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Journal of Functional Analysis 218 (2005) 318–346
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Cauchy–Schwarz norm inequalities for weak*-integrals of operator valued functions

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Received 14 January 2004; revised 10 June 2004; accepted 11 June 2004

Available online 18 August 2004

Communicated by G. Pisier

Abstract

For a σ-finite measures μ on Ω and μ-weakly*-measurable families $\{\mathcal{A}_t\}_{t\in\Omega}$ and $\{\mathcal{B}_t\}_{t\in\Omega}$ of Hilbert space operators we have the non-commutative Cauchy–Schwarz inequalities in Schatten p-ideals

$$\left\|\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu\right\|_p \leqslant \left\|\sqrt[2q]{\int_\Omega \mathcal{A}^*\left(\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu\right)^{q-1}\mathcal{A}\,d\mu}\;X\;\sqrt[2r]{\int_\Omega \mathcal{B}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\right)^{r-1}\mathcal{B}^*\,d\mu}\right\|_p$$

for all $X\in\mathfrak{C}_p(\mathcal{H})$ and for all $p,q,r\geqslant 1$ such that $\frac{1}{q}+\frac{1}{r}=\frac{2}{p}$. If both $\{\mathcal{A}_t\}_{t\in\Omega}$ and $\{\mathcal{B}_t\}_{t\in\Omega}$ consists of commuting normal operators, then

$$\left|\left|\left|\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu\right|\right|\right| \leqslant \left|\left|\left|\sqrt{\int_\Omega \mathcal{A}^*\mathcal{A}\,d\mu}\,X\sqrt{\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu}\right|\right|\right|$$

for all unitarily invariant norms $|||\cdot|||$ and all $X\in\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$. If additionally $\int_\Omega \mathcal{A}^*\mathcal{A}\,d\mu\leqslant I$ and $\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\leqslant I$, then $\sqrt{I-\int_\Omega \mathcal{A}^*\mathcal{A}\,d\mu}\,X\sqrt{I-\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu}\in\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ and

$$\left|\left|\left|\sqrt{I-\int_\Omega \mathcal{A}^*\mathcal{A}\,d\mu}X\sqrt{I-\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu}\right|\right|\right| \leqslant \left|\left|\left|X-\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu\right|\right|\right|.$$

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doi:10.1016/j.jfa.2004.06.003

Applications include Young's and arithmetic–geometric–logarithmic means inequalities for operators and the mean value theorem for operator monotone functions.
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Keywords: Schatten ideals; Operator mean inequalities; Unitarily invariant norms

MSC: 46B70; 46E40; 47A30; 47A55; 47A63; 47B10; 47B47; 47B49; 47D50

1. Introduction and preliminaries

In recent years different norm inequalities for operators and unitarily invariant norms have been obtained based on the majorization and some complementary techniques. It is the arithmetic–geometric–logarithmic (56),(74) and Young (66) inequalities that have been under especially intensive investigation and a very good account can be found in [2,6,16]. Power and other various natural mean operator inequalities are investigated in [18,20,21,16], perturbation and generalized derivations norm inequalities in [1,11,17], and different Cauchy–Schwarz inequalities in [9,18,19], with the numerous references therein.

Hadamard product and positive definite functions play also an important role in those inequalities for finite matrices, greatly contributing to their more systematic understanding (see [10]). For infinite-dimensional Hilbert space operators the adoption of those proofs usually requires tedious adaptation or other procedures. Some proofs (see [11,16]) are also carried via double operator integrals, which were developed earlier by Birman and Solomyak. As pointed occasionally, it is the appropriate integral representation of one elementary operator by another, that usually leads to the shorter and more insightful proof for their norm inequality. Accompanying practical difficulties are mainly related to the convergence properties of those integrals, so the appropriate concept of integration is also important.

The goal of this work is to fully develop discrete results of [19] to show the practical potential of the Cauchy–Schwarz inequalities based on the Gel'fand's integration for the more systematic and unifying approach to the above mentioned, as well as some new inequalities and related problems.

Let $\mathcal{B}(\mathcal{H})$ and $\mathfrak{C}_\infty(\mathcal{H})$ denote, respectively, spaces of all bounded and all compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space $\mathcal{H}$. Each "symmetric gauge function" Φ on sequences gives rise to a symmetric norm or a unitarily invariant norm on operators defined by $||X||_\Phi = \Phi(\{s_n(X)\}_{n=1}^\infty)$, with $s_1(X) \geqslant s_2(X) \geqslant \cdots$ being the singular values of X. We will denote by the symbol $|||\cdot|||$ any such norm, which is therefore defined on a naturally associated norm ideal $\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ of $\mathfrak{C}_\infty(\mathcal{H})$ and satisfies the invariance property $|||UXV||| = |||X|||$ for all $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ and for all unitary operators U, V. Another property of those norms says that $|||X||| = \sup \Phi(\{\langle Xe_n, f_n\rangle\}_{n=1}^\infty)$, with supremum ranging over orthonormal sets $\{e_n\}_{n=1}^\infty, \{f_n\}_{n=1}^\infty$ in $\mathcal{H}$, actually attaining its maximum at eigenvectors $\{e_n\}_{n=1}^\infty$ of $|X|$ and $f_n = Ue_n$, where U is partial isometry in the polar decomposition

$X = U|X|$. Each norm $|||\cdot|||$ is lower semi-continuous, i.e., $|||w - \lim_{n\to\infty} X_n||| \leqslant \liminf_{n\to\infty} |||X_n|||$. This follows from the well-known representation formula $|||X||| = \sup\{\frac{|\mathrm{tr}(XY)|}{|||Y|||_{\mathrm{d}}} : Y \text{ is finite dimensional}\}$, where $|||\cdot|||_{\mathrm{d}}$ stands for the dual norm of $|||\cdot|||$ (see (d) [25, Theorem 2.7]).

Specially well known among unitarily invariant norms are the Schatten p-norms defined as $||X||_p = \sqrt[p]{\sum_{i=1}^{\infty} s_i^p(X)}$ for $1 \leqslant p < \infty$, and $||X||_\infty = ||X|| = s_1(X)$ coincides with the $\mathcal{B}(\mathcal{H})$ norm $||X||$. Minimal and maximal unitarily invariant norm are among Schatten norms, i.e., $||X||_\infty \leqslant |||X||| \leqslant ||X||_1$ for all $X \in \mathfrak{C}_1(\mathcal{H})$ (see inequality in [6, IV.38]). For $f, g \in \mathcal{H}$, we will denote by $g^* \otimes f$ one-dimensional operators $g^* \otimes f(h) = \langle h, g \rangle f$ for all $h \in \mathcal{H}$, known to have their linear span dense in each of $\mathfrak{C}_p(\mathcal{H})$ for $1 \leqslant p \leqslant \infty$.

The Ky–Fan norms defined as $||A||_{(k)} = \sum_{i=1}^{k} s_i(A)$, $k = 1, 2, \ldots$, represent another interesting family of unitarily invariant norms. The property saying that for all $X \in \mathfrak{C}_\infty(\mathcal{H})$ and $Y \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ with $||X||_{(k)} \leqslant ||Y||_{(k)}$ for all $k \geqslant 1$, we have $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ with $|||X||| \leqslant |||Y|||$ is known as the Ky–Fan dominance property. For a complete account of the theory of norm ideals, the reader is referred to [14,25]

2. Integration of operator valued functions

Following [12, p. 41], if $(\Omega, \mathfrak{M}, \mu)$ is a measure space, a mapping $\mathcal{A} : \Omega \to \mathcal{B}(\mathcal{H})$ will be called weakly*-measurable if the scalar function $t \to \mathrm{tr}(\mathcal{A}_t Y)$ is measurable for any $Y \in \mathfrak{C}_1(\mathcal{H})$. In addition, if all those functions are in $L^1(\Omega, d\mu)$, then according to the fact that $\mathcal{B}(\mathcal{H})$ is the dual space of $\mathfrak{C}_1(\mathcal{H})$, for any $E \in \mathfrak{M}$ there will be a unique $\mathcal{I}_E \in \mathcal{B}(\mathcal{H})$, called the Gel'fand (Gel'fand; see [12, p. 53] for details) or weak*-integral of $\mathcal{A}$ over E, such that

$$\mathrm{tr}(\mathcal{I}_E Y) = \int_E \mathrm{tr}(\mathcal{A}_t Y)\, d\mu(t) \quad \text{for all } Y \in \mathfrak{C}_1(\mathcal{H}). \tag{1}$$

We will denote it by $\int_E \mathcal{A}_t\, d\mu(t)$, $\int_E \mathcal{A}\, d\mu$ or exceptionally by $\oint_E \mathcal{A}\, d\mu$, if the context requires to distinguish this one from other types of integration. A practical tool for this type of integrability to deal with is that weak*-measurability (resp. weak*-integrability) of a o.v. function $\mathcal{A}_t$ is measurability (resp. integrability) of all scalar functions $t \to \langle \mathcal{A}_t f, f \rangle$, with f ranging throughout $\mathcal{H}$. A substantial argument for the non-trivial part of this principle relies on the closed graph theorem, which assures the boundedness of the operator (valued integral) related sesquilinear form. So if $\langle \mathcal{A} f, f \rangle \in L^1(E, d\mu)$ for all $f \in \mathcal{H}$, for some $E \in \mathfrak{M}$ and a $\mathcal{B}(\mathcal{H})$ valued function $\mathcal{A}$ on E, then there is a (unique) bounded operator (denoted by) $\oint_E \mathcal{A}\, d\mu$ (rightfully called the Gel'fand integral of $\mathcal{A}$ over E) satisfying

$$\left\langle \left(\oint_E \mathcal{A}\, d\mu \right) f, f \right\rangle = \int_E \langle \mathcal{A}_t f, f \rangle\, d\mu(t) \quad \text{for all } f \in \mathcal{H}. \tag{2}$$

The monotonicity and other standard o.v. integral properties for the Gel'fand integral are easily derivable from this formula.

The following example is a "weak" version of Pearson's theorem and shows the practical value of the above tool.

Example 1. Let $AJ - JB \in \mathfrak{C}_1(\mathcal{H})$ for some $J \in \mathcal{B}(\mathcal{H}), A = A^*$ and $B = B^*$, and let $P_{ac}(A)$ and $P_{ac}(B)$ stand for the orthogonal projections on the absolutely continuous subspaces of A and B, respectively. Then $\Omega^{\mp}_{A,B}(J) = w-\lim_{t \to \pm\infty} P_{ac}(A)e^{itA}Je^{-itB}P_{ac}(B)$ exist, and

$$\Omega^{\mp}_{A,B}(J) = P_{ac}(A)JP_{ac}(B) \pm i \int_{\mathbb{R}^{\pm}} P_{ac}(A)e^{itA}(AJ - JB)e^{-itB}P_{ac}(B)\, dt.$$

Proof. $P_{ac}(A)(e^{itA}Je^{-itB} - J)P_{ac}(B) = iP_{ac}(A)\int_0^t e^{isA}(AJ - JB)e^{-isB}\, ds\, P_{ac}(B)$ and therefore, according to the Banach–Steinhaus theorem, it will be enough to show that $\langle P_{ac}(A)e^{isA}(AJ - JB)e^{-isB}P_{ac}(B)g, h\rangle \in L^1(\mathbb{R}, dx)$ for all f and g belonging to some dense subsets of $\mathcal{H}$. This is because the left-hand side of the last identity guarantees that the integral of the above function will define a bounded sesquilinear form on $\mathcal{H}$. For a self-adjoint A and his spectral measure E_A let $\mathcal{M}(A)$ denote all $f \in \mathcal{H}$ such that $\frac{d\langle E_\lambda f, f\rangle}{d\lambda}$ exists a.e. and belongs to $L^\infty(\mathbb{R})$. The norm $|||\cdot|||$ on $\mathcal{M}(A)$ is given by $|||f||| = \left\|\frac{d\langle E_\lambda f, f\rangle}{d\lambda}\right\|_\infty$, and it is known [24, Lemma XI.3] that $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are $L^2(\mathbb{R})$ norm dense in $P_{ac}(A)\mathcal{H}$ and $P_{ac}(B)\mathcal{H}$, respectively; moreover

$$\sqrt{\int_{\mathbb{R}} |\langle e, e^{isA}g\rangle|^2\, ds} \leqslant \sqrt{2\pi}||e||\, |||g||| \quad \text{and} \quad \sqrt{\int_{\mathbb{R}} |\langle f, e^{isB}h\rangle|^2\, ds} \leqslant \sqrt{2\pi}||f||\, |||h|||$$

for all $e, f \in \mathcal{H}$, $g \in \mathcal{M}(A)$ and $h \in \mathcal{M}(B)$. If $AJ - JB = \sum_{n=1}^\infty s_n e_n^* \otimes f_n$ is the singular value expansion of $AJ - JB$, with all $||e_n|| = ||f_n|| = 1$, then

$$\begin{aligned}
&\int_{\mathbb{R}} |\langle e^{isA}(AJ - JB)e^{-isB}g, h\rangle|\, ds \\
&\quad \leqslant \sum_{n=1}^{\infty} s_n \int_{\mathbb{R}} |\langle e^{isA}f_n, h\rangle \langle g, e^{isB}e_n\rangle|\, ds \\
&\quad \leqslant \sum_{n=1}^{\infty} s_n \sqrt{\int_{\mathbb{R}} |\langle e^{isA}f_n, h\rangle|^2\, ds}\sqrt{\int_{\mathbb{R}} |\langle g, e^{isB}e_n\rangle|^2\, ds} \\
&\quad \leqslant 2\pi \sum_{n=1}^{\infty} s_n||e_n||\, ||f_n||\, |||g|||\, |||h||| = 2\pi||AJ - JB||_1|||g|||\, |||h|||. \qquad \square
\end{aligned}$$

