Parallelogram Law give that

$$\begin{aligned} 4\|Tx\|^2 &= 2(Tx\,|\,Tx) + 2(TTx\,|\,x) = 2s((Tx\,|\,y) + (Ty\,|\,x)) \\ &= s((T(x+y)\,|\,x+y) - (T(x-y)\,|\,x-y)) \\ &\le sM_T(\|x+y\|^2 + \|x-y\|^2) \\ &= 2sM_T(\|x\|^2 + \|y\|^2) = 4sM_T\|x\|^2. \end{aligned} \tag{6.1.3}$$

Therefore $\|Tx\| \le M_T\|x\|$ for all x, so $\|T\| \le M_T$. □

6.2. Compact operators

6.2.1. Preliminaries. The next result is often important; it states that for subspaces X of finite dimension one need only consider the coordinates with respect to a fixed basis of X (as we would like to), *even* when it comes to topological questions.

LEMMA 6.2.1. *Let X be a finite-dimensional subspace of a normed vector space V over $\mathbb{F}$, say with $\dim X = n \in \mathbb{N}$. Then every linear bijection $\Phi\colon \mathbb{F}^n \to X$ is a homeomorphism, and X is closed in V.*

PROOF. Any Φ of the mentioned type has the form $(\alpha_1,\dots,\alpha_n) \mapsto \alpha_1x_1+\dots+\alpha_nx_n$ for some basis $(x_1,\dots,x_n)$. However, using continuity of the vector operations, induction after n gives that Φ is continuous.

Φ^{-1} is continuous if and only if $\Phi(O)$ is open in X for every open set $O \subset \mathbb{F}^n$. By the linearity it suffices to see that $\Phi(B)$ is a neighbourhood of 0, when B is the open unit ball of $\mathbb{F}^n$. But $S := \{\,\alpha \in \mathbb{F}^n \mid \alpha_1^2+\dots+\alpha_n^2 = 1\,\}$ is compact, and so is $\Phi(S)$ by the continuity of Φ. Combining this with the Hausdorff property of X gives a ball C centered at 0 such that $C \cap \Phi(S) = \emptyset$. Now $C \subset \Phi(B)$ follows, for if $C \ni c = \Phi(\alpha)$ with $\|\alpha\| \ge 1$, the continuous map $t \mapsto \|t\alpha\|$ attains the value 1 for some $t_0 \in]0,1]$, so the convexity of C entails the contradiction $t_0c \in C \cap \Phi(S)$.

Using that V is Hausdorff, a sequence in X cannot converge to a point in $V \setminus X$, for its image under Φ^{-1} converges in $\mathbb{F}^n$. Consequently $\overline{X} = X$. □

Notice that for $X = V = \mathbb{F}^n$ the lemma gives that all the norms $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$ with $1 \le p < \infty$ and the sup-norm $\|x\|_\infty$ give the same topology (which can also be seen directly), and that moreover the same is true for *any* norm on $\mathbb{F}^n$.

The result of the lemma holds in a much wider context too, for it suffices to presuppose only that V is a Hausdorff topological vector space. (The proof only needs to have the ball C replaced by another type of neighbourhood of 0 in which $tC \subset C$ for scalars with $|t| \le 1$.)

The *rank* of a linear map $T\colon V \to W$ is defined as $\operatorname{rank} T = \dim T(V)$. In analogy with Lemma 6.2.1, one could wonder whether operators of finite rank are necessarily bounded. But this is not the case, for counterexamples exist already when $\operatorname{rank} T = 1$ as seen in the specific construction given below (this also elucidates why a basis U is required to fulfil $V = \overline{\operatorname{span}}\, U$ rather than $V = \operatorname{span} U$):

Let (e_n) denote the canonical orthonormal basis in $\ell^2(\mathbb{N})$; then $\ell^2 \setminus \operatorname{span}(e_n) \neq \emptyset$ because it contains eg $(n^{-1})_{n\in\mathbb{N}}$, and by Zorn's lemma there exists a maximal linearly independent set of the form $\{\, e_n \mid n \in \mathbb{N}\,\} \cup \{\, v_i \mid i \in I\,\}$; this is a Hamel basis of ℓ^2, cf Remark 3.0.5, so that every $v \in \ell^2$

may be written

$$v = \sum_{n\in\mathbb{N}} \lambda_n e_n + \sum_{i\in I} \mu_i v_i. \tag{6.2.1}$$

Defining $\varphi\colon \ell^2 \to \mathbb{C}$ by letting $\varphi v = \sum \mu_i$, it is clear that φ is a linear functional which is nonzero on every v_i. Moreover, $Z(\varphi) \supset \operatorname{span}(e_n)$, and since the latter set is dense, φ is *discontinuous* on ℓ^2.

6.2.2. Compact operators. A linear operator is bounded if and only if it maps the unit ball to a bounded set or (the reader should verify that it is equivalent) if and only if T maps every bounded set to a bounded set. To get a subclass of operators with stronger properties one could therefore require that every bounded set should be sent into a compact set:

DEFINITION 6.2.2. Let $T\colon V \to W$ be a linear operator between normed spaces V and W. Then T is said to be *compact* if every bounded set $A \subset V$ has an image with compact closure (ie if $\overline{T(A)}$ is compact in W).

Notice that T is bounded and hence continuous, if it is compact. (It is a rather stronger fact that a compact operator, after restriction to say a ball of its domain, is continuous not only with respect to the induced norm topology,(as just observed), but also with respect to the so-called *weak* topology. Perhaps for these reasons, compact operators are synonymously called *completely continuous*.)

As an example, the identity I is *not* a compact operator in any infinite dimensional Hilbert space H, for an orthonormal sequence is never a Cauchy sequence, hence cannot have convergent subsequences. But the inclusion operator $C^1([0,1]) \hookrightarrow C([0,1])$ is compact (although this requires too many efforts to be shown here). Similarly, $H^1(\Omega) \subset L^2(\Omega)$ is a compact embedding when $\Omega \subset \mathbb{R}^n$ is bounded; cf Example 5.0.4–5.2.3 and the proof further below.

A simpler example concerns the operators $T\colon V \to W$ of *finite rank*.

LEMMA 6.2.3. *Let $T \in \mathbb{B}(V,W)$ be an operator of finite rank between normed spaces V, W. Then T is compact.*

PROOF. There is a linear homeomorphism $\Phi\colon T(V) \to \mathbb{F}^n$ for some $n \in \mathbb{N}$, and $T(V)$ is closed in W. Given a bounded set $A \subset V$ it follows that $\overline{T(A)} \subset T(V)$; hence $\Phi(\overline{T(A)})$ is well defined, it is bounded since $T(A)$ is bounded, and closed in $\mathbb{F}^n$, ie compact. Since Φ^{-1} is continuous this yields the compactness of $\overline{T(A)}$, and eventually also of T. □

More generally, there is a convenient way to write down numerous operators, in fact, one for each sequence (λ_n). Indeed, let (e_n) be an orthonormal basis for a Hilbert space H, and consider for each sequence (λ_n) in $\mathbb{F}$ the operator T in H given by the expression

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x \,|\, e_n) e_n, \tag{6.2.2}$$

and by its 'maximal' domain

$$D(T) = \{\, x \in H \mid \sum_{n=1}^{\infty} \lambda_n (x \,|\, e_n) e_n \text{ converges in } H \,\}. \tag{6.2.3}$$

Notice by insertion of $x = e_n$ that every λ_n is an eigenvalue of the defined T. Moreover, simple properties such as boundedness and compactness are also easy to verify:

THEOREM 6.2.4. *Under the above hypotheses, the operator T given by the formulae* (6.2.2)–(6.2.3) *is densely defined and linear, and it holds that*

$$T \in \mathbb{B}(H) \iff (\lambda_n) \in \ell^\infty \tag{6.2.4}$$

$$T \text{ is compact} \iff \lambda_n \to 0. \tag{6.2.5}$$

In the affirmative case, $\|T\|_{\mathbb{B}(H)} = \|(\lambda_n)\|_{\ell^\infty}$.

PROOF. To see that $D(T)$ is dense, notice that T clearly is defined on any finite linear combination of the e_n, hence on the dense set $\operatorname{span}(e_n)$; linearity follows from the calculus of limits.

If $|\lambda_n| \le C$ for every n, then $(\lambda_n (x \,|\, e_n))$ is in ℓ^2 for all $x \in H$, so $D(T) = H$ by Proposition 4.2.3; and T is bounded with $\|T\| \le \sup |\lambda_n|$ because

$$\|Tx\| \le \Big(\sum_{n=1}^{\infty} |C(x \,|\, e_n)|^2\Big)^{1/2} \le C\|x\|. \tag{6.2.6}$$

Conversely, if $T \in \mathbb{B}(H)$, insertion of $x = e_n$ shows that $|\lambda_n| \le \|T\|$.

Given that $\lambda_n \to 0$, there is a sequence of compact operators (they have finite rank)

$$T_k x = \sum_{n \le k} \lambda_n (x \,|\, e_n) e_n. \tag{6.2.7}$$

T is compact because $T_k \to T$ in $\mathbb{B}(H)$,

$$\|(T - T_k)\|^2 = \sup_{\|x\| \le 1} \sum_{n > k} |\lambda_n|^2 |(x \,|\, e_n)|^2 \le \sup_{n > k} |\lambda_n|^2 \searrow 0. \tag{6.2.8}$$

If $\lambda_n \not\to 0$ there exist an $\varepsilon > 0$ and $n_1 < n_2 < \dots$ such that $|\lambda_{n_k}| > \varepsilon$ for all k. Since (e_{n_k}) is orthonormal,

$$\|Te_{n_j} - Te_{n_k}\|^2 = \|\lambda_{n_j} e_{n_j} - \lambda_{n_k} e_{n_k}\|^2 = |\lambda_{n_j}|^2 + |\lambda_{n_k}|^2 \ge 2\varepsilon^2, \tag{6.2.9}$$

so (Te_{n_k}) is not a Cauchy sequence. Therefore T's image of the unit ball in H does not have compact closure, and T is thus not compact. □

Notice that T given by (6.2.2) is diagonalised in the sense that the coefficient in front of e_n only contains $(x \,|\, e_n)$, the n^{th} coordinate of x with respect to to basis (e_k).