Remark 1. With the same argument applied to another $\mathfrak{C}_1(\mathcal{H})$ operator $2i\Im(J^*P_{ac}(A)(AJ-JB)) = J^*P_{ac}(A)(AJ-JB) - (AJ-JB)^*P_{ac}(A)J$, we get

$$\begin{aligned}
&\lim_{t\to\pm\infty} \|P_{ac}(A)e^{itA}Je^{-itB}P_{ac}(B)f\|^2 \\
&\quad = \lim_{t\to\pm\infty} \langle P_{ac}(B)e^{itB}J^*P_{ac}(A)Je^{-itB}P_{ac}(B)f, f\rangle = \|P_{ac}(A)JP_{ac}(B)f\|^2 \\
&\qquad \pm i\int_{\mathbb{R}^\pm} \langle P_{ac}(B)e^{itB}(BJ^*P_{ac}(A)J - J^*P_{ac}(A)JB)e^{-itB}P_{ac}(B)f, f\rangle\, dt \\
&\quad = \langle P_{ac}(B)J^*P_{ac}(A)JP_{ac}(B)f, f\rangle \\
&\qquad \mp 2\int_{\mathbb{R}^\pm} \langle P_{ac}(B)e^{itB}\Im(J^*P_{ac}(A)(AJ-JB))e^{-itB}P_{ac}(B)f, f\rangle\, dt
\end{aligned}$$

exists, and once we show it coincides with $\|\Omega^\mp(J)f\|^2$, the strong convergence will follow from the weak one. Now, the full Pearson theorem relies on formula

$$\begin{aligned}
&\left|P_{ac}(A)JP_{ac}(B) \pm i\int_{\mathbb{R}^\pm} P_{ac}(A)e^{itA}(AJ-JB)e^{-itB}P_{ac}(B)\, dt\right|^2 \\
&\quad = |P_{ac}(A)JP_{ac}(B)|^2 \mp 2\int_{\mathbb{R}^\pm} P_{ac}(B)e^{itB}\Im(J^*P_{ac}(A)(AJ-JB))e^{-itB}P_{ac}(B)\, dt.
\end{aligned}$$

Now, we turn our attention to the spaces of integrable vector and operator valued functions. Let $L^2(\Omega, d\mu, \mathcal{H})$ denote the space of all (weakly) measurable functions $f : \Omega \to \mathcal{H}$ such that $\int_\Omega \|f(t)\|^2\, d\mu(t) < \infty$ (see [4] for an illustrative example), and similarly, let $L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ denote the space of all μ-weak*-measurable functions $\mathcal{F} : \Omega \to \mathcal{B}(\mathcal{H})$ such that $\int_\Omega \|\mathcal{F}_t f\|^2\, d\mu(t) < \infty$ for all $f \in \mathcal{H}$. Note that $t \to \|\mathcal{F}_t f\|$ and $t \to \|\mathcal{F}_t\|$ are measurable for all $f \in \mathcal{H}$ as $\|\mathcal{F}_t f\| = \sqrt{\sum_{n=1}^\infty |\langle \mathcal{F}_t f, e_n\rangle|^2}$ and $\|\mathcal{F}_t\| = \sup \frac{\|\mathcal{F}_t g_n\|}{\|g_n\|}$ for a dense set $\{g_n\}$ in $\mathcal{H}$. Clearly $\mathcal{F} \in L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ iff $\mathcal{F}f \in L^2(\Omega, d\mu, \mathcal{H})$ for all $f \in \mathcal{H}$, but there is another simple characterization.

Example 2. $\mathcal{F}^*\mathcal{F}$ is Gel'fand integrable iff $\mathcal{F} \in L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$; in that case

$$\left\langle \oint\!\!\!\!G_\Omega \mathcal{F}^*\mathcal{F}\, d\mu f, f \right\rangle = \int_\Omega \|\mathcal{F}f\|^2\, d\mu (= \|\mathcal{F}f\|^2_{L^2(\Omega, d\mu, \mathcal{H})}) \quad \text{for all } f \in \mathcal{H}. \tag{3}$$

Before we proceed with a natural norming of $L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$, we need some refined approximation in this space, as well as in $L^2(\Omega, d\mu, \mathcal{H})$. First, let us note that the proof of Proposition 4.1 in [4] actually shows that simple vector valued functions are dense in $L^2(\Omega, d\mu, \mathcal{H})$ for all σ-finite measures μ. So for all $f \in \mathcal{H}$ and a given $\varepsilon > 0$ there are some disjoint sets $\{\delta_k\}_{k=1}^K$ of finite measure and some $f_1, \dots, f_K$ belonging to (the even prescribed) dense subset of $\mathcal{H}$ such that $\int_\Omega | f(t)$

$-\sum_{k=1}^{K} f_k \chi_{\delta_k}(t)|^2\, d\mu(t) < \varepsilon^2$. When the partition (division) $\{\delta_k\}_{k=1}^{K}$ is fixed, then among such simple functions the best approximation of f will be obtained by $\sum_{k=1}^{K} \frac{\chi_{\delta_k}}{\mu(\delta_k)} \int_{\delta_K} f\, d\mu$. This follows immediately from the identity

$$\int_\Omega \left\| f - \sum_{k=1}^{K} f_k \chi_{\delta_k} \right\|^2 d\mu$$
$$= \int_\Omega \left\| f - \sum_{k=1}^{K} \frac{\chi_{\delta_k}}{\mu(\delta_k)} \int_{\delta_k} f\, d\mu \right\|^2 d\mu + \sum_{k=1}^{K} \mu(\delta_k) \left\| f_k - \frac{1}{\mu(\delta_k)} \int_{\delta_k} f\, d\mu \right\|^2, \quad (4)$$

where $\int_{\delta_k} f\, d\mu$ stands for a suitable Pettis integral, or equivalently to the unique element of $\mathcal{H}$ representing a bounded linear functional $g \to \int_{\delta_k} \langle g, f(t) \rangle\, d\mu(t)$ on $\mathcal{H}$. In the space $L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ the best approximant is described by

Lemma 2.1. (a) *For a given $\mathcal{F} \in L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ its best approximation among simple functions (related to the prescribed partition $\mathcal{P} = \{\delta_k\}_{k=1}^{K}$) is achieved by*

$$\mathcal{F}_{\mathcal{P}} = \sum_{k=1}^{K} \frac{\chi_{\delta_k}}{\mu(\delta_k)} \oint\!\!\!\!\!\!G_{\delta_k} \mathcal{F}\, d\mu, \quad (5)$$

which also satisfies

$$\oint\!\!\!\!\!\!G_{\Omega} |\mathcal{F}|^2\, d\mu - \oint\!\!\!\!\!\!G_{\Omega} |\mathcal{F}_{\mathcal{P}}|^2\, d\mu = \oint\!\!\!\!\!\!G_{\Omega} |\mathcal{F} - \mathcal{F}_{\mathcal{P}}|^2\, d\mu \geqslant 0. \quad (6)$$

(b) *If μ is σ-finite, then for every function $\mathcal{F} \in L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ there is a simple function sequence $\{\mathcal{F}_n\}$ in $L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ such that*

$$\oint\!\!\!\!\!\!G_{\Omega} |\mathcal{F}|^2\, d\mu - \oint\!\!\!\!\!\!G_{\Omega} |\mathcal{F}_n|^2\, d\mu = \oint\!\!\!\!\!\!G_{\Omega} |\mathcal{F} - \mathcal{F}_n|^2\, d\mu \to 0 \quad (7)$$

strongly and monotonically decreasing.

Proof. (a) The proof is straightforward (like in (4)) and left to the reader.

(b) Let $\{f_i, i = 1, 2, \ldots\}$ be a dense subset of $\mathcal{H}$. As we know, every $\mathcal{F} f_i \in L^2(\Omega, d\mu, \mathcal{H})$ and so it is approximable by simple functions. The known form of the best approximant assures the existence of partitions $\{\delta_{k_{n,i}}^{(n,i)}\}_{k_{n,i}=1}^{K_{n,i}}$ such that

$$\int_\Omega \left\| \mathcal{F} f_i - \sum_{k_{n,i}=1}^{K_{n,i}} \frac{\chi_{\delta_{k_{n,i}}^{(n,i)}}}{\mu(\delta_{k_{n,i}}^{(n,i)})} \int_{\delta_{k_{n,i}}^{(n,i)}} \mathcal{F} f,\, d\mu \right\|^2 d\mu < \frac{1}{n^2}$$

for any $n = 1, 2, \ldots$ and all $i = 1, \ldots, n$. Superposing those partitions $\{\delta_{k_{n,i}}^{(n,i)}\}_{k_{n,i}=1}^{K_{n,i}}$ for $i = 1, \ldots, n$, and taking account that more "refined" partitions offer better approximation, we could find some non-negative integers K_n and a partitions $\{\delta^{(n)}_{k_n}\}^{K_n}_{k_n=1}$ such that

$$\left\langle \oint_\Omega \mathcal{F}^*\mathcal{F} - \mathcal{F}^*_n\mathcal{F}_n \, d\mu f_i, f_i \right\rangle = \int_\Omega ||\mathcal{F} f_i - \mathcal{F}_n f_i||^2 \, d\mu < \frac{1}{n^2} \tag{8}$$

for all $i = 1, \ldots, n$ and $n \in \mathbb{N}$. Here $\mathcal{F}_n$ stands for

$$\sum_{k_n=1}^{K_n} \frac{\chi_{\delta^{(n)}_{k_n}}}{\mu(\delta^{(n)}_{k_n})} \int_{\delta^{(n)}_{k_n}} \mathcal{F} \, d\mu.$$

More than that, the same superposition argument assures that for every $n \in \mathbb{N}$ the partition $\{\delta^{(n)}_{k_n}\}^{K_n}_{k_n=1}$ could be chosen as a subpartition of the proceeding one. In this way, we get a monotonically increasing and uniformly bounded from above by $\oint_\Omega \mathcal{F}^*\mathcal{F} \, d\mu$ sequence $\{\oint_\Omega \mathcal{F}^*_n\mathcal{F}_n \, d\mu\}^\infty_{n=1}$ in $\mathcal{B}(\mathcal{H})$, which therefore strongly converge to some $C \in \mathcal{B}(\mathcal{H})$ by Proposition 1.1 of [4]. By (8) $\langle Cf_i, f_i \rangle = \langle \oint_\Omega \mathcal{F}^*\mathcal{F} \, d\mu f_i, f_i \rangle$ for $i = 1, 2, \ldots$; therefore $C = \oint_\Omega \mathcal{F}^*\mathcal{F} \, d\mu$ as required. $\square$

Definition 1. For $\mathcal{F} \in L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ we define by $||\mathcal{F}||_2 = || \int_\Omega \mathcal{F}^*\mathcal{F} \, d\mu||^{\frac{1}{2}}$ to be its norm. If additionally $\int_\Omega \mathcal{F}^*\mathcal{F} \, d\mu \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$, we define

$$|||\mathcal{F}|||_2 = \left|\left|\left| \int_\Omega \mathcal{F}^*\mathcal{F} \, d\mu \right|\right|\right|^{\frac{1}{2}} \tag{9}$$

and denote by $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$ the space of all measurable $\mathcal{B}(\mathcal{H})$ valued functions with its finite (i.e. normed by) $||| \cdot |||_2$.

Theorem 2.1. *Every* $||| \cdot |||_2$ *is a norm on a Banach space* $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$.

Proof. For any $0 < \alpha < 1$ and measurable $\mathcal{A}$ and $\mathcal{B}$ there holds

$$|\mathcal{A} + \mathcal{B}|^2 + \alpha(1-\alpha)\left|\frac{\mathcal{A}}{1-\alpha} + \frac{\mathcal{B}}{\alpha}\right|^2 = \frac{|\mathcal{A}|^2}{1-\alpha} + \frac{|\mathcal{B}|^2}{\alpha}$$

from which we deduce

$$\begin{aligned} |||\mathcal{A} + \mathcal{B}|||_2^2 &= \left|\left|\left| \int_\Omega |\mathcal{A} + \mathcal{B}|^2 \, d\mu \right|\right|\right| \leqslant \left|\left|\left| \frac{\int_\Omega |\mathcal{A}|^2 \, d\mu}{1-\alpha} + \frac{\int_\Omega |\mathcal{B}|^2 \, d\mu}{\alpha} \right|\right|\right|^2 \\ &\leqslant \frac{||| \int_\Omega |\mathcal{A}|^2 \, d\mu |||}{1-\alpha} + \frac{||| \int_\Omega |\mathcal{B}|^2 \, d\mu |||}{\alpha} = \frac{|||\mathcal{A}|||_2^2}{1-\alpha} + \frac{|||\mathcal{B}|||_2^2}{\alpha} \\ &= (|||\mathcal{A}|||_2 + |||\mathcal{B}|||_2)^2 \end{aligned}$$

for optimally chosen $\alpha = \frac{|||\mathcal{B}|||_2}{|||\mathcal{A}|||_2 + |||\mathcal{B}|||_2}$. Thus $||| \cdot |||_2$ is norm on $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$.

Note here that a special case when μ is atomic (concentrated on a single point) shows that the function $\Phi_2(\{s_i\}_{i=1}^{\infty}) = \Phi^{\frac{1}{2}}(\{s_i^2\}_{i=1}^{\infty})$, known also as 2-convexization of Φ, is indeed a symmetric gauge function associated with the norm $|||\,|\cdot|^2|||^{\frac{1}{2}} = |||\cdot|||_{\Phi_2}$. This fact will be needed in the sequel.

In order to prove completeness of $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$ we can confine ourselves to deduce $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$ summability of $\sum_{n=1}^{\infty}\mathcal{A}_n$ from $\sum_{n=1}^{\infty}|||\mathcal{A}_n|||_2 < \infty$. For every $f \in \mathcal{H}$ a general Minkowski inequality in $L^2(\Omega, d\mu)$ implies

$$\begin{aligned}\sqrt{\int_{\Omega}\Big(\sum_{n=1}^{\infty}||\mathcal{A}_n f||\Big)^2 d\mu} &= \left|\left|\sum_{n=1}^{\infty}||\mathcal{A}_n f||\right|\right|_{L^2(\Omega,d\mu)} \\ &\leqslant \sum_{n=1}^{\infty}||\,||\mathcal{A}_n f||\,||_{L^2(\Omega,d\mu)} = \sum_{n=1}^{\infty}\sqrt{\int_{\Omega}||\mathcal{A}_n f||^2\, d\mu} \\ &= \sum_{n=1}^{\infty}\left|\left|\sqrt{\int_{\Omega}\mathcal{A}_n^*\mathcal{A}_n\, d\mu}\, f\right|\right| \\ &\leqslant \sum_{n=1}^{\infty}\sqrt{\left|\left|\int_{\Omega}\mathcal{A}_n^*\mathcal{A}_n\, d\mu\right|\right|}\,||f|| \end{aligned} \tag{10}$$

$$\leqslant \sum_{n=1}^{\infty}\sqrt{\left|\left|\left|\int_{\Omega}\mathcal{A}_n^*\mathcal{A}_n\, d\mu\right|\right|\right|}\,||f|| = \sum_{n=1}^{\infty}|||\mathcal{A}_n|||_2||f||. \tag{11}$$

To get (11) we used the minimality of $||\cdot||$ among u.i. norms. This assures the absolute summability of $\sum_{n=1}^{\infty}\mathcal{A}_n f[\mu]$ a.e on Ω and therein it defines strongly summable $\mathcal{A} = \sum_{n=1}^{\infty}\mathcal{A}_n$, an obviously a weak*-measurable function. From (10) one can easily derive that $\sum_{n=1}^{\infty}\mathcal{A}_n \in L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ and $||\sum_{n=1}^{\infty}\mathcal{A}_n||_2 \leqslant \sum_{n=1}^{\infty}||\mathcal{A}_n||_2$, which is fairly enough to deduce the completeness of $L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$. Similarly, a general case $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$ requires only the proof of $|||\sum_{n=1}^{\infty}\mathcal{A}_n|||_2 \leqslant \sum_{n=1}^{\infty}|||\mathcal{A}_n|||_2$. So let $\{s_i\}_{i=1}^{\infty}$ and $\{e_i\}_{i=1}^{\infty}$ be the singular values and eigenvectors of $\sqrt{\int_{\Omega}\mathcal{A}^*\mathcal{A}\, d\mu}$. By the monotonicity of Φ and (10) it follows

$$\begin{aligned}\left|\left|\left|\sum_{n=1}^{\infty}\mathcal{A}_n\right|\right|\right|_2 &= \left|\left|\left|\int_{\Omega}\left|\sum_{n=1}^{\infty}\mathcal{A}_n\right|^2 d\mu\right|\right|\right|^{\frac{1}{2}} = \Phi^{\frac{1}{2}}(\{s_i^2\}_{i=1}^{\infty}) \\ &= \Phi^{\frac{1}{2}}\left(\left\{\left\langle\int_{\Omega}\left|\sum_{n=1}^{\infty}\mathcal{A}_n\right|^2 d\mu\, e_i, e_i\right\rangle\right\}_{i=1}^{\infty}\right)\end{aligned}$$

$$= \Phi^{\frac{1}{2}}\left(\left\{\int_\Omega \left\|\sum_{n=1}^\infty \mathcal{A}_n e_i\right\|^2 d\mu\right\}_{i=1}^\infty\right)$$

$$\leqslant \Phi^{\frac{1}{2}}\left(\left\{\left(\sum_{n=1}^\infty \left\|\sqrt{\int_\Omega |\mathcal{A}_n|^2\, d\mu}\, e_i\right\|\right)^2\right\}_{i=1}^\infty\right)$$

$$= \Phi_2\left(\left\{\sum_{n=1}^\infty \left\|\sqrt{\int_\Omega |\mathcal{A}_n|^2\, d\mu}\, e_i\right\|\right\}_{i=1}^\infty\right).$$

As we noticed above, Φ_2 is another symmetric gauge function, and therefore

$$\left|\left|\left|\sum_{n=1}^\infty \mathcal{A}_n\right|\right|\right|_2 \leqslant \sum_{n=1}^\infty \Phi_2\left(\left\{\left\|\sqrt{\int_\Omega |\mathcal{A}_n|^2\, d\mu}\, e_i\right\|\right\}_{i=1}^\infty\right)$$

$$= \sum_{n=1}^\infty \Phi^{\frac{1}{2}}\left(\left\{\left\|\sqrt{\int_\Omega |\mathcal{A}_n|^2\, d\mu}\, e_i\right\|^2\right\}_{i=1}^\infty\right)$$

$$= \sum_{n=1}^\infty \Phi^{\frac{1}{2}}\left(\left\{\left\langle \int_\Omega |\mathcal{A}_n|^2\, d\mu\, e_i, e_i\right\rangle\right\}_{i=1}^\infty\right) \leqslant \sum_{n=1}^\infty \left|\left|\left|\int_\Omega |\mathcal{A}_n|^2\, d\mu\right|\right|\right|^{\frac{1}{2}}$$

$$= \sum_{n=1}^\infty |||\mathcal{A}_n|||_2$$

by Proposition 2.6 of [25]. This ends the proof. □

3. Inequalities for factorable operator valued functions

If $\mathcal{A}$ and $\mathcal{B}$ are measurable, the same is true for $t \to X\mathcal{A}_t Y\mathcal{B}_t Z$ if X, Y and Z are in $\mathcal{B}(\mathcal{H})$. Indeed, it follows by the Parseval identity that $\langle X\mathcal{A}_t Y\mathcal{B}_t Zf, g\rangle = \sum_{n=1}^\infty \langle \mathcal{B}_t Zf, Y^* e_n\rangle \langle \mathcal{A}_t e_n, X^* g\rangle$ for an orthonormal basis $\{e_n\}$ of $\mathcal{H}$, and thus the pointwise limit of measurable functions is also a measurable one. We will usually refer to this class of function as factorable operator valued (o.v.) functions and we investigate them in the sequel. Thus, we start with some integrability properties of such functions.

Lemma 3.1. *Let $\mathcal{A}, \mathcal{B}, : \Omega \to \mathcal{B}(H)$ be weakly* measurable and let $X \in \mathcal{B}(\mathcal{H})$.*

(a) *If $\int_\Omega ||\mathcal{A}_t^* f||^2 + ||\mathcal{B}_t f||^2\, d\mu(t) < \infty$ for all $f \in \mathcal{H}$, then*
  (a1) *$t \to \mathcal{A}_t X \mathcal{B}_t$ is weakly*-integrable and there holds*

$$\left\|\int_\Omega \mathcal{A}X\mathcal{B}\, d\mu\right\| \leqslant \sqrt{\left\|\int_\Omega \mathcal{A}\mathcal{A}^*\, d\mu\right\| \left\|\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu\right\|}\, ||X||. \tag{12}$$

(a2) *If* $\{\mathcal{A}^*\}_{n=1}^{\infty}$ *(resp.* $\{\mathcal{B}\}_{n=1}^{\infty}$*) is a simple o.v. sequence for* $\mathcal{A}^*$ *(resp.* $\mathcal{B}$*) guaranteed by Lemma* 2.1, *then* $\int_\Omega \mathcal{A}_n X \mathcal{B}_n \, d\mu \to \int_\Omega \mathcal{A} X \mathcal{B} \, d\mu$ *weakly as* $n \to \infty$.

(b) *If* $\int_\Omega ||\mathcal{A}_t f||^2 + ||\mathcal{B}_t^* f||^2 \, d\mu(t) < \infty$ *for all* $f \in \mathcal{H}$, *then*

$$\left|\left|\int_\Omega \mathcal{A} X \mathcal{B} \, d\mu\right|\right|_1 \leqslant \sqrt{\left|\left|\int_\Omega \mathcal{A}^* \mathcal{A} \, d\mu\right|\right| \left|\left|\int_\Omega \mathcal{B}\mathcal{B}^* \, d\mu\right|\right|} ||X||_1. \tag{13}$$

for all $X \in \mathfrak{C}_1(\mathcal{H})$.

(c) *If* $\int_\Omega ||\mathcal{A}_t f||^2 + ||\mathcal{A}_t^* f||^2 + ||\mathcal{B}_t f||^2 + ||\mathcal{B}_t^* f||^2 \, d\mu(t) < \infty$ *for all* $f \in \mathcal{H}$, *then*

$$\left|\left|\left|\int_\Omega \mathcal{A} X \mathcal{B} \, d\mu\right|\right|\right| \leqslant \max\left\{\sqrt{\left|\left|\int_\Omega \mathcal{A}^* \mathcal{A} \, d\mu\right|\right| \left|\left|\int_\Omega \mathcal{B}\mathcal{B}^* \, d\mu\right|\right|}, \sqrt{\left|\left|\int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu\right|\right| \left|\left|\int_\Omega \mathcal{B}^* \mathcal{B} \, d\mu\right|\right|}\right\} |||X||| \tag{14}$$

for all $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$. *Particularly, for all* $X \in \mathfrak{C}_p(\mathcal{H})$

$$\left|\left|\int_\Omega \mathcal{A} X \mathcal{B} \, d\mu\right|\right|_p \leqslant \left|\left|\int_\Omega \mathcal{A}^* \mathcal{A} \, d\mu\right|\right|^{\frac{1}{p}} \left|\left|\int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu\right|\right|^{\frac{p-1}{p}} \left|\left|\int_\Omega \mathcal{B}\mathcal{B}^* \, d\mu\right|\right|^{\frac{1}{p}} \left|\left|\int_\Omega \mathcal{B}^* \mathcal{B} \, d\mu\right|\right|^{\frac{p-1}{p}} ||X||_p. \tag{15}$$

Proof. (a) As $|\langle \mathcal{A}_t X \mathcal{B}_t f, f \rangle| \leqslant ||X|| \, ||\mathcal{A}_t^* f|| \, ||\mathcal{B}_t f||$ for all $t \in \Omega$ and $f \in \mathcal{H}$, then $t \to \langle \mathcal{A}_t X \mathcal{B}_t \rangle$ is integrable, securing the weak*-integrability of $\mathcal{A}_t X \mathcal{B}_t$ and

$$\begin{aligned}\left|\int_\Omega \langle \mathcal{A} X \mathcal{B} f, g \rangle \, d\mu\right| &\leqslant ||X|| \int_\Omega ||\mathcal{A}_t^* g|| \, ||\mathcal{B}_t f|| d\mu(t) \\ &\leqslant ||X|| \sqrt{\int_\Omega ||\mathcal{A}_t^* g||^2 \, d\mu} \sqrt{\int_\Omega ||\mathcal{B}_t g||^2 \, d\mu} \\ &= ||X|| \sqrt{\left\langle \int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu g, g \right\rangle} \sqrt{\left\langle \int_\Omega \mathcal{B}^* \mathcal{B} \, d\mu f, f \right\rangle}\end{aligned} \tag{16}$$

from which the conclusion (a1) follows. To prove (a2) we realize that for all $f, g \in \mathcal{H}$ there holds

$$\begin{aligned}&\left|\int_\Omega \langle \mathcal{A}_n X \mathcal{B}_n f - \mathcal{A} X \mathcal{B} f, g \rangle \, d\mu\right| \\ &\leqslant ||X|| \int_\Omega ||\mathcal{A}_n^* g - \mathcal{A}^* g|| ||\mathcal{B}_n f|| + ||\mathcal{A}^* g|| \, ||\mathcal{B}_n f - \mathcal{B} f|| \, d\mu\end{aligned}$$

$$\leqslant ||X||\sqrt{\int_\Omega ||\mathcal{A}^*g||^2+||\mathcal{B}_nf||^2\,d\mu}$$

$$\times\sqrt{\int_\Omega ||\mathcal{A}_n^*g-\mathcal{A}^*g||^2+||\mathcal{B}_nf-\mathcal{B}f||^2\,d\mu}$$

$$\leqslant ||X||\sqrt{\left\|\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu\right\|||g||^2+\left\|\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\right\|||f||^2}$$

$$\times\sqrt{\int_\Omega\langle\mathcal{A}\mathcal{A}^*-\mathcal{A}_n\mathcal{A}_n^*\,d\mu g,g\rangle+\left\langle\int_\Omega \mathcal{B}^*\mathcal{B}-\mathcal{B}_n^*\mathcal{B}_n\,d\mu f,f\right\rangle}\to 0 \tag{17}$$

as $n\to\infty$, as required.