It will be seen later in the so-called Spectral Theorem, that every *self-adjoint*, compact operator actually has the particularly nice form in (6.2.2).

As direct application of Theorem 6.2.4, this chapter is concluded with a useful construction of a compact operator.

EXAMPLE 6.2.5. Consider the Hilbert space $\ell^2(\mathbb{N})$ and the subspace

$$h^1(\mathbb{N}) = \{\, (x_k) \in \ell^2(\mathbb{N}) \mid \sum (1+k^2)|x_k|^2 < \infty \} \tag{6.2.10}$$

(met in connection with Fourier series in Example 5.2.3). It is straightforward to see that h^1 is a Hilbert space with respect to the norm

$$\|(x_k)\|_{h^1} = (\sum (1+k^2)|x_k|^2)^{1/2}. \tag{6.2.11}$$

Clearly the injection $h^1 \hookrightarrow \ell^2$ is continuous, for $\|(x_k)\|_{\ell^2} \le \|(x_k)\|_{h^1}$; but the identity $I\colon h^1 \to \ell^2$ is actually even *compact*, and for this reason h^1 is said to be compactly embedded into ℓ^2.

The compactness follows from Theorem 6.2.4; indeed K given by

$$K(x_k) = ((1+k^2)^{-1/2} x_k), \tag{6.2.12}$$

is compact in ℓ^2 because $(1+k^2)^{-1/2} \to 0$ for $k \to \infty$; and K is an isometry onto h^1, so $K^{-1}\colon h^1 \to \ell^2$ is bounded. So, to any bounded sequence v_n of vectors in h^1, there is $B > 0$ for which

$$\|K^{-1} v_n\|_{\ell^2} \le B \quad \text{for every} \quad n, \tag{6.2.13}$$

and because $v_n = KK^{-1} v_n$, where K is compact, there exists a subsequence (v_{n_p}) converging in ℓ^2. It follows that I is compact.

CHAPTER 7

Basic Spectral Theory

The idea behind spectral theory is that by representing the elements of $\mathbb{B}(H)$ by a suitable subset of $\mathbb{C}$, called the *spectrum*, one can get a useful overview of the complicated behaviour such operators may have. This is in analogy with the spectral lines used to describe the wavelengths entering various (whitish) light rays.

However, in Linear Algebra an $n \times n$-matrix is usually seen as an operator $\mathbb{C}^n \to \mathbb{C}^n$ and its spectrum consists of its *complex* eigenvalues (in order that characteristic roots in $\mathbb{C} \setminus \mathbb{R}$ do not require special treatment). Hence the above-mentioned idea is only really fruitful if spectra of complex numbers are allowed; and since only normal matrices are unitarily equivalent to diagonal matrices, strong results can only be expected for certain subclasses of $\mathbb{B}(H)$. Indeed, this leads one to the Spectral Theorem for self-adjoint, compact operators in Theorem 7.2.3 below.

At no extra cost, the general definitions and basic results, even for unbounded operators, will be given first.

7.1. On spectra and resolvents

Let in the following H be a complex Hilbert space and T be a linear operator in H, that is $D(T) \subset H$. Recall that $Z(T)$ denotes the null-space of T whilst $R(T)$ stands for its range.

It is a central notion to study the following operator $R_\lambda(T)$, which is parametrised by certain $\lambda \in \mathbb{C}$:

$$R_\lambda(T) = (T - \lambda I)^{-1}. \tag{7.1.1}$$

More precisely, this is defined whenever it makes sense, so λ should be such that $T - \lambda I$ is injective and then $D(R_\lambda(T)) = R(T - \lambda I)$.

Since T need not be everywhere defined, it might be worthwhile to write out (7.1.1) in all details: the requirement is that

$$R_\lambda(T)(Tx - \lambda x) = x \quad \text{for every} \quad x \in D(T) \tag{7.1.2}$$

$$(T - \lambda I)R_\lambda(T)y = y \quad \text{for every} \quad y \in R(T - \lambda I). \tag{7.1.3}$$

The operator $R_\lambda(T)$ is called the *resolvent* of T, because it (re)solves the problem of finding, for given data $y \in H$, those $x \in H$ for which

$$Tx - \lambda x = y. \tag{7.1.4}$$

Indeed, provided that λ is such that $T-\lambda I$ is injective, any solution to this equation is unique, and it exists if and only if $y \in D(R_\lambda(T))$; in the affirmative case it is given by $x = R_\lambda(T)y$; cf (7.1.3).

For simplicity $R_\lambda := R_\lambda(T)$ when T is fixed. It is clear from the above that R_λ exists if and only if λ is not an eigenvalue of T. However, in order to have a name for those λ for which R_λ has nice properties, it is customary to introduce two sets:

DEFINITION 7.1.1. 1°. A complex number λ belongs to the resolvent *set* of T, denoted by $\rho(T)$, if R_λ exists, is densely defined and bounded (on its domain $R(T-\lambda I)$).

2°. The *spectrum* of T is the complement of $\rho(T)$, ie $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

EXAMPLE 7.1.2. Even a simple case may be instructive: consider the injection of a subspace $I_V\colon V \hookrightarrow H$, where V is dense in H (an often met situation). Then it is clear that $\lambda = 1$ is an eigenvalue of I_V because $I_V - \lambda I \subset 0$. In sharp contrast to this, any $\lambda \neq 1$ is in the resolvent set, $R_\lambda(I_V)$ being multiplication by $(1-\lambda)^{-1}$ on the dense set V (clearly R_λ is then restriction to V of an element in $B(H)$).

For clarity it should be emphasised that R_λ for each $\lambda \in \rho(T)$ necessarily has an extension by continuity to an operator in $\mathbb{B}(H)$. But when T is closed, then R_λ itself is in $\mathbb{B}(H)$:

LEMMA 7.1.3. *Let T be a closed linear operator in H and let $\lambda \in \rho(T)$. Then R_λ is everywhere defined, ie $D(R_\lambda) = H$.*

PROOF. By the definition of resolvent set, it suffices to show that $D(R_\lambda)$ is closed. Let $y_n := (T-\lambda I)x_n \to y$. Since $x_n = R_\lambda y_n$ and R_λ is bounded, clearly (x_n) is a Cauchy sequence. Hence $x_n \to x$ for some $x \in H$. Since $T-\lambda I$ is closed too, $x \in D(T-\lambda I)$ with $(T-\lambda I)x = y$. Ie $y \in D(R_\lambda)$. □

Some authors specify $\rho(T)$ by the requirement that $R_\lambda(T)$ should belong to $\mathbb{B}(H)$; since most operators in the applications are closed (if not bounded), this usually gives the same subset of $\mathbb{C}$ by the above lemma. However, the present definition is slightly more general and flexible.

In view of Definition 7.1.1 there are three different reasons why a given number λ could belong to $\sigma(T)$.

- Either $T-\lambda I$ is not injective.
- Or $T-\lambda I$ is injective but far from surjective, in the sense that $\overline{D(R_\lambda)} \neq H$.
- Or, finally, $T-\lambda I$ is injective with dense range, so that R_λ is densely defined; but R_λ is unbounded.

In the third case the criterion for boundedness of R_λ is whether there exists some constant $c_\lambda > 0$ such that

$$\|(T-\lambda I)x\| \geq c_\lambda \|x\| \quad \text{for all} \quad x \in D(T). \tag{7.1.5}$$

Corresponding to these three possibilities, one says that λ is an eigenvalue of T, or belongs to $\sigma_{\mathrm{p}}(T)$, the so-called *point spectrum* of T; or that

λ is in the *residual* spectrum of T written $\sigma_{\text{res}}(T)$; respectively that λ is in the *continuous* spectrum of T, ie$\sigma_{\text{cont}}(T)$.

This gives a *disjoint* decomposition of $\sigma(T)$ as

$$\sigma(T) = \sigma_{\text{p}}(T) \cup \sigma_{\text{res}}(T) \cup \sigma_{\text{cont}}(T). \tag{7.1.6}$$

One should observe that $R_\lambda(T)$ is defined on the set $\mathbb{C} \setminus \sigma_{\text{p}}(T)$, so that it also makes sense as an operator in H for λ in $\sigma_{\text{res}}(T) \cup \sigma_{\text{cont}}(T)$. The resolvent set $\rho(T)$ is the smaller set where R_λ is densely defined and (7.1.5) holds.

To demystify the notion of spectrum, it is shown now that one can read off immediately what $\sigma(T)$ is when T is *diagonalisable*:

PROPOSITION 7.1.4. *Let T be an operator in a Hilbert space H, with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, defined from a (not necessarily bounded) sequence (λ_n) in $\mathbb{F}$ as in Theorem 6.2.4; that is*

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x \,|\, e_n) e_n. \tag{7.1.7}$$

Then $\Lambda = \{\, \lambda_n \mid n \in \mathbb{N} \}$ equals the point spectrum of T, ie $\sigma_{\text{p}}(T) = \Lambda$; the residual spectrum is empty; and $\sigma_{\text{cont}}(T) = \bar{\Lambda} \setminus \Lambda$. Consequently $\sigma(T) = \bar{\Lambda}$.

PROOF. Clearly $\Lambda \subset \sigma_p(T)$, so let $\lambda \in \mathbb{C} \setminus \Lambda$. For every $x \in D(T)$

$$Tx - \lambda x = \sum_{n=1}^{\infty} (\lambda_n - \lambda)(x \,|\, e_n) e_n. \tag{7.1.8}$$

So if $Tx - \lambda x = 0$, then $(\lambda - \lambda_n)(x \,|\, e_n) = 0$ for every n, and this entails $x \perp \operatorname{span}(e_n)$, hence $x = 0$; so λ is not an eigenvalue, ie $\sigma_{\text{p}}(T) = \Lambda$. Using this for T^*, it follows for $\lambda \in \mathbb{C} \setminus \Lambda$ that $\bar{\lambda} \notin \sigma_{\text{p}}(T^*)$, ie $Z(T^* - \bar{\lambda} I) = (0)$; whence $H = \overline{R(T - \lambda I)}$. This means that $\sigma_{\text{res}}(T) = \emptyset$.