(b) Follows from (a) by duality.

(c) First we note that due to (b) $\mathcal{I}(X)=\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu\in\mathfrak{C}_1(\mathcal{H})\subset\mathfrak{C}_\infty(\mathcal{H})$ whenever X is a finite rank operator. Due to (a) $\mathcal{I}$ is bounded on $\mathcal{B}(\mathcal{H})$ and therefore it leaves $\mathfrak{C}_\infty(\mathcal{H})$ invariant. Ky–Fan norms are interpolations of $||\cdot||_1$ and $||\cdot||_\infty$ as $||X||_{(k)}=\inf_{X=Y+Z}||Y||_1+k||Z||_\infty$ for all $k\in\mathbb{N}$, so (c) follows by applying (b) and (a) to $Y=\sum_{n=1}^{k-1}(s_n(X)-s_{n+1}(X))\sum_{j=1}^n e_j^*\otimes f_j$ and $Z=s_k(X)\sum_{j=1}^k e_j^*\otimes f_j+\sum_{j=k+1}^\infty s_j(X)e_j^*\otimes f_j$, respectively. Finally, (15) follows from (a) and (b) by interpolation. □

An immediate consequence of this lemma is

Theorem 3.1. *Let $\mathcal{C}_t|\mathcal{X}_t^*|^{1-\theta}$ and $\mathcal{V}_t|\mathcal{X}_t|^\theta\mathcal{D}_t$ be in $L_G^2(\Omega,d\mu,\mathcal{B}(\mathcal{H}))$ for some $\theta\in[0,1]$ and o.v. functions $\mathcal{C}_t,\mathcal{D}_t$ and $\mathcal{X}_t$, where $\mathcal{X}_t=\mathcal{V}_t|\mathcal{X}_t|$ is the polar decomposition of $\mathcal{X}_t$ for each $t\in\Omega$. Then*

(a)

$$\begin{bmatrix}\int_\Omega \mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu & \int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu\\ \left(\int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu\right)^* & \int_\Omega \mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu\end{bmatrix}\geqslant 0,$$

(b) *there is a contraction C such that*

$$\int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu=\sqrt{\int_\Omega \mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu}\;C\sqrt{\int_\Omega \mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu}, \tag{18}$$

(c) *for every $n\in\mathbb{N}$ there holds*

$$\prod_{k=1}^n s_k\left(\int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu\right)\leqslant\prod_{k=1}^n s_k^{\frac{1}{2}}\left(\int_\Omega \mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu\right)s_k^{\frac{1}{2}}\left(\int_\Omega \mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu\right), \tag{19}$$

(d) *for every* $n\in\mathbb{N}$ *and* $\alpha>0$ *there holds*

$$\sum_{k=1}^{n} s_k^{\alpha}\left(\int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu\right)\leqslant \sum_{k=1}^{n} s_k^{\frac{\alpha}{2}}\left(\int_\Omega \mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu\right)s_k^{\frac{\alpha}{2}}\left(\int_\Omega \mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu\right), \tag{20}$$

(e) *for every* $\alpha>0$ *there holds the following abstract Hölder inequality*

$$\left|\left|\left|\,\left|\int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu\right|^{\alpha}\right|\right|\right|_{\Phi_1}\leqslant\left|\left|\left|\sqrt{\int_\Omega \mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu}^{\,\alpha}\right|\right|\right|_{\Phi_2}\left|\left|\left|\sqrt{\int_\Omega \mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu}^{\,\alpha}\right|\right|\right|_{\Phi_3}, \tag{21}$$

whenever symmetric gauge functions Φ_1,Φ_2,Φ_3 *satisfy*

$$\Phi_1(\{s_nt_n\}_{n=1}^{\infty})\leqslant\Phi_2(\{s_n\}_{n=1}^{\infty})\Phi_3(\{t_n\}_{n=1}^{\infty})$$

for all sequences $\{s_n\}_{n=1}^{\infty},\{t_n\}_{n=1}^{\infty}$ *with finite number of non-zero elements,*

(f) *for all* $p,q>0$ *there holds*

$$\left|\left|\int_\Omega \mathcal{C}\mathcal{X}\mathcal{D}\,d\mu\right|\right|_{\frac{pq}{p+q}}\leqslant\left|\left|\sqrt{\int_\Omega \mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu}\right|\right|_p\left|\left|\sqrt{\int_\Omega \mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu}\right|\right|_q. \tag{22}$$

Proof. (a) As $|\mathcal{X}_t^*|^{1-\theta}\mathcal{V}_t=\mathcal{V}_t|\mathcal{X}_t|^{1-\theta}$ (follows by polynomial approximation from $\mathcal{X}_t(\mathcal{X}_t^*\mathcal{X}_t)^n=(\mathcal{X}_t\mathcal{X}_t^*)^n\mathcal{X}_t$ for all $n\in\mathbb{N}$), then $\mathcal{X}_t=\mathcal{V}_t|\mathcal{X}_t|=|\mathcal{X}_t^*|^{1-\theta}\mathcal{V}_t|\mathcal{X}_t|^{\theta}$ for all $t\in\Omega$. If $\mathcal{A}_t=\mathcal{C}_t|\mathcal{X}_t^*|^{1-\theta}$ and $\mathcal{B}_t=\mathcal{V}_t|\mathcal{X}_t|^{\theta}\mathcal{D}_t$, then $\mathcal{A},\mathcal{B}\in L_G^2(\Omega,d\mu,\mathcal{B}(\mathcal{H}))$, $\int_\Omega\mathcal{A}\mathcal{B}\,d\mu=\int_\Omega\mathcal{C}\mathcal{X}\mathcal{D}\,d\mu$, $\int_\Omega\mathcal{A}\mathcal{A}^*\,d\mu=\int_\Omega\mathcal{C}|\mathcal{X}^*|^{2-2\theta}\mathcal{C}^*\,d\mu$ and $\int_\Omega\mathcal{B}^*\mathcal{B}\,d\mu=\int_\Omega\mathcal{D}^*|\mathcal{X}|^{\theta}\mathcal{V}_t^*\mathcal{V}_t|\mathcal{X}|^{\theta}\mathcal{D}\,d\mu=\int_\Omega\mathcal{D}^*|\mathcal{X}|^{2\theta}\mathcal{D}\,d\mu$. For arbitrary $[g,f]^T\in\mathcal{H}\oplus\mathcal{H}$ the special case $X=I$ of (16) can be reread as the non-negativity of

$$\begin{vmatrix}\langle\int_\Omega\mathcal{A}\mathcal{A}^*\,d\mu g,g\rangle & \langle\int_\Omega\mathcal{A}\mathcal{B}\,d\mu f,g\rangle\\ \langle(\int_\Omega\mathcal{A}\mathcal{B}\,d\mu)^*g,f\rangle & \langle\int_\Omega\mathcal{B}^*\mathcal{B}\,d\mu f,f\rangle\end{vmatrix}$$

what is just the non-negativity of

$$\left\langle\begin{bmatrix}\int_\Omega\mathcal{A}\mathcal{A}^*\,d\mu & \int_\Omega\mathcal{A}\mathcal{B}\,d\mu\\ (\int_\Omega\mathcal{A}\mathcal{B}\,d\mu)^* & \int_\Omega\mathcal{B}^*\mathcal{B}\,d\mu\end{bmatrix}\begin{bmatrix}g\\ f\end{bmatrix},\begin{bmatrix}g\\ f\end{bmatrix}\right\rangle,$$

what was in fact required by (a).

(b) follows immediately from (a) by Theorem IX.5.9 of [6].

(c) follows from (b) by Horn inequality Theorem 1.13 in [25].

(d) follows from (c) by Horn and Weil Theorem 1.15 in [25].

(e) follows from (d) by abstract Hölder inequality Theorem 2.8 in [25].

(f) is a special case of (e) with $|||\cdot|||_{\Phi_1}=||\cdot||_1$, $|||\cdot|||_{\Phi_2}=||\cdot||_{\frac{p+q}{q}}$, $|||\cdot|||_{\Phi_3}=||\cdot||_{\frac{p+q}{p}}$ and $\alpha=\frac{pq}{p+q}$. $\square$

Remark 2. As seen from the proof of this theorem, the special case $\mathcal{X} = I$ implies the general one.

Remark 3. A special case of (e) of this theorem is [23] inequality (I.4) for $\mathcal{X} = I, \alpha = 1$ and semi-discrete o.v. functions $\mathcal{A}_t = \sum_1^K \varphi_k(t) A_k$ and $\mathcal{B}_t = \sum_1^K \varphi_k(t) B_k$, where $A_k, B_k \in \mathfrak{C}_\infty(\mathcal{H})$ and $\varphi_k \in L^2(\Omega, d\mu)$ for all $k = 1, \dots, K$. Another special case of (e) is the discrete Schwarz–Cauchy inequality of Kittaneh in [20] for $\theta = \frac{1}{2}, \Phi_2(\{s_n\}_{n=1}^\infty) = \Phi_3(\{s_n\}_{n=1}^\infty) = \Phi_1^{\frac{1}{2}}(\{s_n^2\}_{n=1}^\infty)$.

It is now easy to get a non-discrete version of Theorem 2.2 of Jocić [18], which will be the one mostly applied in this paper.

Theorem 3.2. *Let $\{\mathcal{A}_t\}_{t\in\Omega}$ and $\{\mathcal{B}_t\}_{t\in\Omega}$ be μ measurable families of normal commuting operators such that $\int_\Omega ||\mathcal{A}_t f||^2 + ||\mathcal{B}_t f||^2 d\mu(t) < \infty$ for all $f \in \mathcal{H}$. then*

$$\left|\left|\left|\int_\Omega \mathcal{A} X \mathcal{B}\, d\mu\right|\right|\right| \leqslant \left|\left|\left|\sqrt{\int_\Omega \mathcal{A}^* \mathcal{A}\, d\mu}\, X \sqrt{\int_\Omega \mathcal{B}^* \mathcal{B}\, d\mu}\right|\right|\right| \tag{23}$$

for all unitarily invariant norms $|||\cdot|||$ and all $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$.

Proof. Theorem 2.2 of [18] covers the case of simple o.v. functions, so if $\{\mathcal{A}_n\}_{n=1}^\infty, \{\mathcal{B}_n\}_{n=1}^\infty$ are chosen as in Lemma 2.1(b), then

$$\begin{aligned}\left|\left|\left|\int_\Omega \mathcal{A}_n X \mathcal{B}_n\, d\mu\right|\right|\right| &\leqslant \left|\left|\left|\sqrt{\int_\Omega |\mathcal{A}_n|^2\, d\mu}\, X \sqrt{\int_\Omega |\mathcal{B}_n|^2\, d\mu}\right|\right|\right| \\ &\leqslant \left|\left|\left|\sqrt{\int_\Omega |\mathcal{A}|^2\, d\mu}\, X \sqrt{\int_\Omega |\mathcal{B}|^2\, d\mu}\right|\right|\right|,\end{aligned}$$

according to the monotonicity of (singular values and) u.i. norms based on (6). By virtue of (a2) case of Lemma 3.1, the lower semi-continuity of the norm will secure the passage to the general case. □

Now, we present an improvement of our Lemma 3.1(c), which also improves (discrete) Theorem 2.1 of [19] and enhances it to the non-discrete case as well.

Theorem 3.3. *If μ is a σ-finite measure on Ω and $\{\mathcal{A}_t\}_{t\in\Omega}$ and $\{\mathcal{B}_t\}_{t\in\Omega}$ are μ-weak*-measurable families of bounded Hilbert space operators such that $\int_\Omega ||\mathcal{A}_t f||^2 +$*

$||\mathcal{A}_t^* f||^2 + ||\mathcal{B}_t f||^2 + ||\mathcal{B}_t^* f||^2 \, d\mu(t) < \infty$ *for all* $f \in \mathcal{H}$, *then*

$$\left|\left|\int_\Omega \mathcal{A}X\mathcal{B} \;\, d\mu\right|\right|_p \leqslant \left|\left|\sqrt[2q]{\int_\Omega \mathcal{A}^*\left(\int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu\right)^{q-1} \mathcal{A}\, d\mu}\; X \sqrt[2r]{\int_\Omega \mathcal{B}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu\right)^{r-1} \mathcal{B}^* \, d\mu}\right|\right|_p \tag{24}$$

for all $X \in \mathcal{C}_p(\mathcal{H})$ *and for all* $p, q, r \geqslant 1$ *such that* $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$.

Proof. In order to apply the (two variable) complex interpolation method, we will first turn our attention to the finite sums and the cases $p = 1, 2$ and $+\infty$, using later the approximations to carry on the general case.

So let $\{A_n\}_{n=1}^N$ and $\{B_n\}_{n=1}^N$ be some families of bounded operators and for a given $\varepsilon > 0$ let $A_0 = B_0 = \varepsilon I$. Define $A_{\diamond\varepsilon} = \sqrt{\sum_{n=0}^N A_n^* A_n}$, $A_{*\varepsilon} = \sqrt{\sum_{n=0}^N A_n A_n^*}$, $B_{\diamond\varepsilon} = \sqrt{\sum_{n=0}^N B_n^* B_n}$ and $B_{*\varepsilon} = \sqrt{\sum_{n=0}^N B_n B_n^*}$, and then obviously $A_{\diamond\varepsilon}, A_{*\varepsilon}, B_{\diamond\varepsilon}, B_{*\varepsilon} \geqslant \varepsilon I$, so that they are all invertible. Cases $p = 1$ and $p = \infty$ are covered by (2) and (1) of Theorem 2.1 of [19] (or by a discrete case of (12)), and therefore

$$\left|\left|\sum_{n=0}^N A_n X B_n\right|\right|_1 \leqslant ||A_{\diamond\varepsilon} X B_{*\varepsilon}||_1, \tag{25}$$

$$\left|\left|\sum_{n=0}^N A_n X B_n\right|\right| \leqslant \sqrt{\left|\left|\sum_{n=0}^N B_n^* B_n\right|\right|} \sqrt{\left|\left|\sum_{n=0}^N A_n A_n^*\right|\right|}\, ||X||. \tag{26}$$

In the case $p = 2$ for all $X \in \mathcal{C}_2(\mathcal{H})$ we will have

$$\begin{aligned}\left|\left|\sum_{n=0}^N A_n X B_n\right|\right|_2^2 &= \left|\left|\begin{bmatrix} A_0 & A_1 & \cdots & A_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} XB_0 & 0 & \cdots & 0 \\ XB_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ XB_N & 0 & \cdots & 0 \end{bmatrix}\right|\right|_2^2 \\ &\leqslant \left|\left|\begin{bmatrix} A_0 & A_1 & \cdots & A_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right|\right|^2 \left|\left|\begin{bmatrix} XB_0 & 0 & \cdots & 0 \\ XB_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ XB_N & 0 & \cdots & 0 \end{bmatrix}\right|\right|_2^2\end{aligned}$$

$$= \left\| \sum_{n=0}^{N} A_n A_n^* \right\| \operatorname{tr}\left(\sum_{n=0}^{N} B_n^* X^* X B_n \right)$$
$$= \left\| \sum_{n=0}^{N} A_n A_n^* \right\| \left\| X \sqrt{\sum_{n=0}^{N} B_n B_n^*} \right\|_2^2 .$$

So we have

$$\left\| \sum_{n=0}^{N} A_n X B_n \right\|_2 \leqslant \sqrt{\left\| \sum_{n=0}^{N} A_n A_n^* \right\|} \left\| X \sqrt{\sum_{n=0}^{N} B_n B_n^*} \right\|_2 \tag{27}$$

and similarly,

$$\left\| \sum_{n=0}^{N} A_n X B_n \right\|_2 \leqslant \sqrt{\left\| \sum_{n=0}^{N} B_n^* B_n \right\|} \left\| \sqrt{\sum_{n=0}^{N} A_n^* A_n}\, X \right\|_2 . \tag{28}$$

Now consider a general case. So take $w_\circ \notin \Omega$, let $\tilde{\Omega} = \{w_\circ\} \sqcup \Omega$, $\tilde{\mu} = \delta_{w_\circ} + \mu$, where $\delta_{w_\circ}$ is the Dirac atomic probabilistic measure on $w_\circ$ ($\delta_{w_\circ}(E) = \chi_E(w_\circ)$ for arbitrary $E \in \mathfrak{M}$.) Furthermore, let $\tilde{\mathcal{A}} = \varepsilon I \chi_{\{w_\circ\}} + \mathcal{A}\chi_\Omega$ and $\tilde{\mathcal{B}} = \varepsilon I \chi_{\{w_\circ\}} + \mathcal{B}\chi_\Omega$, and calculate

$$\tilde{\mathcal{A}}_\diamond = \sqrt{\int_{\tilde{\Omega}} \tilde{\mathcal{A}}^* \tilde{\mathcal{A}}\, d\tilde{\mu}} = \sqrt{\varepsilon^2 I + \mathcal{A}_\diamond^2} \geqslant \varepsilon I, \quad \tilde{\mathcal{A}}_* = \sqrt{\varepsilon^2 I + \mathcal{A}_*^2},$$

$$\tilde{\mathcal{B}}_\diamond = \sqrt{\varepsilon^2 I + \mathcal{B}_\diamond^2} \quad \text{and} \quad \tilde{\mathcal{B}}_* = \sqrt{\varepsilon^2 I + \mathcal{B}_*^2}. \tag{29}$$

Define the new families

$$\tilde{\mathcal{C}} = \tilde{\mathcal{A}}_*^{q-1} \tilde{\mathcal{A}} \quad \text{and} \quad \tilde{\mathcal{D}} = \tilde{\mathcal{B}} \tilde{\mathcal{B}}_\diamond^{r-1}. \tag{30}$$

Clearly $\tilde{\mathcal{C}}, \tilde{\mathcal{C}}^*, \tilde{\mathcal{D}}, \tilde{\mathcal{D}}^* \in L^2(\tilde{\Omega}, \tilde{\mu}, \mathcal{B}(\mathcal{H}))$, there holds $\tilde{\mathcal{C}}_\diamond, \tilde{\mathcal{C}}_* \geqslant \varepsilon^q$ and $\tilde{\mathcal{D}}_\diamond, \tilde{\mathcal{D}}_* \geqslant \varepsilon^r$, so they are all invertible. Therefore $\tilde{\mathcal{A}}_* = \tilde{\mathcal{C}}_*^{\frac{1}{q}}, \tilde{\mathcal{B}}_\diamond = \tilde{\mathcal{D}}_\diamond^{\frac{1}{r}}$, so by (30)

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_*^{1-q} \tilde{\mathcal{C}} = \tilde{\mathcal{C}}_*^{\frac{1}{q}-1} \tilde{\mathcal{C}}, \quad \tilde{\mathcal{B}} = \tilde{\mathcal{D}} \tilde{\mathcal{D}}_\diamond^{\frac{1}{r}-1}. \tag{31}$$

Now, based on Lemma 2.1(b), take approximating sequences of simple o.v. functions $\tilde{\mathcal{A}}_n$ and $\tilde{\mathcal{B}}_n$, as well as the corresponding $\tilde{\mathcal{C}}_n$ and $\tilde{\mathcal{D}}_n$ satisfying the analogs of (30) and (31). Thus, for an arbitrary $Y \in \mathfrak{C}_\infty(\mathcal{H})$ we are going to consider the following operator valued function:

$$\varphi(z, w) = \sum_{k_n=0}^{K_n} \tilde{\mathcal{C}}_*^{z-1} C_{k_n}^{(n)} \tilde{\mathcal{C}}_\diamond^{-z} Y \tilde{\mathcal{D}}_*^{-w} D_{k_n}^{(n)} \tilde{\mathcal{D}}_\diamond^{w-1}.$$

If $z = x + iy, w = s + it$, then in the direct strips product $0 \leqslant \Re(z), \Re(w) \leqslant 1$ this is a holomorphic in each variable, bounded function. For all $Y \in \mathfrak{C}_1(\mathcal{H})$ it satisfies the following boundary value estimates:

$$\begin{aligned}
\|\varphi(1+iy,1+it)\|_1 &= \left\| \sum_{K_n=0}^{K_n} \tilde{\mathcal{C}}_*^{iy} C_{k_n}^{(n)} \tilde{\mathcal{C}}_\diamond^{-iy-1} Y \tilde{\mathcal{D}}_*^{-it-1} D_{k_n}^{(n)} \tilde{\mathcal{D}}_\diamond^{it} \right\|_1 \\
&\leqslant \left\| \sqrt{\sum\nolimits_{k_n=0}^{K_n} \tilde{\mathcal{C}}_\diamond^{-1} C_{k_n}^{(n)*} C_{k_n}^{(n)} \tilde{\mathcal{C}}_\diamond^{-1}}\ \tilde{\mathcal{C}}_\diamond^{-iy} Y \tilde{\mathcal{D}}_*^{-it} \right. \\
&\quad \left. \times \sqrt{\sum\nolimits_{k_n=0}^{K_n} \tilde{\mathcal{D}}_*^{-1} D_{k_n}^{(n)} D_{k_n}^{(n)*} \tilde{\mathcal{D}}_*^{-1}} \right\|_1 \\
&\leqslant \|\tilde{\mathcal{C}}_{n\diamond} \tilde{\mathcal{C}}_\diamond^{-1}\| \|\tilde{\mathcal{C}}_\diamond^{-iy} Y \tilde{\mathcal{D}}_*^{-it}\|_1 \|\tilde{\mathcal{D}}_{n*} \tilde{\mathcal{D}}_*^{-1}\| \leqslant \|Y\|_1
\end{aligned} \tag{32}$$

according to (25), combined with the fact that $\tilde{\mathcal{C}}_{n\diamond} \tilde{\mathcal{C}}_\diamond^{-1}$ and $\tilde{\mathcal{D}}_{n*} \tilde{\mathcal{D}}_*^{-1}$ are contractions due to $\tilde{\mathcal{C}}_{n\diamond}^2 \leqslant \tilde{\mathcal{C}}_\diamond^2$ and $\tilde{\mathcal{D}}_{n*}^2 \leqslant \tilde{\mathcal{D}}_*^2$. For all $Y \in \mathfrak{C}_2(\mathcal{H})$ we also have

$$\|\varphi(1+iy, 0+it)\|_2 \leqslant \|Y\|_2 \tag{33}$$

according to the previously proven (28) and $\tilde{\mathcal{D}}_{n\diamond}^2 \leqslant \tilde{\mathcal{D}}_\diamond^2$. Similarly, from (27) and $\tilde{\mathcal{C}}_{n*}^2 \leqslant \tilde{\mathcal{C}}_*^2$ we get

$$\|\varphi(0+iy, 1+it)\|_2 \leqslant \|Y\|_2. \tag{34}$$

Also, by virtue of (26) for all $Y \in \mathfrak{C}_\infty(\mathcal{H})$ there holds

$$\|\varphi(0+iy, 0+it)\|_\infty \leqslant \|Y\|_\infty. \tag{35}$$

Thus $\varphi(1+it, w)$ is holomorphic in the strip $0 \leqslant \Re(w) \leqslant 1$ and satisfies (32) and (33), so we can invoke the "three line" Theorem 3.13.1. in [14] to get

$$\|\varphi(1+iy, 1/r)\|_{\frac{2r}{1+r}} \leqslant \|Y\|_{\frac{2r}{1+r}} \tag{36}$$

for all $Y \in \mathcal{C}_{\frac{2r}{1+r}}(\mathcal{H})$, as $(1-\frac{1}{r})0 + \frac{1}{r} \cdot 1 = \frac{1}{r}$ and $(1-\frac{1}{r})\frac{1}{2} + \frac{1}{r} \cdot 1 = \frac{1+r}{2r}$.

Another application of the three line theorem based on boundary estimates (32) and (33) to the function $\varphi(it, w)$ gives

$$\|\varphi(0+iy, 1/r)\|_{2r} \leqslant \|Y\|_{2r} \tag{37}$$

for all $Y \in \mathcal{C}_{2r}(\mathcal{H})$, as $(1-\frac{1}{r})0+\frac{1}{r}\cdot 1=\frac{1}{r}$ and $(1-\frac{1}{r})\frac{1}{\infty}+\frac{1}{r}\frac{1}{2}=\frac{1}{2r}$. As $\varphi(z,\frac{1}{r})$ is holomorphic in the strip $0\leqslant \Re(z)\leqslant 1$, the final application of the same interpolation theorem based on (36) and (37) gives

$$||\varphi(1/q,1/r)||_p=\left|\left|\sum_{k_n=0}^{K_n}\tilde{\mathcal{C}}_*^{\frac{1}{q}-1}C_{k_n}^{(n)}\tilde{\mathcal{C}}_\diamond^{-\frac{1}{q}}Y\tilde{\mathcal{D}}_*^{-\frac{1}{r}}D_{k_n}^{(n)}\tilde{\mathcal{D}}_\diamond^{\frac{1}{r}-1}\right|\right|_p\leqslant ||Y||_p \tag{38}$$

for all $Y\in\mathfrak{C}_p(\mathcal{H})$, as $(1-\frac{1}{q})0+\frac{1}{q}\cdot 1=\frac{1}{q}$ and $(1-\frac{1}{q})\frac{1}{2q}+\frac{1}{q}\frac{1+r}{2r}=\frac{1}{2q}+\frac{1}{2r}=\frac{1}{p}$.