Let $c_\mu = \inf\{\, |\mu - \lambda| \mid \lambda \in \Lambda \}$ for $\mu \in \mathbb{C}$. Notice that $c_\mu > 0$ is equivalent to $\mu \notin \bar{\Lambda}$. For all $x \in D(T)$ it is seen from (7.1.8) that

$$\|Tx - \mu x\| = \Big(\sum |\lambda_n - \mu|^2 |(x \,|\, e_n)|^2\Big)^{1/2} \geq c_\mu \|x\|. \tag{7.1.9}$$

Clearly this property cannot hold for any constant $c > c_\mu$. So if $\mu \in \bar{\Lambda} \setminus \Lambda$ it follows that μ belongs to neither $\sigma_{\text{p}}(T)$ nor $\sigma_{\text{res}}(T)$, but since $c_\mu = 0$ it holds that $\mu \in \sigma_{\text{cont}}(T)$ (cf (7.1.5)). Conversely, if μ is an element of the continuous spectrum, then by (7.1.5) it holds that $c_\mu = 0$, so $\mu \in \bar{\Lambda}$; and $\mu \notin \Lambda$ since it is not an eigenvalue. □

It is a fascinating programme of spectral theory to prove that the spectrum of an operator "behaves the like the operator does". To explain this, consider the following types of operators in $\mathbb{B}(H)$:

$$T \quad \text{self-adjoint}, \qquad T^* = T \tag{7.1.10}$$
$$U \quad \text{unitary}, \qquad U^*U = UU^* = I \tag{7.1.11}$$
$$P \quad \text{projection}, \qquad P^2 = P \tag{7.1.12}$$
$$T \quad \text{positive}, \qquad (Tx\,|\,x) \geq 0 \quad \text{for every } x \in H. \tag{7.1.13}$$

The idea is to make replacements $T \rightsquigarrow \lambda$ and $T^* \rightsquigarrow \bar{\lambda}$, whereby $\lambda \in \sigma(T)$ can be arbitrary. For the four cases above this would give

$$\bar{\lambda} = \lambda, \quad \text{ie } \sigma(T) \subset \mathbb{R} \tag{7.1.14}$$
$$\bar{\lambda}\lambda = 1, \quad \text{ie } \sigma(U) \subset \{\, z \in \mathbb{C} \mid |z| = 1 \,\} \tag{7.1.15}$$
$$\lambda^2 - \lambda = 0, \quad \text{ie } \sigma(P) \subset \{\, 0, 1 \,\} \tag{7.1.16}$$
$$\lambda \geq 0, \quad \text{ie } \sigma(T) \subset [0, \infty[. \tag{7.1.17}$$

These inferences are actually true, but in this chapter only the first case will be treated, for simplicity's sake.

REMARK 7.1.5. The four types above are all *normal* operators; an operator $N \in \mathbb{B}(H)$ is normal if it commutes with its adjoint, ie if $N^*N = NN^*$. At first sight, it is surprising that the above replacements for a normal operator gives $\bar{\lambda}\lambda = \lambda\bar{\lambda}$, which is a tautology in all of $\mathbb{C}$. But if N is normal so is $N + zI$ for all $z \in \mathbb{C}$ so that the class of normal operators can have spectra everywhere in $\mathbb{C}$ (and intuitively it is clear that if an operator class $\mathfrak{C}$ does not have this property, then $\mathfrak{C}$ is not a maximal class to develop a spectral theory for). However, for simplicity focus will be restrained to the much smaller class of self-adjoint compact operators here.

7.1.1. The self-adjoint case. For an operator T in a Hilbert space H to be self-adjoint it is necessary that the adjoint should be defined, whence that $D(T)$ should be dense in H. Denseness of $D(T)$ assumed throughout this section; clearly it then holds that

$$R(T - \lambda I)^\perp = Z(T^* - \bar{\lambda} I) \quad \text{for} \quad \lambda \in \mathbb{C}. \tag{7.1.18}$$

For spectra one has the elementary observation that any eigenvalue of a self-adjoint operator T is real, ie $\sigma_{\mathrm{p}}(T) \subset \mathbb{R}$. Indeed, if $Tx = \lambda x$ for a non-trivial x, say with $\|x\| = 1$,

$$\bar{\lambda} = (x\,|\,\lambda x) = (x\,|\,T^* x) = (Tx\,|\,x) = \lambda. \tag{7.1.19}$$

Moreover, for $T = T^*$ the eigenspaces are orthogonal; ie $Z(T - \lambda I) \perp Z(T - \mu I)$ for $\lambda \neq \mu$. For if $Tx = \lambda x$ and $Ty = \mu y$, then $\lambda(x\,|\,y) = (x\,|\,T^* y) = \mu(x\,|\,y)$, so that $x \perp y$.

Furthermore, for $T = T^*$ the right hand side of (7.1.18) equals $Z(T - \lambda I)$ for every eigenvalue. This implies the fundamental facts in

PROPOSITION 7.1.6. *For a densely defined operator T in H*

$$T = T^* \implies \begin{cases} \sigma_{\text{res}}(T) = \emptyset, \\ \sigma(T) \subset \mathbb{R}. \end{cases} \tag{7.1.20}$$

PROOF. For $\lambda \notin \sigma_{\text{p}}(T)$ it follows from (7.1.18) that $R(T-\lambda I)$ is dense, whence $\sigma_{\text{res}}(T) = \emptyset$. Since $T = T^*$ the number $(Tx\,|\,x)$ is always real, so (4.1.9) gives for real β

$$\|Tx - \mathrm{i}\beta x\|^2 = \|Tx\|^2 + \|\beta x\|^2 + 2\operatorname{Re}\mathrm{i}\beta(Tx\,|\,x) \geq |\beta|^2\|x\|^2. \tag{7.1.21}$$

This formula also applies to $T - \alpha I$ for $\alpha \in \mathbb{R}$, since this is self-adjoint; therefore $T - (\alpha + \mathrm{i}\beta)I$ with $\alpha \in \mathbb{R}$, $\beta \neq 0$ is injective and has dense range (since $\sigma_{\text{res}}(T) = \emptyset$ has just been proved) and bounded inverse. Hence $\rho(T) \supset (\mathbb{C} \setminus \mathbb{R})$. □

If $\lambda \in \mathbb{C}$ is such that there exists a sequence (x_n) in H with $\|x_n\| = 1$ for every n and such that $\|Tx_n - \lambda x_n\| \to 0$, the x_n are called *approximate* eigenvectors corresponding to λ, although λ need not be an eigenvalue. But in the self-adjoint case, the approximate eigenvectors characterise the spectrum:

PROPOSITION 7.1.7. *Let T be a self-adjoint operator in a Hilbert space H. Then $\lambda \in \sigma(T)$ if and only if there is a sequence (x_n) of approximate eigenvectors corresponding to λ.*

PROOF. If such a sequence exists, then either $\lambda \in \sigma_{\text{p}}(T)$ or (7.1.2) implies that R_λ is unbounded, so $\lambda \in \sigma(T)$. Conversely, given λ in $\sigma_{\text{p}}(T)$, the claim is trivial for the sequence may be taken constantly equal to a normalised eigenvector. Otherwise $\lambda \in \sigma_{\text{cont}}(T)$ (cf Proposition 7.1.6), and

$$0 = \inf\{\, \|Tx - \lambda x\| \,\big|\, x \in H,\ \|x\| = 1 \,\} \tag{7.1.22}$$

by (7.1.5); hence there exists (x_n) as desired. □

7.1.2. Examples. First a perspective is put on linear algebra from the present point of view. Secondly it will be seen that eg differential operators can have spectra that are much larger sets than the spectra met in linear algebra; indeed even $\sigma(T) = \mathbb{C}$ is possible. Lastly, also bounded operators may have uncountable spectra.

EXAMPLE 7.1.8. Any linear map $T\colon \mathbb{C}^n \to \mathbb{C}^n$ may be represented by a matrix, say with respect to the natural basis in $\mathbb{C}^n$; the eigenvalues of T are precisely the characteristic roots of the matrix. Repeating eigenvalues according to the multiplicities, $\sigma_{\text{p}}(T) = \{\lambda_1, \dots, \lambda_n\}$. When λ is not an eigenvalue, $T - \lambda I$ is injective and hence a surjection; moreover, $R_\lambda(T)$ is in $\mathbb{B}(\mathbb{C}^n)$, so λ is in the resolvent set then. Altogether T has *pure* point spectrum and $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$ whilst $\rho(T) = \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_n\}$.

EXAMPLE 7.1.9. Consider $\partial = \frac{d}{dt}$ with domain $C^1([0,1])$ as an operator in $H = L^2(]0,1[)$. Clearly $(\partial - \lambda I)e^{\lambda t} = 0$ for every $\lambda \in \mathbb{C}$; therefore $\sigma_p(\partial) = \mathbb{C}$ so that ∂ has pure point spectrum. The resolvent set is empty, $\rho(\partial) = \emptyset$, for the spectrum of ∂ fills the entire complex plane.

EXAMPLE 7.1.10 (The one-sided shift operator). In $\mathbb{B}(\ell^2(\mathbb{N}))$ there is an operator T given by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots). \tag{7.1.23}$$

For every $\lambda \in \mathbb{C}$ and $x = (x_n) \in \ell^2$, the equation $(T - \lambda I)x = 0$ is equivalent to the system where $x_{j+1} = \lambda x_j$ for every $j \in \mathbb{N}$. If only those x with $x_1 = 1$ are considered, then this is equivalent to

$$x_{j+1} = \lambda^j \quad \text{for every} \quad j \in \mathbb{N}; \tag{7.1.24}$$

the sequence defined by this is in ℓ^2 if and only if $\sum_{j=1}^{\infty} |\lambda^{j-1}|^2 < \infty$, which is the case precisely when $|\lambda| < 1$. It follows from this analysis that λ is an eigenvalue of T if and only if $|\lambda| < 1$; hence $\sigma_{\mathrm{p}}(T)$ is the *open* unit disk in $\mathbb{C}$.

Because $\|Tx\| = \|x\|$ holds if $x_1 = 0$, it follows that $\|T\| = 1$. As a consequence of results proved below, $\sigma(T)$ is a closed set contained in $\{ z \in \mathbb{C} \mid |z| \le 1 \}$. It was found above that $\sigma(T)$ is dense in this, so $\sigma(T)$ equals the closed unit disk.