For a given $X\in\mathfrak{C}_p(\mathcal{H})$ we apply (38) to $Y=\tilde{\mathcal{C}}_\diamond^{\frac{1}{q}}X\tilde{\mathcal{D}}_*^{\frac{1}{r}}\in\mathfrak{C}_p(\mathcal{H})$, to get

$$\begin{aligned}\left|\left|\sum_{k_n=1}^{K_n}A_{k_n}^{(n)}XB_{k_n}^{(n)}\right|\right|_p&=\left|\left|\sum_{k_n=0}^{K_n}\tilde{\mathcal{C}}_*^{\frac{1}{q}-1}C_{k_n}^{(n)}\tilde{\mathcal{C}}_\diamond^{-\frac{1}{q}}Y\tilde{\mathcal{D}}_*^{-\frac{1}{r}}D_{k_n}^{(n)}\tilde{\mathcal{D}}_\diamond^{\frac{1}{r}-1}\right|\right|_p\\&\leqslant ||Y||_p=\left|\left|\tilde{\mathcal{C}}_\diamond^{\frac{1}{q}}X\tilde{\mathcal{D}}_*^{\frac{1}{r}}\right|\right|_p\\&=\Bigg|\Bigg|\sqrt[2q]{\int_{\tilde{\Omega}}\tilde{\mathcal{A}}^*\left(\int_{\Omega}\tilde{\mathcal{A}}\tilde{\mathcal{A}}^*\,d\tilde{\mu}\right)^{q-1}\tilde{\mathcal{A}}\,d\tilde{\mu}}\,X\\&\quad\times\sqrt[2r]{\int_{\Omega}\tilde{\mathcal{B}}\left(\int_{\Omega}\tilde{\mathcal{B}}^*\tilde{\mathcal{B}}\,d\tilde{\mu}\right)^{r-1}\tilde{\mathcal{B}}^*\,d\tilde{\mu}}\Bigg|\Bigg|_p\end{aligned} \tag{39}$$

in view of (30). According to Lemma 3.1(a2) we have that $\sum_{k_n=1}^{K_n}A_{k_n}^{(n)}XB_{k_n}^{(n)}=\int_{\tilde{\Omega}}\tilde{\mathcal{A}}_nX\tilde{\mathcal{B}}_n\,d\tilde{\mu}$ converges weakly to $\int_{\tilde{\Omega}}\tilde{\mathcal{A}}X\tilde{\mathcal{B}}\,d\tilde{\mu}$ as $n\to\infty$. According to (17) an appeal to the lower semi-continuity of $||\cdot||_p$ proves Theorem 3.3 with $\tilde{\mathcal{A}},\tilde{\mathcal{B}}$ instead of $\mathcal{A},\mathcal{B}$. In other words, we now have

$$\begin{aligned}&\left|\left|\varepsilon^2X+\int_{\Omega}\mathcal{A}X\mathcal{B}\,d\mu\right|\right|_p\\&\quad\leqslant\left|\left|\sqrt[2q]{\int_{\tilde{\Omega}}\tilde{\mathcal{A}}^*\left(\int_{\tilde{\Omega}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}^*\,d\tilde{\mu}\right)^{q-1}\tilde{\mathcal{A}}\,d\tilde{\mu}}\,X\sqrt[2r]{\int_{\tilde{\Omega}}\tilde{\mathcal{B}}\left(\int_{\tilde{\Omega}}\tilde{\mathcal{B}}^*\tilde{\mathcal{B}}\,d\tilde{\mu}\right)^{r-1}\tilde{\mathcal{B}}^*\,d\tilde{\mu}}\right|\right|_p.\end{aligned} \tag{40}$$

Furthermore, we estimate

$$\begin{aligned}&\left|\left|\int_{\tilde{\Omega}}\tilde{\mathcal{A}}^*\left(\int_{\tilde{\Omega}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}^*\,d\tilde{\mu}\right)^{q-1}\tilde{\mathcal{A}}\,d\tilde{\mu}-\int_{\Omega}\mathcal{A}^*\left(\int_{\Omega}\mathcal{A}\mathcal{A}^*\,d\mu\right)^{q-1}\mathcal{A}\,d\mu\right|\right|\\&\quad\leqslant\varepsilon^2(\varepsilon^2+||\mathcal{A}_*||^2)^{q-1}+||(\varepsilon^2+\mathcal{A}_*^2)^{q-1}-(\mathcal{A}_*^2)^{q-1}||\left|\left|\int_{\Omega}\mathcal{A}^*\mathcal{A}\,d\mu\right|\right|\end{aligned} \tag{41}$$

and, by analogy, the corresponding inequality for $\tilde{\mathcal{B}}$ and $\mathcal{B}$ instead of $\tilde{\mathcal{A}}$ and $\mathcal{A}$. Now, to conclude the proof it will be enough to let $\varepsilon \to 0$, by invoking the continuity of all mappings $A \mapsto A^\alpha$, defined on positive operators in $\mathcal{B}(\mathcal{H})$ and any $\alpha \geqslant 0$. $\square$

Theorems 3.1 and 3.3 offer us some immediate estimates for integral transformers (induced by factorable o.v. functions) from one Schatten class to another. The simplest among elementary mappings is the multiplication transformer $\mathcal{M}_{A,B}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given by $\mathcal{M}_{A,B}(X) = AXB$ for fixed $A, B \in \mathcal{B}(\mathcal{H})$ and every $X \in \mathcal{B}(\mathcal{H})$. It was thoroughly investigated by Fialkow and Loebl who proved in [13] that the range of $\mathcal{M}_{A,B}$ is contained in a proper two-sided ideal $\mathfrak{C}_\Phi$ if and only if $\Phi(\{s_n(A)s_n(B)\}) < \infty$ and for the induced operator from $B(H)$ to $\mathfrak{C}_p(\mathcal{H})$ they proved

$$||\mathcal{M}_{A,B}||_{\mathcal{B}(\mathcal{H}) \to \mathfrak{C}_p(\mathcal{H})} = ||s_n(A)s_n(B)||_{\ell_p}. \tag{42}$$

For measurable o.v. functions $\mathcal{A}^*, \mathcal{B} \in L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ define $\mathcal{I}_{\mathcal{A},\mathcal{B}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\mathcal{I}_{\mathcal{A},\mathcal{B}}(X) = \int_\Omega \mathcal{A}X\mathcal{B}\, d\mu \quad \text{for all } X \in \mathcal{B}(\mathcal{H}). \tag{43}$$

In the sequel, we always assume $\mathcal{A}, \mathcal{B}^* \in L^2_G(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ as well, so that Lemma 3.1(c) assures that $\mathcal{I}_{\mathcal{A},\mathcal{B}}$ leaves every u.i. norm ideal invariant. So we have

Theorem 3.4. (a) *If $\int_\Omega \mathcal{A}\mathcal{A}^*\, d\mu$ and $\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu$ belong to $\mathfrak{C}_p(\mathcal{H})$ for some $1 \leqslant p \leqslant \infty$, then $\int_\Omega \mathcal{A}X\mathcal{B}\, d\mu \in \mathfrak{C}_p(\mathcal{H})$ for all $X \in \mathcal{B}(\mathcal{H})$ and*

$$||\mathcal{I}_{\mathcal{A},\mathcal{B}}||_{\mathcal{B}(\mathcal{H}) \to \mathfrak{C}_p(\mathcal{H})} \leqslant \left|\left| s_i^{\frac{1}{2}}\left(\int_\Omega \mathcal{A}\mathcal{A}^*\, d\mu\right) s_i^{\frac{1}{2}}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu\right)\right|\right|_{\ell_p} \tag{44}$$

$$\leqslant \sqrt{\left|\left|\int_\Omega \mathcal{A}\mathcal{A}^*\, d\mu\right|\right|_p \left|\left|\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu\right|\right|_p} = ||\mathcal{A}_*||_{2p}||\mathcal{B}_\diamond||_{2p}, \tag{45}$$

as well as

$$||\mathcal{I}_{\mathcal{A},\mathcal{B}}||^p_{\mathcal{B}(\mathcal{H}) \to \mathfrak{C}_p(\mathcal{H})} \leqslant \sum_{i=1}^\infty s_i^{\frac{1}{2}}\left(\int_\Omega \mathcal{A}^*\left(\int_\Omega \mathcal{A}\mathcal{A}^*\, d\mu\right)^{p-1} \mathcal{A}\, d\mu\right) s_i^{\frac{1}{2}}\left(\int_\Omega \mathcal{B}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu\right)^{p-1} \mathcal{B}^*\, d\mu\right). \tag{46}$$

Specially, if $\mathcal{B}^* = \mathcal{A}$ then

$$||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}||_{\mathcal{B}(\mathcal{H}) \to \mathfrak{C}_p(\mathcal{H})} = ||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}(I)||_p = \left|\left|\int_\Omega \mathcal{A}\mathcal{A}^*\, d\mu\right|\right|_p.$$

(b) *If* $\int_\Omega \mathcal{A}^*\mathcal{A}\,d\mu, \int_\Omega \mathcal{B}\mathcal{B}^*\,d\mu \in \mathfrak{C}_{\frac{q}{q-1}}(\mathcal{H})$ *for some* $1 \leqslant q < \infty$ *then* $\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu \in \mathfrak{C}_1(\mathcal{H})$ *for all* $X \in \mathcal{B}(\mathcal{H})$ *and*

$$||\mathcal{I}_{\mathcal{A},\mathcal{B}}||_{\mathfrak{C}_q(\mathcal{H})\to\mathfrak{C}_1(\mathcal{H})} \leqslant ||\mathcal{A}_\diamond||_{\frac{2q}{q-1}} ||\mathcal{B}_*||_{\frac{2q}{q-1}}. \tag{47}$$

If additionally $\mathcal{B}^* = \mathcal{A}$ *then*

$$||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}||_{\mathfrak{C}_q(\mathcal{H})\to\mathfrak{C}_1(\mathcal{H})} = ||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}(I)||_{\frac{q}{q-1}} = \left|\left|\int_\Omega \mathcal{A}^*\mathcal{A}\,d\mu\right|\right|_{\frac{q}{q-1}}.$$

(c) *Under conditions* (a) *and* (b) $\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu \in \mathfrak{C}_{\frac{pr}{pq-q+r}}(\mathcal{H})$ *for every* $r \geqslant q$ *and* $X \in \mathfrak{C}_r(\mathcal{H})$; *moreover*

$$||\mathcal{I}_{\mathcal{A},\mathcal{B}}||_{\mathfrak{C}_r(\mathcal{H})\to\mathfrak{C}_{\frac{pr}{pq-q+r}}} \leqslant ||\mathcal{A}_*||_{2p}^{1-\frac{q}{r}} ||\mathcal{B}_\diamond||_{2p}^{1-\frac{q}{r}} ||\mathcal{A}_\diamond||_{\frac{2q}{q-1}}^{\frac{q}{r}} ||\mathcal{B}_*||_{\frac{2q}{q-1}}^{\frac{q}{r}}. \tag{48}$$

Proof. Eq. (44) follows by case $\alpha = p$ of Theorem 3.1(d), together with $\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu$ in $\mathfrak{C}_p(\mathcal{H})$ for all $X \in \mathcal{B}(\mathcal{H})$, while $\theta = 1$ case (with $2p$ instead of p and q) of Theorem 3.1(f) gives

$$\begin{aligned}\left|\left|\int_\Omega \mathcal{A}X\mathcal{B}\,d\mu\right|\right|_p &\leqslant \left|\left|\sqrt{\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu}\right|\right|_{2p} \left|\left|\sqrt{\int_\Omega \mathcal{B}^*X^*X\mathcal{B}\,d\mu}\right|\right|_{2p} \\ &\leqslant \left|\left|\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu\right|\right|_p^{\frac{1}{2}} \left|\left|\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\right|\right|_p^{\frac{1}{2}} ||X||,\end{aligned}$$

establishing (45). Similar range inclusion argument is applicable in (46) as the trace of a positive operator $\int_\Omega \mathcal{A}^*(\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu)^{p-1}\mathcal{A}\,d\mu$ coincides with the trace of $(\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu)^p$. So by Fialkow and Loebl's inequality (42) there holds

$$\begin{aligned}&\left|\left|\left(\int_\Omega \mathcal{A}^*\left(\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu\right)^{p-1}\mathcal{A}\,d\mu\right)^{\frac{1}{2p}} X \left(\int_\Omega \mathcal{B}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\right)^{p-1}\mathcal{B}^*\,d\mu\right)^{\frac{1}{2p}}\right|\right|_p \\ &\leqslant \left|\left| s_i^{\frac{1}{2p}}\left(\int_\Omega \mathcal{A}^*\left(\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu\right)^{p-1}\mathcal{A}\,d\mu\right) s_i^{\frac{1}{2p}}\left(\int_\Omega \mathcal{B}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\right)^{p-1}\mathcal{B}^*\,d\mu\right)\right|\right|_{\ell_p} ||X|| \\ &= \sqrt[p]{\sum_{i=1}^\infty s_i^{\frac{1}{2}}\left(\int_\Omega \mathcal{A}^*\left(\int_\Omega \mathcal{A}\mathcal{A}^*\,d\mu\right)^{p-1}\mathcal{A}\,d\mu\right) s_i^{\frac{1}{2}}\left(\int_\Omega \mathcal{B}\left(\int_\Omega \mathcal{B}^*\mathcal{B}\,d\mu\right)^{p-1}\mathcal{B}^*\,d\mu\right)}\,||X||,\end{aligned}$$

which together with Theorem 3.3 proves (46). Note that the last estimate offered by (45) can easily be obtained from (46) as well. Now, if $\mathcal{B} = \mathcal{A}^*$ then

$$||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}||^p \leqslant \sum_{i=1}^{\infty} s_i \left(\int_\Omega \mathcal{A}^* \left(\int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu \right)^{p-1} \mathcal{A} \, d\mu \right)$$

$$= \operatorname{tr} \left(\int_\Omega \mathcal{A}^* \left(\int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu \right)^{p-1} \mathcal{A} \, d\mu \right)$$

$$= \left|\left| \int_\Omega \mathcal{A}\mathcal{A}^* \, d\mu \right|\right|_p^p = ||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}(I)||_p^p \leqslant ||\mathcal{I}_{\mathcal{A},\mathcal{A}^*}||^p,$$

from which we get the desired conclusion.