7.1.3. Spectral theory for $\mathbb{B}(H)$. For an operator $T \in \mathbb{B}(H)$, where H is a Hilbert space, there are a few facts on spectra and resolvent sets that may be established without any further assumptions. Such results are very convenient eg for the determination of specific spectra, as seen in Example 7.1.10 above.

Notice that since any $T \in \mathbb{B}(H)$ is closed, $R_\lambda(T) \in \mathbb{B}(H)$ for every $\lambda \in \rho(T)$ because of Lemma 7.1.3.

PROPOSITION 7.1.11. *Let H be a Hilbert space and $T \in \mathbb{B}(H)$. Then the resolvent set of T is an* open *subset of $\mathbb{C}$ and the map $\rho(T) \to \mathbb{B}(H)$ given by $\lambda \mapsto R_\lambda(T)$ is continuous in the norm topology of $\mathbb{B}(H)$.*

PROOF. If $\rho(T) = \emptyset$, it is open; so let $\mu \in \rho(T)$. Then $R_\mu \in \mathbb{B}(H)$ and $(T - \mu I)R_\mu x = x$ for every $x \in H$. Therefore every $\lambda \in \mathbb{C}$ gives

$$T - \lambda I = T - \mu I - (\lambda - \mu)I = (T - \mu I)(I - (\lambda - \mu)R_\mu). \tag{7.1.25}$$

Here the right hand side has an inverse in $\mathbb{B}(H)$ if both factors have that; by the Neumann series this is the case if

$$\|(\lambda - \mu)R_\mu\| < 1. \tag{7.1.26}$$

This holds for all λ such that $|\lambda - \mu| < \|R_\mu\|^{-1}$, ie in a ball around μ. Thus $\rho(T)$ is shown to consist of interior points only.

When $|\lambda-\mu| < \|R_\mu\|^{-1}$, one can invert both sides of the identity above and subtract R_μ; in this way,

$$\|R_\lambda - R_\mu\| = \|\sum_{k=1}^{\infty}(\lambda-\mu)^k R_\mu^{k+1}\| \le \frac{|\lambda-\mu|\|R_\mu\|^2}{1-|\lambda-\mu|\|R_\mu\|}. \tag{7.1.27}$$

This implies that $\|R_\lambda - R_\mu\| \to 0$ for $\lambda \to \mu$, as claimed. □

One can also prove that $\lambda \mapsto R_\lambda$ is holomorphic (in a specific sense), but details are omitted here.

It is easy to imagine that boundedness of an operator T on H should have consequences for the spectrum of T; eg it would be natural to expect that $\sigma(T)$ must be bounded for bounded T. But more than that holds:

PROPOSITION 7.1.12. *Let $T \in \mathbb{B}(H)$ for some Hilbert space H. Then $\sigma(T)$ is a* compact *set in $\mathbb{C}$ and it is contained in the closed ball of radius $\|T\|$ and centre* 0.

PROOF. In view of Proposition 7.1.11, compactness of $\sigma(T)$ follows if it can be shown to be bounded. So it suffices to show that every λ with $|\lambda| > \|T\|$ is in $\rho(T)$. But for such λ the operator $T-\lambda I = -\lambda(I-\frac{1}{\lambda}T)$ has a bounded inverse, since $\frac{1}{\lambda}T$ has norm less than 1. □

The ball referred to in this result is often called the *norm ball* of T. There is another natural ball in $\mathbb{C}$ to consider for $T \in \mathbb{B}(H)$, namely the *smallest* ball centred at 0, which contains $\sigma(T)$. To make this precise we need

DEFINITION 7.1.13. For an operator T in a Hilbert space H, the *spectral radius* of T is the number

$$r(T) = \sup\{\, |\lambda| \mid \lambda \in \sigma(T)\,\}. \tag{7.1.28}$$

For $T \in \mathbb{B}(H)$ it is seen from Proposition 7.1.12 that $r(T) \le \|T\|$. Moreover, the supremum is attained because $\sigma(T)$ is compact, so $\overline{B}(0, r(T)) \supset \sigma(T)$; no smaller ball has this property, whence $\overline{B}(0, r(T))$ is *the* smallest ball containing the spectrum of T, as desired. Ie

$$r(T) = \inf\{\, \mu > 0 \mid \sigma(T) \subset B(0,\mu)\,\}. \tag{7.1.29}$$

REMARK 7.1.14. For an operator T in $\mathbb{B}(H)$ that is normal, ie $T^*T = TT^*$, it is a cornerstone of the theory that the two numbers are equal:

$$T \quad \text{is normal} \implies r(T) = \|T\|. \tag{7.1.30}$$

This result has many applications, eg in the spectral theorem of normal operators (and also in applied mathematics).

It would however lead too far to prove this here. But for self-adjoint *compact* operators, it will be proved as a substitute in the next section that either $\pm\|T\|$ is an eigenvalue (implying the above formula for such operators). Using this it is possible to give a relatively elementary proof of the spectral theorem for such operators anyway.

7.2. Spectra of compact operators

The main goal of this section is to prove the Spectral Theorem of compact, self-adjoint operators on Hilbert spaces.

It will be clear further below that compact self-adjoint operators have spectra consisting mainly of eigenvalues. Therefore it is natural to observe already now that these (except possibly for 0) always have finite multiplicity.

PROPOSITION 7.2.1. *Let T be a compact operator on a Hilbert space H. For every eigenvalue $\lambda \neq 0$ the corresponding eigenspace*

$$H_\lambda = \{\, x \in H \mid Tx = \lambda x \,\} \tag{7.2.1}$$

has finite dimension, ie $\dim H_\lambda < \infty$.

PROOF. Assuming that H_λ has a sequence of linearly independent vectors, there is even an orthonormal sequence (x_n) in H_λ. By Pythagoras, $\|x_{n+k} - x_n\| = \sqrt{2}$, and since $T|_{H_\lambda}$ just multiplies by $\lambda \neq 0$, the sequence (Tx_n) has no fundamental subsequences. Therefore T is not compact. □

The next result is essential for the proof of the Spectral Theorem. It holds quite generally, cf Remark 7.1.14, but in the context of compact operators there is a rather elementary proof.

PROPOSITION 7.2.2. *When T is a compact, self-adjoint operator on a Hilbert space, then the spectral radius formula is valid, that is*

$$r(T) = \|T\|, \tag{7.2.2}$$

for either $\lambda = \|T\|$ or $\lambda = -\|T\|$ is an eigenvalue of T. Moreover,

$$\|T\| = \sup\{\, |(Tx\,|\,x)| \mid x \in H,\ \|x\| = 1 \,\} \tag{7.2.3}$$

and the supremum is attained for an eigenvector in (at least) one of the spaces $H_{\pm\|T\|}$.

PROOF. The expression for $\|T\|$ was shown in Lemma 6.1.2, and it suffices to show that the supremum is attained in the claimed way, for then $\sigma(T)$ contains one of $\pm\|T\|$, so that $\|T\| \le r(T)$.

Take first a normalised sequence (x_n) such that $|(Tx_n\,|\,x_n)| \to \|T\|$. Then $(Tx_n\,|\,x_n)$ has an accumulation point in $\{-\|T\|, \|T\|\}$. Denoting any of these by λ and extracting a subsequence (y_n) for which $(Ty_n\,|\,y_n) \to \lambda$, it is seen that

$$\begin{aligned} \|Ty_n - \lambda y_n\|^2 &= \|Ty_n\|^2 + |\lambda|^2 - 2\operatorname{Re}\lambda(Ty_n\,|\,y_n) \\ &\le 2\lambda^2 - 2\lambda(Ty_n\,|\,y_n) \searrow 0. \end{aligned} \tag{7.2.4}$$

Therefore (y_n) is a sequence of approximate eigenvectors corresponding to λ, whence $\lambda \in \sigma(T)$.

It follows that λ is an eigenvalue; for $T = 0$ this is trivial, so assume that $\lambda > 0$. Because T is compact, it may be assumed that (y_n) is such that (Ty_n) converges. However, $\lim(Ty_n - \lambda y_n) = 0$ so also (y_n) converges, say to some $z \in H$. By continuity $\|z\| = 1$ and $Tz = \lambda z$, so λ is an eigenvalue; and $(Tz\,|\,z) = \lambda$ so that the supremum is a maximum in the claimed way. □

THEOREM 7.2.3 (Spectral Theorem for Compact Self-adjoint Operators). *Let H be a separable Hilbert space and $T \in \mathbb{B}(H)$ a compact, self-adjoint operator. Then H has an orthonormal basis $(e_j)_{j\in J}$, with index set $J \subset \mathbb{N}$, of eigenvectors for T with corresponding eigenvalues $\lambda_j \in \mathbb{R}$. This means that*

$$\forall x \in H\colon \quad x = \sum_j (x\,|\,e_j)e_j \;\wedge\; Tx = \sum_j \lambda_j (x\,|\,e_j)e_j. \tag{7.2.5}$$

In the affirmative case either $\dim H < \infty$, *or it holds that $\lambda_j \to 0$ and $\sigma(T) = \{0\} \cup \{\lambda_j \mid j \in \mathbb{N}\}$.*

For H of finite dimension, the statement is clearly that any $T = T^*$ has a diagonal matrix $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ with respect to a certain basis.

REMARK 7.2.4. When H is infinite dimensional, T can either have finite rank or the non-zero λ_j form a sequence which may be numbered such that

$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_j| \ge \cdots > 0. \tag{7.2.6}$$

With *this* convention (7.2.5) would not be valid if $Z(T) \ne (0)$ (the eigenvalue $\lambda = 0$ will not be counted by (7.2.6)). As a remedy one can use $H = Z(T) \oplus Z(T)^\perp$ to add a vector $x_0 \in Z(T)$ to the expansion of x, for in $Tx = \sum_j \lambda_j (x\,|\,e_j)e_j$ only the $\lambda_j \ne 0$ need enter.

PROOF. 1°. The last claim is a consequence of Proposition 7.1.4.

2°. Notice that if $Q \subset H$ is a closed, T-invariant subspace, then $T|_Q$ is both self-adjoint and compact in $\mathbb{B}(Q)$. Indeed, $(Tx\,|\,y) = (x\,|\,Ty)$ holds in particular for x, $y \in Q$, and if $B \subset Q$ is a bounded set then $T(B) \subset Q \cap K$ for some compact set $K \subset H$; and $Q \cap K$ is compact since Q is closed.

3°. Consider the case in which, for some $n \in \mathbb{N}$, there are eigenvalues $|\lambda_1| \ge \cdots \ge |\lambda_n|$ with orthonormalised eigenvectors $e_1, \ldots, e_n$ together with closed, T-invariant subspaces $Q_1 \supset \cdots \supset Q_n$ fulfilling $Q_k = \operatorname{span}(e_1, \ldots, e_k)^\perp$; and moreover, for $k = 1, \ldots, n$,

$$|\lambda_k| = \max\bigl\{\, |(Tx\,|\,x)| \;\big|\; x \in Q_{k-1}, \quad \|x\| = 1 \,\bigr\}. \tag{7.2.7}$$

Observe that with $Q_0 = H$ this actually holds for $n = 1$, since firstly Proposition 7.2.2 shows that (λ_1, e_1) exists and fulfils (7.2.7), secondly $Q_1 = \{e_1\}^\perp$ is T-invariant because $(Tq\,|\,e_1) = \lambda_1(q\,|\,e_1) = 0$ holds for $q \in Q_1$.

Now $Q_n = \{0\}$ would imply $H = \operatorname{span}(e_1, \ldots, e_n)$, and then (7.2.5) would be evident. And if $Q_n \ne \{0\}$, Proposition 7.2.2 applies to $T|_{Q_n}$ in view of 2°, and this gives a pair (λ_{n+1}, e_{n+1}) in $\mathbb{R} \times Q_n$ fulfilling (7.2.7) for $k = n+1$ and $Te_{n+1} = \lambda_{n+1}e_{n+1}$, $\|e_{n+1}\| = 1$. Then the subspace $Q_{n+1} = \operatorname{span}(e_1, \ldots, e_{n+1})^\perp$ is closed and T-invariant, and (7.2.7) implies that $|\lambda_{n+1}| \le |\lambda_n|$ while $(e_1, \ldots, e_{n+1})$ is orthonormal (since $e_{n+1} \in Q_n$).

4°. For $\dim H = \infty$ one may by 3° define λ_n, e_n inductively so that $(|\lambda_n|)$ is a decreasing, non-negative hence convergent sequence. But $\lambda_n = \|Te_n\| \to 0$, because T is compact. $\sigma(T)$ is closed so it contains the limit 0.

5°. The main case is when $|\lambda_n| > 0$ for all $n \in \mathbb{N}$. Then $H = M \oplus M^\perp$ for $M = \overline{\operatorname{span}}\{ e_n \mid n \in \mathbb{N} \}$. Here $M^\perp = Z(T)$, for since $M^\perp \subset \cap_n Q_n$ it holds for any $y \in M^\perp$ that

$$\forall n \in \mathbb{N}\colon \|Ty\| \leq \|T\|_{\mathbb{B}(Q_n)} \|y\| \leq |\lambda_{n+1}| \|y\| \searrow 0, \tag{7.2.8}$$

so $T|_{M^\perp} = 0$; conversely any $z \in Z(T)$ equals $m + m^\perp$ for $m \in M$ and $m^\perp \in M^\perp \subset Z(T)$, and here $m = 0$ because $m = \sum \alpha_n e_n$ yields $0 = Tz = \sum \lambda_n \alpha_n e_n$ so that $\lambda_n \alpha_n = 0$ for all n.

Since $M^\perp$ is separable (it is closed), it has a countable orthonormal basis $\{ f_1, f_2, \dots \}$. The orthonormal set $\{ e_1, f_1, e_2, f_2, \dots \}$ is a basis for H, for if $x = m + z$ with $m \in M$ and $z \in Z(T)$, then $m = \sum \alpha_n e_n$ and $z = \sum \beta_n f_n$; then the triangle inequality gives

$$x = m + z = \lim_{n \to \infty} \sum_{j=1}^{n} (\alpha_j e_j + \beta_j f_j). \tag{7.2.9}$$

(It is understood that the terms $\beta_j f_j$ only occur for $j \leq \dim M^\perp$.) Corresponding to this basis there are the eigenvalues $\{ \lambda_1, 0, \lambda_2, 0, \dots \}$. Renumerating both this and the basis for H one obtains $(\lambda_n)_{n \in \mathbb{N}}$ and $(e_n)_{n \in \mathbb{N}}$. The first part of (7.2.5) has just been proved above, but for $x \in H$,

$$T\Big(\sum_{j=1}^{\infty} (x \,|\, e_j) e_j\Big) = \lim_{n \to \infty} \sum_{j=1}^{n} \lambda_j (x \,|\, e_j) e_j = \sum_{j=1}^{\infty} \lambda_j (x \,|\, e_j) e_j, \tag{7.2.10}$$

so also the second part of (7.2.5) holds.

6° Finally, T has finite rank if and only if there is some N such that $\lambda_n = 0$ for $n > N$. One may then proceed as in 5° with the modification that M should equal $\operatorname{span}\{ e_1, \dots, e_N \}$; details are left for the reader. □

It is clear now that (if the case $\dim H < \infty$ is excluded) a self-adjoint, compact operator T on H *always* has 0 as a very special point of it spectrum: indeed, the eigenvalues $\lambda \neq 0$ are isolated and have finite-dimensional eigenspaces H_λ, by Proposition 7.2.1 — but 0 has infinite multiplicity if $\dim Z(T) = \infty$, eg if $\operatorname{rank} T$ is finite, and in any case it is an accumulation point since $\lambda_n \to 0$, also if $Z(T) = (0)$. Therefore the point 0 has a character rather different from the rest of $\sigma(T)$ (it belongs to $\sigma_{\text{ess}}(T)$, the so-called *essential* spectrum of T).

In view of this the Spectral Theorem conveys two messages, one about the structure of $\sigma(T)$ and the second being that such T may be diagonalised; cf (7.2.5).

The Spectral Theorem has various generalisations, eg a version for normal operators $T \in \mathbb{B}(H)$, but in such cases $\sigma(T)$ is usually uncountable, so that the sum in (7.2.5) needs to be replaced by certain integrals. The reader may consult the literature for this.

EXAMPLE 7.2.5. As an application of the Spectral theorem, one can for a compact, self-adjoint operator $T \in \mathbb{B}(H)$ discuss the solvability of

$$(T - \lambda I)x = y \tag{7.2.11}$$

for given data $y \in H$. The interesting case is $\dim H = \infty$, and additionally $\lambda \neq 0$ is assumed (for even if T^{-1} exists it is unbounded).

In the notation of Theorem 7.2.3, (7.2.11) is equivalent to

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda)(x \,|\, e_n)e_n = \sum_{n=1}^{\infty} (y \,|\, e_n)e_n, \tag{7.2.12}$$

hence to

$$\forall n \in \mathbb{N}\colon\ (\lambda_n - \lambda)(x \,|\, e_n) = (y \,|\, e_n). \tag{7.2.13}$$

Since $(\frac{1}{\lambda_n - \lambda})$ is a bounded sequence for $\lambda \notin \sigma(T)$ and $((y \,|\, e_n)) \in \ell^2$, equation (7.2.11) is therefore uniquely solved by

$$x = \sum_{\lambda_n \neq \lambda} \tfrac{(y \,|\, e_n)}{\lambda_n - \lambda} e_n. \tag{7.2.14}$$

This reflects the solution formula $x = R_\lambda(T)y$, valid for $\lambda \in \rho(T)$.

For $\lambda \in \sigma(T) \setminus \{0\}$ it is necessary for solvability of (7.2.11) that $y \in \overline{R(T - \lambda I)}$, ie $y \in Z(T - \lambda I)^\perp$; with $Z(T - \lambda I) = \operatorname{span}(e_{i_1}, \dots, e_{i_N})$ this means

$$(y \,|\, e_{i_j}) = 0 \quad \text{for} \quad j = 1, \dots, N. \tag{7.2.15}$$

This condition is also sufficient, for the right hand side of (7.2.12) is then a sum over $n \notin \{i_1, \dots, i_N\}$, ie over $\lambda_n \neq \lambda$, so that (7.2.14) also defines a solution of (7.2.11) in this case (seen by simple insertion). This reflects the invertibility of $T - \lambda I$ on $Z(T - \lambda I)^\perp$.

The result in (7.2.14) is remarkable because it is a solution formula for the "infinitely many equations with infinitely many unknowns" in (7.2.11). (Notice that the discussion does not carry over to $\lambda = 0$, because the sequence $(\frac{1}{\lambda_n - \lambda})$ is unbounded then.)

The reader may have noticed that the question of the closedness of $R(T - \lambda I)$ not only appeared implicitly above, but also disappeared again. This indicates that the next statement should be true.

LEMMA 7.2.6. *Let $T = T^*$ be a compact operator on a Hilbert space H and let $\lambda \neq 0$ be an eigenvalue. Then $R(T - \lambda I)$ is closed in H.*

PROOF. $c_\lambda := \min\{\, |\mu - \lambda| \mid \mu \in \sigma(T) \setminus \{\lambda\} \,\} > 0$ since λ is not an accumulation point of $\sigma(T)$. For a Cauchy sequence (y_k) in $R(T - \lambda I)$, let (x_k) be defined by means of (7.2.14). Then

$$\|x_k - x_m\|^2 \leq c_\lambda^{-2} \sum_{n=1}^{\infty} |(y_k - y_m \,|\, e_n)|^2 = c_\lambda^{-2} \|y_k - y_m\|^2, \tag{7.2.16}$$

so that x_k converges to some x in H and $y_n \to (T - \lambda I)x$. □

Using this lemma and that $H = R(T) \oplus Z(T^*)$, one can now most easily derive a famous result.

EXAMPLE 7.2.7 (Fredholm's Alternative). Let T be a self-adjoint, compact operator on a separable Hilbert space H. For given data $y \in H$ and $\lambda \neq 0$, uniqueness of the solutions to

$$(T - \lambda I)x = y \tag{7.2.17}$$

implies the *existence* of a solution $x \in H$. (This is the case if $\lambda \in \rho(T)$.)