(b) follows by duality argument applied to the conjugate transformer $\mathcal{I}^*_{\mathcal{A},\mathcal{B}}$: $\mathfrak{C}_{\frac{q}{q-1}}(\mathcal{H}) \to \mathfrak{C}_1(\mathcal{H})$ by formula $\mathcal{I}^*_{\mathcal{A},\mathcal{B}}(X) = \int_\Omega \mathcal{B}X\mathcal{A} \, d\mu = \mathcal{I}_{\mathcal{B},\mathcal{A}}(X)$ for $X \in \mathfrak{C}_1(\mathcal{H})$, as $\mathfrak{C}_{\frac{q}{q-1}}(\mathcal{H}) = (\mathfrak{C}_q(\mathcal{H}))^*$, $\mathfrak{C}_1(\mathcal{H}) = (\mathcal{B}(\mathcal{H}))^*$ and

$$\operatorname{tr}(\mathcal{I}_{\mathcal{A},\mathcal{B}}(X)Y) = \int_\Omega \operatorname{tr}(\mathcal{A}X\mathcal{B}Y) \, d\mu = \int_\Omega \operatorname{tr}(X\mathcal{B}Y\mathcal{A}) \, d\mu = \operatorname{tr}(X\mathcal{I}^*_{\mathcal{A},\mathcal{B}}(Y))$$

for all $X \in \mathfrak{C}_1(\mathcal{H})$ and $Y \in \mathfrak{C}_{\frac{q}{q-1}}(\mathcal{H})$, according to the fact that the norms of conjugate operators (transformers) coincides.

(c) follows (a) and (b) by three line interpolation Theorem 3.13.1 of [14] as

$$\frac{1}{q} = \left(1 - \frac{q}{r}\right)\frac{1}{\infty} + \frac{q}{r}\frac{1}{q} \quad \text{and} \quad \frac{pq - q + r}{pr} = \left(1 - \frac{q}{r}\right)\frac{1}{p} + \frac{q}{r}\frac{1}{1}. \qquad \square$$

Here, we have finally arrived at a natural interpretation of $L^2_G(\Omega, d\mu, \mathfrak{C}_{|||\cdot|||}(\mathcal{H}))$ space's 2-convexization norms $|||\cdot|||_2$ from definition (9) as (the square root of) the norms of integral transformers from $\mathcal{B}(\mathcal{H})$ to $\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$.

Operator integrals play an important role throughout Mathematics and, as desirable, they often comply to criteria of some additional types of integrability, not just the Gel'fand's one. At the end of this section, we give as a brief example the Lyapunov equation $AX + XA^* = -W$, with the spectrum of a given bounded operator A contained in the open left half plane. A solution of this equation is expressed by (even Bochner integral) $X = \mathcal{I}_A(W) = \int_0^\infty e^{tA} W e^{tA^*} \, dt$, and with the known values of $||\mathcal{I}_A||_{\mathfrak{C}_p(\mathcal{H}) \to \mathfrak{C}_p(\mathcal{H})}$ for $p = 1$ and ∞, Bhatia in his work [7] raised the question of the exact value of $||\mathcal{I}_A||_{\mathfrak{C}_p(\mathcal{H}) \to \mathfrak{C}_p(\mathcal{H})}$ for $1 < p < \infty$. In this respect, we note here that Theorem 3.3 offers an ameliorated estimate

$$||\mathcal{I}_A||_{\mathfrak{C}_p(\mathcal{H})\to\mathfrak{C}_p(\mathcal{H})} \leqslant \left\| \int_0^\infty e^{tA} \left(\int_0^\infty e^{tA^*} e^{tA}\, dt \right)^{p-1} e^{tA^*}\, dt \right\|^{\frac{1}{p}},$$

sharper then the standard interpolation one offered by [7] or (15).

4. Applications to operator norm inequalities

Once we have the appropriate integral representation of the transformer from one type of elementary mapping to another (usually different generalized derivations), the above presented Cauchy–Schwarz inequalities may offer quite sharp estimates on their correlation. We will try to present some quite new examples, as well as some standard ones but in a new light, so the potential of those inequalities becomes more apparent. The first one will be the Aczel–Bellman inequality for operator integrals; it complements basic Theorem 3.2 and generalizes Theorem 2.3 of [18] in two directions.

Theorem 4.1. *Let μ be a σ-finite, positive measure on Ω and let each of measurable families (o.v. functions) $\{\mathcal{A}_t\}_{t\in\Omega}$ and $\{\mathcal{B}_t\}_{t\in\Omega}$ consists of commuting normal operators such that $\int_\Omega \mathcal{A}^*\mathcal{A}\, d\mu \leqslant I$ and $\int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu \leqslant I$. Then*

$$\left|\left|\left| \sqrt{I - \int_\Omega \mathcal{A}^*\mathcal{A}\, d\mu}\, X \sqrt{I - \int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu} \right|\right|\right| \leqslant \left|\left|\left| X - \int_\Omega \mathcal{A}X\mathcal{B}\, d\mu \right|\right|\right| \tag{49}$$

for every $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$.

Proof. Following notations from Theorem 3.4 we now have $\mathcal{A}_\diamond$ and $\mathcal{B}_\diamond$ to be contractions, and therefore $\mathcal{I}_{\mathcal{A},\mathcal{B}}$ is contractive on $\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ and leaves this ideal invariant (providing $X - \mathcal{I}_{\mathcal{A},\mathcal{B}}X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$). Given $m \in \mathbb{N}$, we have

$$\begin{aligned} &\left|\left|\left| \sqrt{I - \int_\Omega \mathcal{A}^*\mathcal{A}\, d\mu}\, X \sqrt{I - \int_\Omega \mathcal{B}^*\mathcal{B}\, d\mu} \right|\right|\right| \\ &= \left|\left|\left| \sqrt{I - \mathcal{A}_\diamond^2}\, X \sqrt{I - \mathcal{B}_\diamond^2} \right|\right|\right| \\ &\leqslant \left|\left|\left| \sqrt{I - \mathcal{A}_\diamond^2}\, (X - \mathcal{I}_{\mathcal{A},\mathcal{B}}^m X) \sqrt{I - \mathcal{B}_\diamond^2} \right|\right|\right| \end{aligned} \tag{50}$$

$$
\begin{aligned}
&+ \left|\left|\left| \sqrt{I - \mathcal{A}_\diamond^2}\, \mathcal{I}_{\mathcal{A},\mathcal{B}}^m X \sqrt{I - \mathcal{B}_\diamond^2} \right|\right|\right| \\
&= \left|\left|\left| \sum_{l=0}^{m-1} \mathcal{I}_{\mathcal{A},\mathcal{B}}^l \sqrt{I - \mathcal{A}_\diamond^2} (X - \mathcal{I}_{\mathcal{A},\mathcal{B}} X) \sqrt{I - \mathcal{B}_\diamond^2} \right|\right|\right| \\
&+ \left|\left|\left| \mathcal{I}_{\mathcal{A},\mathcal{B}}^m \sqrt{I - \mathcal{A}_\diamond^2}\, X \sqrt{I - \mathcal{B}_\diamond^2} \right|\right|\right| \\
&\leqslant \left|\left|\left| \sqrt{\sum_{l=0}^{m-1} \mathcal{A}_\diamond^{2l}} \sqrt{I - \mathcal{A}_\diamond^2} (X - \mathcal{I}_{\mathcal{A},\mathcal{B}} X) \sqrt{I - \mathcal{B}_\diamond^2} \sqrt{\sum_{l=0}^{m-1} \mathcal{B}_\diamond^{2l}} \right|\right|\right| \\
&+ \left|\left|\left| \mathcal{A}_\diamond^m \sqrt{I - \mathcal{A}_\diamond^2}\, X \sqrt{I - \mathcal{B}_\diamond^2}\, \mathcal{B}_\diamond^m \right|\right|\right| \qquad (51) \\
&\leqslant \left|\left|\left| \sqrt{I - \mathcal{A}_\diamond^{2m}} \big(X - \mathcal{I}_{\mathcal{A},\mathcal{B}} X\big) \sqrt{I - \mathcal{B}_\diamond^{2m}} \right|\right|\right| \\
&+ \left\| \sqrt{\mathcal{A}_\diamond^{2m} - \mathcal{A}_\diamond^{2m+2}} \right\| \left\| \sqrt{\mathcal{B}_\diamond^{2m} - \mathcal{B}_\diamond^{2m+2}} \right\| |||X||| \\
&\leqslant |||X - \mathcal{I}_{\mathcal{A},\mathcal{B}} X||| + \frac{1}{m+1} |||X||| = \left|\left|\left| X - \int_\Omega \mathcal{A} X \mathcal{B}\, d\mu \right|\right|\right| + \frac{1}{m+1} |||X|||. \qquad (52)
\end{aligned}
$$

Here, we have applied again Theorem 3.2 to $\sum_{l=0}^m \mathcal{I}_{\mathcal{A},\mathcal{B}}^l$ in (50), as well as to $\mathcal{I}_{\mathcal{A},\mathcal{B}}^m$ in (51), keeping in mind in (50) that, due to the commutativity of $\{\mathcal{A}_t\}_{t\in\Omega}$, $\sqrt{\sum_{l=0}^{m-1} \int_\Omega \int_\Omega \cdots \int_\Omega |\mathcal{A}_{t_1}\mathcal{A}_{t_2}\cdots\mathcal{A}_{t_l}|^2\, d\mu(t_1)\, d\mu(t_2) \cdots d\mu(t_l)} = \sqrt{\sum_{l=0}^{m-1} \mathcal{A}_\diamond^{2l}}$ and $\sqrt{\sum_{l=0}^{m-1} \int_\Omega \int_\Omega \cdots \int_\Omega |\mathcal{B}_{t_1}\mathcal{B}_{t_2}\cdots\mathcal{B}_{t_l}|^2\, d\mu(t_1)\, d\mu(t_2) \cdots d\mu(t_l)} = \sqrt{\sum_{l=0}^{m-1} \mathcal{B}_\diamond^{2l}}$ as well. Eq. (52) is a consequence of the fact that

$$
||\mathcal{A}_\diamond^{2m} - \mathcal{A}_\diamond^{2m+2}|| \leqslant \max_{t\in[0,1]} |t^{2m} - t^{2m+2}| = \frac{m^m}{(m+1)^{m+1}} \leqslant \frac{1}{m+1} \qquad (53)
$$

and $||\mathcal{B}_\diamond^{2m} - \mathcal{B}_\diamond^{2m+2}|| \leqslant \frac{1}{m+1}$ as well. Being $m \in \mathbb{N}$ arbitrary, the final conclusion follows. □

The arithmetic–geometric mean inequality for operators (see [8]) has had considerable applications in Operator Theory (see [9,17]) and different proofs are known. Here, we present one via operator integrals.

Theorem 4.2. *For all bounded* $A, B \geqslant 0$ *and all real* $t \geqslant 0$ *we have*

$$
\begin{aligned}
&2 \left|\left|\left| \sqrt{A}(X - e^{-tA} X e^{-tB}) \sqrt{B} \right|\right|\right| \\
&\qquad \leqslant \left|\left|\left| \sqrt{I - e^{-2tA}} (AX + XB) \sqrt{I - e^{-2tB}} \right|\right|\right| \qquad (54)
\end{aligned}
$$

$$\leqslant \left|\left|\left| AX + XB - e^{-tA}(AX + XB)e^{-tB} \right|\right|\right| \tag{55}$$

for all $X \in \mathcal{B}(\mathcal{H})$ *and all unitarily invariant norms* $\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$.