Alternatively there are non-trivial solutions to the homogeneous equation $(T - \lambda I)z = 0$, and then there exist solutions $x \in H$ of (7.2.17) if and only if $y \perp Z(T - \lambda I)$. In the affirmative case the complete solution equals $x_0 + Z(T - \lambda I)$ for some particular solution x_0 of (7.2.17). (This holds for $\lambda \in \sigma(T)$.)

Nowadays this conclusion is rather straightforward, but it was established by Fredholm for integral operators around 1900, decades before the notion of operators (not to mention their spectral theory) was coined in the present concise form.

7.3. Functional Calculus of compact operators

Using the Spectral Theorem, it is now easy to give a precise meaning to functions $f(T)$ of certain operators.

In order to do so, let $B(\sigma(T))$ denote the sup-normed space of bounded functions $\sigma(T) \to \mathbb{C}$.

THEOREM 7.3.1. *Let T be a self-adjoint, compact operator on a separable Hilbert space H with an orthonormal basis (e_n) of H consisting of eigenvectors of T, corresponding to eigenvalues λ_n in $\sigma(T)$.*

Then there is an operator $f(T)$ in $\mathbb{B}(H)$ defined for arbitrary functions $f \in B(\sigma(T))$ by

$$f(T)x = \sum_n f(\lambda_n)(x \,|\, e_n)e_n. \tag{7.3.1}$$

The map $f \mapsto f(T)$ has the properties

$$\|f(T)\|_{\mathbb{B}(H)} = \|f\|_{B(\sigma(T))} \tag{7.3.2}$$

$$f(T)^* = \bar{f}(T) \tag{7.3.3}$$

$$(\lambda f + \mu g)(T) = \lambda f(T) + \mu g(T) \tag{7.3.4}$$

$$f \cdot g(T) = f(T)g(T) \tag{7.3.5}$$

for arbitrary f, $g \in B(\sigma(T))$ and λ, $\mu \in \mathbb{F}$.

For infinite dimensional H and $f \in B(\sigma(T))$,

$$f(T) \text{ is compact} \iff \lim_{t \to 0} f(t) = 0 \wedge [f(0) = 0 \text{ if } \dim Z(T) = \infty]. \tag{7.3.6}$$

Since $B(\sigma(T))$ is a Banach algebra with involution (complex conjugation, $f \mapsto \bar{f}$), the content is that the map $f \mapsto f(T)$ is an isometric $*$-isomorphism of $B(\sigma(T))$ on a subalgebra of $\mathbb{B}(H)$.

PROOF. That $f(T)$ is well defined by (7.3.1) was seen earlier in Theorem 6.2.4. When $\dim H = \infty$ the Spectral Theorem gives $\lambda_n \to 0$ for $n \to \infty$, and the criterion for compactness is that $f(\lambda_n) \to 0$. So if $f(T)$ is compact $\lim_{t\to 0} f(t) = 0$ by the finite multiplicity of eigenvalues $\lambda_n \neq 0$; and $f(0) = 0$ if $\dim Z(T) = \infty$, for $(f(\lambda_n))$ accumulates at $f(0)$ then. Conversely any ball centred at $0 \in \mathbb{C}$ contains $f(\lambda_n)$ eventually, under the stated conditions.

The relation (7.3.2) follows from Theorem 6.2.4, and (7.3.4) is derived from (7.3.1) by the calculus of limits. Concerning (7.3.3), note that Parseval's identity and continuity of the inner product entails

$$(f(T)x \,|\, y) = \sum f(\lambda_n)(x \,|\, e_n)\overline{(y \,|\, e_n)} = (x \,|\, \bar{f}(T)y). \tag{7.3.7}$$

Moreover, since $f(\lambda_n)g(\lambda_n) = f \cdot g(\lambda_n)$,

$$\begin{aligned} f(T)g(T)x &= \sum f(\lambda_n)(g(T)x \,|\, e_n)e_n \\ &= \sum f \cdot g(\lambda_n)(x \,|\, e_n)e_n = f \cdot g(T)x, \end{aligned} \tag{7.3.8}$$

so the multiplicativity follows. □

To elucidate the efficacy of the functional calculus, it should suffice to note that it immediately gives the solution formula (7.2.14). Indeed, for $\lambda \neq 0$ the function

$$f(t) = \begin{cases} \frac{1}{t-\lambda} & \text{for } t \in \sigma(T), t \neq \lambda, \\ 0 & \text{for } t = \lambda \text{ (void for } \lambda \in \rho(T)), \end{cases} \tag{7.3.9}$$

belongs to $f \in B(\sigma(T))$, and if $x = f(T)y$ for some $y \perp Z(T - \lambda I)$, then (7.3.1) amounts to (7.2.14) and in addition

$$(T - \lambda I)x = \sum_{\lambda_n \neq \lambda} f(\lambda_n)(y \,|\, e_n)(\lambda_n - \lambda)e_n = y \tag{7.3.10}$$

so that $x = f(T)y$ solves (7.2.11) (obviously uniquely for $\lambda \in \rho(T)$).

Since it is clear from (7.3.1) that each $f(\lambda_n)$ is an eigenvalue of $f(T)$, it is not surprising that the image of f, that is $f(\sigma(T))$, is closely related to the spectrum of $f(T)$:

COROLLARY 7.3.2 (The Spectral Mapping Theorem). *Under hypotheses as in Theorem 7.3.1,*

$$\sigma(f(T)) = f(\sigma(T)) \tag{7.3.11}$$

for all continuous f, that is for $f \in C(\sigma(T))$.

It should be mentioned that if $\sigma(T)$ is a finite set, any function $\sigma(T) \to \mathbb{C}$ is automatically both bounded and continuous, so that the continuity assumption on f would be void.

PROOF. For $\lambda \notin f(\sigma(T))$ the function $g(t) = (f(t) - \lambda)^{-1}$ belongs to $C(\sigma(T))$, so $\lambda \in \rho(f(T))$ since (7.3.5) gives eg

$$g(T)(f(T) - \lambda I) = g \cdot (f - \lambda)(T) = I. \tag{7.3.12}$$

Together with the observation before the corollary this shows that

$$f(\sigma_{\text{p}}(T)) \subset \sigma(f(T)) \subset f(\sigma(T)). \tag{7.3.13}$$

The case $\sigma(T) = \sigma_{\text{p}}(T)$ is now obvious. Otherwise $\sigma(T) = \{0\} \cup \sigma_{\text{p}}(T)$, in which case $\lambda_n \to 0$. Then $f(\lambda_n) \to f(0)$ by the continuity, whence

$$f(\sigma(T)) = f(\sigma_{\text{p}}(T)) \cup \{f(0)\} = \overline{f(\sigma_{\text{p}}(T))}. \tag{7.3.14}$$

Since $\sigma(f(T))$ is closed, these two formulae imply that $\sigma(f(T)) = f(\sigma(T))$.■ □

It is clear that the assumption that f should be continuous is essential for the Spectral Mapping Theorem, for if

$$T(x_1, x_2, \dots) = (x_1, \dots, \tfrac{x_n}{n}, \dots) \quad \text{on} \quad \ell^2, \tag{7.3.15}$$

then $\sigma(T) = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ so that $f = 1_{]0,\infty[}$ gives

$$f(\sigma(T)) = \{0, 1\} \neq \{1\} = \sigma(I) = \sigma(f(T)). \tag{7.3.16}$$

The theory extends in a natural way to so-called normal compact operators, but it requires more techniques. The interested reader is referred to the literature, eg [**Ped89**].

7.4. The Functional Calculus for Bounded Operators

For a bounded, self-adjoint operator $T \in \mathbb{B}(H)$ there is also a functional calculus as exposed in eg. [**RS80**, Thm. VII.1].

For this one should note that inside $C(\sigma(T))$ the set $\mathscr{P}$, consisting of all restrictions of polynomials to $\sigma(T)$, is a dense set. This was seen in [**Ped00**] in case $\sigma(T)$ is an interval of $\mathbb{R}$; more generally, $\sigma(T) \subset \mathbb{R}$ since T is self-adjoint, and any continuous function f on $\sigma(T)$ can then be extended to an interval (by Tietze's theorem [**Ped89**, 1.5.8]) and thereafter approximated. (One can also apply the general Stone–Weierstrass theorem, although this requires more efforts to establish first.)

In this set-up, the Spectral Mapping Theorem (7.3.2) is still valid, however the proof is omitted in [**RS80**] so one is given here:

PROPOSITION 7.4.1. *For a self-adjoint operator $T \in \mathbb{B}(H)$,*

$$\sigma(f(T)) = f(\sigma(T)) \tag{7.4.1}$$

for all $f \in C(\sigma(T))$.

PROOF. $\sigma(f(T)) \subset f(\sigma(T))$ follows since (7.3.12) also holds here. Given that $\lambda = f(\mu)$ for some $\mu \in \sigma(T)$, approximative eigenvectors are contructed as follows. To each $\varepsilon > 0$ there is a polynomial P such that $|f(x) - P(x)| \le \varepsilon/3$ for all $x \in \sigma(T)$. One can assume that f and P are real-valued for otherwise the following argument applies to the real and imaginary parts. Because (7.4.1) is known to hold for $f = P$, the number $P(\mu)$ is in $\sigma(P(T))$. Since P is real, $P(T)^* = P(T)$ and there is then (cf. the lectures) a unit vector x so that

$$\|(P(T) - P(\mu)I)x\| \le \varepsilon/3. \tag{7.4.2}$$

Since $f \mapsto f(T)$ is isometric, this leads to the conclusion that $\|(f(T) - \lambda I)x\| \le 2\varepsilon/3 + \|(P(T) - P(\mu)I)x\| \le \varepsilon$. Hence $\lambda \in \sigma(f(T))$. □

CHAPTER 8

Unbounded operators

The purpose of this chapter is to take a closer look at the unbounded operators on Hilbert spaces and to point out some features that are useful for the applications to classical problems in Mathematical Analysis.

8.1. Anti-duals

For a topological vector space V, a functional $\varphi\colon V \to \mathbb{F}$ is called conjugate (or anti-) linear if φ is additive and for all $\alpha \in \mathbb{F}$ and $x \in V$,

$$\varphi(\alpha x) = \bar{\alpha}\varphi(x). \tag{8.1.1}$$

The anti-dual space V' consists of all anti-linear functionals on V; it is occasionally handy. Clearly V' is a subspace of the vector space $\mathscr{F}(V,\mathbb{F})$ of all maps $V \to \mathbb{F}$. Instead of redoing functional analysis for the anti-linear case, it is usually simpler to exploit that the involution on $\mathscr{F}(V,\mathbb{F})$ given by $f \mapsto \bar{f}$ (complex conjugation) maps the dual space V^* bijectively onto V'.

Using $\langle \cdot, \cdot \rangle$ to denote also the action of anti-linear functionals, by definition of $\bar{\varphi}$,

$$\overline{\langle v, \varphi \rangle} = \langle v, \bar{\varphi} \rangle \quad \text{for all} \quad v \in V,\ \varphi \in V^*. \tag{8.1.2}$$

However, on the space V' each vector $v \in V$ defines the functional $\varphi \mapsto \varphi(v)$ (so that $V \subset (V')^*$). Hence it is natural to write (with interchanged roles)

$$\varphi(v) = \langle \varphi, v \rangle \quad \text{for} \quad \varphi \in V',\ v \in V. \tag{8.1.3}$$

Using this for a Hilbert space H, it is easily seen that

- H' endowed with $\|\varphi\| = \sup\{\, |\langle x, \varphi \rangle| \mid x \in H,\ \|x\| \le 1 \,\}$ is a Banach space isometrically, but anti-linearly isomorphic to H^*;
- there is a *linear*, surjective isometry $\Phi\colon H \to H'$ fulfilling

$$\langle \Phi(x), y \rangle = (\, x \,|\, y \,) \quad \text{for all} \quad x, y \in H. \tag{8.1.4}$$

- H' is a Hilbert space since $(\, \xi \,|\, \eta \,)_{H'} := (\, \Phi^{-1}(\xi) \,|\, \Phi^{-1}(\eta) \,)_H$ is an inner product inducing the norm. H and H' are unitarily equivalent hereby.

For $T \in \mathbb{B}(H_1, H)$, where H_1 and H are two Hilbert spaces, there is a unique $T' \in \mathbb{B}(H', H_1')$ such that

$$\langle T'x, y \rangle = (\, x \,|\, Ty \,) \quad \text{for} \quad x \in H,\ y \in H_1. \tag{8.1.5}$$

Indeed, $T' = \Phi_1 T^*$ when $T^* \in \mathbb{B}(H, H_1)$ is the usual Hilbert space adjoint of T and Φ_1 is the isomorphism $H_1 \to H_1'$, for $(\, x \,|\, Ty \,) = (\, T^*x \,|\, y \,) = \langle \Phi_1(T^*x), y \rangle$.

8.2. Lax–Milgram's lemma

Although unbounded operators on a Hilbert space in general are difficult to handle, they are manageable when defined by sesqui-linear forms, for there is a bijective correspondence (explained below) between the bounded sesqui-linear forms on H and $\mathbb{B}(H)$; this allows one to exploit the bounded case at the expense of introducing auxiliary Hilbert spaces.

In this direction Lax–Milgram's lemma is the key result. There are, however, a handful of conclusions to be obtained under this name. But it all follows fairly easily with just a little prudent preparation.

Let H be a fixed Hilbert space in the sequel. It is fruitful to commence with the following three observations:

(I) It is necessary to consider Hilbert spaces V *densely injected* into H,

$$V \hookrightarrow H \quad \text{densely,} \tag{8.2.1}$$

meaning that V is a dense subspace of H, and that V is endowed with an inner product $(\cdot\,|\,\cdot)_V$ such that V is complete and that there exists a constant C fulfilling

$$\|v\|_V \geq C\|v\|_H \quad \text{for all} \quad v \in V. \tag{8.2.2}$$

To elucidate the usefulness of this, note that if T is a densely defined, closed operator in H, then $D(T)$ is a Hilbert space densely injected into H.

(II) It is convenient to consider the anti-duals H' and V', for this gives a *linear* isometry $A\colon V \to V'$ such that

$$\langle Av, w\rangle = (v\,|\,w)_V \quad \text{for all} \quad v, w \in V, \tag{8.2.3}$$

identifying any $v \in V$ with a functional in V'. (The anti-linear isometry $V \to V^*$ would be less useful for, say linear differential operators.)

(III) To every sesqui-linear form $s\colon V \times V \to \mathbb{F}$ which is bounded, ie for some constant c

$$|s(v,w)| \leq c\|v\|_V\|w\|_V \quad \text{for all} \quad v, w \in V, \tag{8.2.4}$$

there corresponds a uniquely determined $S \in \mathbb{B}(V, V')$ such that for all v, w in V

$$s(v,w) = \langle Sv, w\rangle. \tag{8.2.5}$$

In addition to (I), note that when $I\colon V \hookrightarrow H$ densely, then

$$H' \hookrightarrow V' \quad \text{densely.} \tag{8.2.6}$$

Indeed, the adjoint I' of the map I in (8.2.1) is injective and has dense range (as the reader should verify) in view of the formula

$$\langle I'x, v\rangle_{V'\times V} = (x\,|\,Iv)_H = (x\,|\,v) \quad \text{for} \quad x \in H,\ v \in V. \tag{8.2.7}$$

Here H' is identified with H for simplicity's sake; this gives also the very important structure

$$V \subset H \subset V'. \tag{8.2.8}$$

One can therefore, to any s as in (III) above and its associated operator $S \in \mathbb{B}(V,V')$ define an operator T in H simply by restriction:

$$\left.\begin{aligned} D(T) &= S^{-1}(H) \\ T &= S|_{D(T)} \end{aligned}\right\} \tag{8.2.9}$$

It is easy to see that (8.2.9) coincides with a definition of T as the operator given by

$$\left.\begin{aligned} &D(T) = \{\, u \in V \mid \exists x \in H \forall v \in V \colon s(u,v) = (x \mid v)_H \,\} \\ &Tu = x. \end{aligned}\right\} \tag{8.2.10}$$

In these lines, the notation in the latter is explained by the former. To check (8.2.10), note that for $u \in D(T)$ there is some $x \in H$ so that $Su = I'x$, whence for $v \in V$, by (8.2.7),

$$s(u,v) = \langle Su, v \rangle = \langle I'x, v \rangle = (x \mid v)_H. \tag{8.2.11}$$

The other inclusion is shown similarly.

In general T above is an unbounded operator in H. It is called the operator associated with the triple (H,V,s), or the *Lax–Milgram-operator* adjoined to (H,V,s). Moreover, T is also said to be variationally defined, because the definition in (8.2.10) occurs naturally in the calculus of variations (where the goal is to find extrema of specific examples of s).

It is customary, when referring to triples (H,V,s), to let it be tacitly assumed that V is densely injected into H and that s is bounded on V.

Thus motivated, a few properties of sesqui-linear forms are recalled. First of all there is to any form s on V an *adjoint* sesqui-linear form s^* defined by

$$s^*(v,w) = \overline{s(w,v)} \quad \text{for} \quad v,w \in V. \tag{8.2.12}$$

s itself is called symmetric if $s \equiv s^*$; for $\mathbb{F} = \mathbb{C}$ this takes place if and only if $s(v,v)$ is real for all $v \in V$ (by polarisation). Moreover, s gives rise to the forms

$$s_{\mathrm{Re}}(v,w) = \tfrac{1}{2}(s(v,w) + s^*(v,w)) \tag{8.2.13}$$

$$s_{\mathrm{Im}}(v,w) = \tfrac{1}{2\mathrm{i}}(s(v,w) - s^*(v,w)) \tag{8.2.14}$$

that are both *symmetric* (but may take complex values outside the diagonal, whence the notation is a little misleading).

Using (III) on s^*, there is a unique $\tilde{S} \in \mathbb{B}(V,V')$ such that for v, $w \in V$,

$$s^*(v,w) = \langle \tilde{S}v, w \rangle. \tag{8.2.15}$$

Applying (8.2.10) to the operator $\tilde{T}$ defined from (H,V,s^*), it follows when T is densely defined that $\tilde{T} \subset T^*$.

The form s is said to be V-*elliptic* if there exists a constant $c_0 > 0$ such that

$$\operatorname{Re} s(v,v) \geq c_0 \|v\|_V^2 \qquad \text{for all } v \in V; \tag{8.2.16}$$

s is V-*coercive* if there exist $c_0 > 0$ and $k \in \mathbb{R}$ such that

$$\operatorname{Re} s(v,v) \geq c_0 \|v\|_V^2 - k\|v\|_H^2 \qquad \text{for all } v \in V \tag{8.2.17}$$

Notice that these properties carry over to the adjoint form s^* and to s_{Re}, with the same constants.

To elucidate the strength of these concepts, note that (8.2.16) implies that

$$\|Tu\|_H \geq c_0 \|u\|_V \quad \text{for } u \in D(T). \tag{8.2.18}$$

So T is necessarily *injective* and the range $R(T)$ is closed in H (as seen from (8.2.10)). Coerciveness gives operators that are only slightly less well behaved, and this class furthermore absorbs most of the perturbations of elliptic forms one naturally meets in the study of partial differential equations.

V-elliptic forms give rise to particularly nice operators:

PROPOSITION 8.2.1. *Let (H,V,s) fulfil that s is V-elliptic. Then the associated operator is a linear homeomorphism $S\colon V \to V'$.*

PROOF. For s elliptic and symmetric, $s(\cdot,\cdot)$ is an inner product on V. This gives a new Hilbert space structure on V, for the norm $\sqrt{s(v,v)}$ is equivalent to $\|\cdot\|_V$ (by the boundedness of s and (8.2.16)) so that V is complete. By construction S is the linear isometry that identifies V and V'.

In the non-symmetric, elliptic case one has

$$\|Sv\|_{V'} \geq c_0 \|v\|_V \quad \text{for all} \quad v \in V, \tag{8.2.19}$$

so that S is injective and has closed range. But since $\tilde{S}$ is injective by a similar argument, the formula

$$\langle Su, v \rangle = \overline{s^*(v,u)} = \overline{\langle \tilde{S}v, u \rangle}, \quad \text{for} \quad u,v \in V, \tag{8.2.20}$$

implies that $R(S)^\perp = \{0\}$. Therefore $R(S) = V$, and by the open mapping theorem S^{-1} is continuous. □

One should observe from the proof, that for a symmetric, elliptic form s, the operator S may be taken as the well-known isometric isomorphism between V and V'; this only requires a change of inner product on V, which leaves the Banach space structure invariant, however.

When discussing the induced unbounded operators on H, the coercive case gives operators with properties similar to those in the elliptic case; cf the below result. Note however, the difference that the latter case yields operators that *extend* to homeomorphisms from V to V' by the above proposition.

The next result is stated as a theorem because of its fundamental importance for the applications of Hilbert space theory to say, partial differential operators. For the same reasons all assumptions are repeated.

THEOREM 8.2.2 (Lax–Milgram's lemma). *Let the triple* (H,V,s) *be given with complex Hilbert spaces* V *and* H*, with* $V \hookrightarrow H$ *densely, and with* s *a bounded sesqui-linear form on* V*. Denote by* T *the associated operator in* H*. When* s *is* V*-coercive, ie fulfils* (8.2.17)*, then* T *is a closed operator in* H *with* $D(T)$ *dense in* V *(hence dense in* H *too) and with lower bound* $m(T) > -k$*; in fact*

$$\{\lambda \mid \operatorname{Re}\lambda \le -k\} \subset \rho(T) \tag{8.2.21}$$

so that $T - \lambda I$ *is a bijection from* $D(T)$ onto H *whenever* $\operatorname{Re}\lambda \le -k$.

Furthermore the operator associated with s^* *equals* T^**, the adjoint on* H*. And if* s *is symmetric, then* T *is self-adjoint and* $\ge -k$.

PROOF. Consider first $k = 0$, the elliptic case, and let $S\colon V \to V'$ be the homeomorphism determined by s. Because H is dense in V', it is carried over to a dense set (=$D(T)$) in V by S^{-1}. Since S extends T, it is straightforward to check that T is closed (using (8.2.6)). Now T^* is well defined and $T^* \supset \tilde{T}$ as seen above. But T^* is injective, since the surjectivity of S entails $R(T) = H$, and $\tilde{T}$ is surjective by the same argument applied to s^*. Therefore $T^* = \tilde{T}$, showing the claim on T^*.

Because $(Tu\,|\,u)_H = s(u,u)$ for $u \in D(T)$, it is clear from (8.2.16) that $m(T)$ and $m(T^*)$ both are numbers in $[c_0C^2, \infty[$ when C is the constant in (8.2.2). Since $c_0C^2 > 0$ this yields the inclusion for the resolvent set in (8.2.21) for the case $k = 0$ (and hence the statement after (8.2.21)).

For $k \neq 0$ the form $s(\cdot,\cdot) + k(\cdot\,|\,\cdot)_H$ is elliptic, so the above applies to the first term in the splitting $T = (T + kI) - kI$. The conclusions on the domain, the closedness, the adjoint and the resolvent set of T are now elementary to obtain. □

EXAMPLE 8.2.3. For $H = L_2(\Omega)$ and $V = H_0^1(\Omega)$ it is straightforward to see that $s(u,v) = \sum_{j=1}^n (D_ju\,|\,D_jv)_{L_2(\Omega)}$ is elliptic on V. The associated operator is the so-called Dirichét realisation $-\Delta_D$ of the Laplace operator; this means that as an unbounded operator in $L_2(\Omega)$ a function u is in $D(-\Delta_D)$ if and only if it belongs to $H_0^1(\Omega)$ and for some f $(= -\Delta_D u)$ in $L_2(\Omega)$ fulfils

$$-\Delta u = f \quad \text{in } \Omega \tag{8.2.22}$$

$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \tag{8.2.23}$$

The solution operator for this boundary problem is $-\Delta_D^{-1}$. By the theory this extends to $H^{-1}(\Omega)$, and in fact $-\Delta_D$ equals the abstract isomorphism between $H_0^1(\Omega)$ and its anti-dual $H^{-1}(\Omega)$, when the Hilbert space structure is suitably chosen.

CHAPTER 9

Further remarks

9.1. On compact embedding of Sobolev spaces

Below follows a proof of the fact that the first-order Sobolev space $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, provided $\Omega \subset \mathbb{R}^n$ is a *bounded* open set — a cornerstone result in the analysis of boundary problems of differential equations. Although one can go much further with results of this type (with the necessary technical preparations), we stick with this single result here, partly because it often suffices, partly because the reader should be well motivated to see a short proof of such an important, non-trivial result.

As a useful preparation, let us show the claim in Example 5.2.3, that functions u in $H^1(\mathbb{T})$, interpreted as the periodic subspace of $H^1(Q)$ for $Q =]-\pi,\pi[^n$, are characterised by their Fourier coefficients.

For a more precise statement, recall first that the Fourier transformation

$$\mathscr{F}u = (c_k)_{k\in\mathbb{Z}^n}, \quad \text{with} \quad c_k = (\,u\,|\,e_k\,), \tag{9.1.1}$$

is an isometry $L^2(Q) \to \ell^2(\mathbb{Z}^n)$. Secondly there is the Hilbert space $h^1(\mathbb{Z}^n)$ of those sequences (x_k) in $\ell^2(\mathbb{Z}^n)$ for which

$$\|(x_k)\|_{h^1} := \Big(\sum_{k\in\mathbb{Z}^n}(1+k_1^2+\dots+k_n^2)|x_k|^2\Big)^{1/2} < \infty; \tag{9.1.2}$$

cf Example 6.2.5. Now the claim is that any (c_k) in ℓ^2 is in $\mathscr{F}(H^1(\mathbb{T}))$ if and only if the sum in (9.1.2) is finite; and this is a consequence of

LEMMA 9.1.1. *There is a commutative diagram*

$$\begin{array}{ccc} H^1(\mathbb{T}) & \xrightarrow{\;I\;} & L^2(Q) \\ {\scriptstyle\mathscr{F}}\big\downarrow & & \big\downarrow{\scriptstyle\mathscr{F}} \\ h^1(\mathbb{Z}^n) & \xrightarrow[\;I\;]{} & \ell^2(\mathbb{Z}^n), \end{array} \tag{9.1.3}$$

where $\mathscr{F}$ is an isometry in both *columns*

PROOF. By repeated use of Parseval's identity,

$$\|u\|_{H^1}^2 = \sum(|c_k|^2 + |(\,D_1u\,|\,e_k\,)|^2 + \dots + |(\,D_nu\,|\,e_k\,)|^2), \tag{9.1.4}$$

so it follows that $\|\mathscr{F}u\|_{h^1} = \|u\|_{H^1}$ if and only if

$$(\,D_ju\,|\,e_k\,) = k_jc_k \quad \text{for all} \quad j=1,\dots,n;\; k\in\mathbb{Z}^n. \tag{9.1.5}$$

To show this, it is clear for $u \in C^\infty(\overline{Q})$ that, with the splitting $x = (x', x_n)$ and $Q' =]-\pi, \pi[^{n-1}$,

$$\begin{aligned} \int_{Q'} \tfrac{-\mathrm{i}(-1)^{k_n}}{e^{\mathrm{i}k'\cdot x'}} (u(x',\pi) - u(x',-\pi))\,dx' &= \int_Q D_n(u\overline{e_k})\,dx \\ &= (\,D_n u\,|\,e_k\,) - (\,u\,|\,k e_k\,) \end{aligned} \tag{9.1.6}$$

If an arbitrary $u \in H^1(\mathbb{T})$ is approximated in $H^1(Q)$ by a sequence u_m in $C^\infty(\overline{Q})$, this identity applies to each u_m; since $u_m \to u$ and $D_n u_m \to D_n u$ in the topology of L^2 one may pass to the limit on the right hand side, and by continuity of the trace operators also the left hand side converges for $m \to \infty$; there the limit is zero. This shows (9.1.5) for $j = n$; the other values of j are analogous.

By the above, $\mathscr{F}$ is isometric and hence injective on $H^1(\mathbb{T})$; but any (c_k) in h^1 defines a function $u \in L^2(Q)$ with $D_j u = \sum k_j c_k e_k$ (by continuity of D_j in $\mathscr{D}'$), and here the right hand side is in L^2. □

The reader should observe that the embedding of $H^1(\mathbb{T})$ into $L^2(Q)$ in the first row of (9.1.3) is *compact*; this follows from the diagram and the earlier result that $h^1 \hookrightarrow \ell^2$ is compact; cf Example 6.2.5.

That also the larger space $H^1(Q)$ is compactly embedded into $L^2(Q)$ is now a consequence of

THEOREM 9.1.2. *For every bounded open set $\Omega \subset \mathbb{R}^n$ the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is a* compact *operator.*

PROOF. Clearly $\Omega \subset]-R, R[^n =: Q_R$ for all sufficiently large $R > 0$. The above carries over to this cube if on the torus $\mathbb{T}_R = \mathbb{R}^n/Q_R$ one considers $e_k(x) = c_R \exp(2\pi\mathrm{i}\,k \cdot x/R)$ for some suitable c_R; so $H^1(\mathbb{T}_R) \hookrightarrow L^2(Q_R)$ is compact also for such R.

Given any bounded sequence in $H^1(\Omega)$ with $\Omega \subset Q_R$, it may be taken as restriction of a sequence in $H^1(Q_{3R})$, for which the supports are contained in Q_{2R}, so that the sequence is in $H^1(\mathbb{T}_{3R})$. Therefore there exists a subsequence converging in $L^2(Q_{3R})$, and a fortiori the restricted subsequence converges in $L^2(\Omega)$.

□

Bibliography

[Ban32] S. Banach, *Théorie des opérations linéaires*, Monografje matematyczne, vol. 1, Warszawa, 1932.

[Hör89] L. Hörmander, *Linear functional analysis*, Technical University of Lund, Sweden, 1989, Lecture notes.

[Ped89] Gert K. Pedersen, *Analysis now*, Springer Verlag, 1989.

[Ped00] Michael Pedersen, *Functional analysis in applied mathematics and engineering*, Chapman & Hall/CRC, Boca Raton, FL, 2000. MR 2000k:46002

[RS80] M. Reed and B. Simon, *Methods of modern mathemtical physics. I: Functional analysis. Revised and enlarged edition*, Academic Press, 1980.

[You01] R. M. Young, *An introduction to nonharmonic Fourier series*, Academic Press, 2001.