Also, for all C, D *and* X *in* $\mathcal{B}(\mathcal{H})$ *there holds the arithmetic–geometric means inequality*

$$2|||CXD||| \leqslant |||C^*CX + XDD^*|||. \tag{56}$$

Proof. As $\frac{d}{dt} e^{-tA} X e^{-tB} = -e^{-tA}(AX + XB)e^{-tB}$, an application of Theorem 3.2 gives

$$\begin{aligned}
&2|||\sqrt{A}(X - e^{-tA}Xe^{-tB})\sqrt{B}|||\\
&\quad = 2\left|\left|\left| \int_0^t \sqrt{A}e^{-sA}(AX + XB)e^{-sB})\sqrt{B}\,ds \right|\right|\right|\\
&\quad \leqslant 2\left|\left|\left| \sqrt{\int_0^t Ae^{-2sA}\,ds}(AX + XB)\sqrt{\int_0^t Be^{-2sB}\,ds} \right|\right|\right|\\
&\quad = \left|\left|\left| \sqrt{I - e^{-2tA}}(AX + XB)\sqrt{I - e^{-2tB}} \right|\right|\right|,
\end{aligned} \tag{57}$$

which proves (54). The next inequality (55) is an application of (4.1), which completes the proof of the first part of theorem.

We note that $\sqrt{A}(X - e^{-tA}Xe^{-tB})\sqrt{B} \to \sqrt{A}X\sqrt{B}$ weakly as $t \to \infty$, as well as the right-hand side of (54) is majorized by $|||AX + XB|||$. Therefore by the lower semi-continuity of $|||\cdot|||$ we conclude that

$$2|||\sqrt{A}X\sqrt{B}||| \leqslant |||AX + XB|||. \tag{58}$$

Finally, (56) follows by an application of (58) to $A = C^*C$ and $B = DD^*$, i.e.,

$$2|||CXD||| = 2||||C|X|D^*|||| \leqslant \left|\left|\left| |C|^2X + X|D^*| \right|\right|\right| = |||C^*CX + XDD^*|||. \tag{59}$$

The following is a generalization of the arithmetic–geometric mean inequality to higher-order generalized derivations.

Theorem 4.3. *Let* A *and* B *be positive*, $n \geqslant 0$ *integer and* $0 < \alpha < n$. *Then*

$$|||A^{\alpha}XB^{n-\alpha}||| \leqslant \frac{\sqrt{\Gamma(2\alpha)\Gamma(2n-2\alpha)}}{2^n(n-1)!}\left|\left|\left| \sum_{k=0}^{n} \binom{n}{k} A^{n-k}XB^k \right|\right|\right| \tag{60}$$

for all unitarily invariant norms $|||\cdot|||$.

Proof. If E_A and E_B are the spectral measures for A and B, respectively, then

$$\left|\left|\left|\int_0^\infty t^{n-1}A^\alpha e^{-\frac{tA}{2}}\left(\sum_{k=0}^n \binom{n}{k} A^{n-k}XB^k\right)e^{-\frac{tB}{2}}B^{n-\alpha}\,dt\right|\right|\right| \tag{61}$$

$$\leqslant \left|\left|\left|\sqrt{\int_0^\infty t^{2\alpha-1}A^{2\alpha}e^{-tA}dt}\left(\sum_{k=0}^n \binom{n}{k} A^{n-k}XB^k\right) \times \sqrt{\int_0^\infty t^{2n-2\alpha-1}B^{2n-2\alpha}e^{-tB}dt}\right|\right|\right| \tag{62}$$

$$= \sqrt{\Gamma(2\alpha)\Gamma(2n-2\alpha)}\left|\left|\left|E_A(0,+\infty)\sum_{k=0}^n \binom{n}{k} A^{n-k}XB^k E_B(0,+\infty)\right|\right|\right|. \tag{63}$$

Step (62) is by Theorem 3.2 and (63) by direct computation from the fact that

$$\begin{aligned}\left\langle \int_0^\infty t^{2\alpha-1}A^{2\alpha}e^{-tA}\,dt f, f\right\rangle &= \int_0^\infty t^{2\alpha-1}\langle A^{2\alpha}e^{-tA}f, f\rangle\,dt\\ &= \int_0^\infty \int_0^\infty t^{2\alpha-1}\lambda^{2\alpha}e^{-t\lambda}\,d\mu_f(\lambda)\,dt\\ &= \int_0^\infty \int_0^\infty (t\lambda)^{2\alpha-1}e^{-t\lambda}\,dt\,d\mu_f(\lambda)\\ &= \int_{0+}^\infty \Gamma(2\alpha)\,d\mu_f(s) = \Gamma(2\alpha)\langle E_A(0,+\infty)f, f\rangle \end{aligned} \tag{64}$$

for all $f \in \mathcal{H}$. The last step follows by change of variable $t\lambda = u$ for $\lambda > 0$ and the very definition of the Gamma function. Similarly $\int_0^\infty t^{2n-2\alpha-1}B^{2n-2\alpha}e^{-tB}\,dt = \Gamma(2n-2\alpha)E_B(0,+\infty)$. By (64) we also validated the Gel'fand integrability in (61), which by the Lebesgue convergence theorem assures the existence of

$$w-\lim_{T\to\infty}\int_0^T t^{n-1}A^\alpha e^{-\frac{tA}{2}}\left(\sum_{k=0}^n \binom{n}{k} A^{n-k}XB^k\right)e^{-\frac{tB}{2}}B^{n-\alpha}\,dt.$$

The same Lebesgue theorem used jointly with spectral theory gives

$$s-\lim_{T\to\infty} T^K E_A(0,+\infty)e^{-\frac{TA}{2}}\left(\sum_{k=0}^K \binom{K}{k} A^{K-k}XB^k\right)e^{-\frac{TB}{2}}E_B(0,+\infty) = 0$$

for all $0 \leqslant K \leqslant n-1$. By partial integration formula

$$\begin{aligned} &2^n(n-1)!A^\alpha XB^{n-\alpha} \\ &\quad = 2^n(n-1)!\sum_{K=0}^{n-1}\frac{T^K}{2^K K!}A^\alpha e^{-\frac{TA}{2}}\left(\sum_{k=0}^{K}\binom{K}{k}A^{K-k}XB^k\right)e^{-\frac{TB}{2}}B^{n-\alpha} \\ &\qquad + \int_0^T t^{n-1}A^\alpha e^{-\frac{tA}{2}}\left(\sum_{k=0}^{n}\binom{n}{k}A^{n-k}XB^k\right)e^{-\frac{tB}{2}}B^{n-\alpha}\,dt, \end{aligned} \tag{65}$$

so letting $T\to\infty$ we recognize $2^n(n-1)!A^\alpha XB^{n-\alpha}$ as the Gel'fand integral in (61). The final conclusion is now obvious from (63). $\square$

Corollary 4.1 (Young's inequality). *For all unitarily invariant norms* $|||\cdot|||$ *and all real* $p>1$ *and* $p'=\frac{p}{p-1}$

$$|||AXB||| \leqslant \frac{\sqrt[p]{p}\,\sqrt[p']{p'}}{2}\sqrt{\frac{(1-\frac{2}{p})\pi}{\sin(1-\frac{2}{p})\pi}}\left|\left|\left|\frac{|A|^p}{p}X + X\frac{|B^*|^{p'}}{p'}\right|\right|\right|. \tag{66}$$

Proof. The special case $n=1$ of the preceding theorem shows that

$$|||\sqrt[p]{C}X\sqrt[p']{D}||| \leqslant \frac{1}{2}\sqrt{\Gamma\left(\frac{2}{p}\right)\Gamma\left(2-\frac{2}{p}\right)}|||CX+XD||| \tag{67}$$

for all positive $C,D\in\mathcal{B}(\mathcal{H})$. Thus for $C=\frac{|A|^p}{p}$ and $D=\frac{|B^*|^{p'}}{p'}$ it follows that

$$|||AXB||| = \sqrt[p]{p}\,\sqrt[p']{p'}\left|\left|\left|\frac{|A|}{\sqrt[p]{p}}X\frac{|B^*|}{\sqrt[p']{p'}}\right|\right|\right| \leqslant \frac{\sqrt[p]{p}\,\sqrt[p']{p'}}{2}\sqrt{\frac{(1-\frac{2}{p})\pi}{\sin(1-\frac{2}{p})\pi}}\left|\left|\left|\frac{|A|^p}{p}X + X\frac{|B^*|^{p'}}{p'}\right|\right|\right|,$$

according to the fact that $\Gamma(\frac{2}{p})\Gamma(2-\frac{2}{p}) = |1-\frac{2}{p}|\frac{\pi}{\sin|1-\frac{2}{p}|\pi}$. $\square$

Remark 4. As pointed out in [2] it cannot be expected to reduce the constant in (66) to 1 for all unitarily invariant norms. Some special cases as $|||\cdot||| = ||\cdot||_2$ do (see [3]), as well as $p=p'=2$, which is the arithmetic–geometric means inequality. The constant C_p in (66) depends only on p, just as the analogous constant K_p in [21] does. However C_p have the advantage in factor $C_p/K_p \leqslant \sqrt{\frac{\pi\cos(1-\frac{2}{p})}{8}} < 0.7$, as well as the better asymptotic $C_p \sim \sqrt{\frac{p}{8}}$ as $p\to\infty$ and $C_p\sim\sqrt{\frac{1}{8p-8}}$ as $p\to 1+$, so that compared to K_p it diverges at a slower rate as $p\to 1+$ or $p\to\infty$.

Let us recall that a function $f: I\to\mathbb{R}$ is said to be operator monotone on the interval $I\subset\mathbb{R}$ if $A\leqslant B$ implies $f(A)\leqslant f(B)$ for all bounded A,B with spectra contained in I. Now we present the following mean value theorem for such functions.

Theorem 4.4 (Mean value theorem for o.m. functions). *Let $A, B \geqslant 0$ be a bounded operators and let $f : [0, +\infty) \to \mathbb{R}$ be operator monotone. Then for all bounded X and all unitarily invariant norms.*

$$\left|\left|\left|\frac{1}{\sqrt{f'(A)}}(f(A)X - Xf(B))\frac{1}{\sqrt{f'(B)}}\right|\right|\right| \leqslant |||AX - XB|||. \tag{68}$$

If additionally $\sqrt{f'(A)}(AX - XB)\sqrt{f'(B)} \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$, then also

$$|||f(A)X - Xf(B)||| \leqslant \left|\left|\left|\sqrt{f'(A)}(AX - XB)\sqrt{f'(B)}\right|\right|\right|. \tag{69}$$

Proof. Every operator monotone (increasing) function has its integral representation

$$f(A) = \alpha + \beta A + \int_0^\infty A(s + A)^{-1} s \, d\mu(s) \tag{70}$$

(see [5,1] or [22]), with $\beta \geqslant 0$ and μ being a positive Borel measure on $[0, \infty)$, such that the integral in the identity converges. So its derivative will be $f'(A) = \beta + \int_0^\infty (s + A)^{-2} s^2 \, d\mu(s)$. Given $\varepsilon > 0$, let $C = A + \varepsilon$ and $D = B + \varepsilon$, and note that $f'(C)^{-1} \leqslant f'(\|C\|)^{-1}$ and $f'(D)^{-1} \leqslant f'(\|D\|)^{-1}$ are bounded operators as f' is decreasing. Then for $Y = \sqrt{f'(C)^{-1}} X \sqrt{f'(D)^{-1}}$ there holds

$$\begin{aligned}
&\left|\left|\left|\frac{1}{\sqrt{f'(A+\varepsilon)}}(f(A+\varepsilon)X - Xf(B+\varepsilon))\frac{1}{\sqrt{f'(B+\varepsilon)}}\right|\right|\right| \\
&\quad = |||f(C)Y - Yf(D)||| \\
&\quad = \left|\left|\left|\beta(CY - YD) + \int_0^\infty (C(s+C)^{-1}Y - YD(s+D)^{-1}) s \, d\mu(s)\right|\right|\right| \\
&\quad = \left|\left|\left|\beta(CY - YD) + \int_0^\infty (s+C)^{-1}(CY - YD)(s+D)^{-1} s^2 \, d\mu(s)\right|\right|\right| \\
&\quad \leqslant \left|\left|\left|\sqrt{\beta + \int_0^\infty (s+C)^{-2} s^2 \, d\mu(s)}(CY - YD)\sqrt{\beta + \int_0^\infty (s+D)^{-2} s^2 \, d\mu(s)}\right|\right|\right| \qquad (71) \\
&\quad = \left|\left|\left|\sqrt{f'(C)}(CY - YD)\sqrt{f'(D)}\right|\right|\right| = |||AX - XB|||, \qquad (72)
\end{aligned}$$

with Theorem 3.2 applied in (71), where we added to the measure μ an additional atomic mass of weight β. Now the continuity of f and $\frac{1}{f'}$ on $[0,+\infty)$ assures that

$$s-\lim_{\varepsilon\to 0+} \frac{1}{\sqrt{f'(A+\varepsilon)}}(f(A+\varepsilon)X - Xf(B+\varepsilon))\frac{1}{\sqrt{f'(B+\varepsilon)}} = \frac{1}{\sqrt{f'(A)}}(f(A)X - Xf(B))\frac{1}{\sqrt{f'(B)}},$$

therefore (68) follows from (72) by the lower semi-continuity of $|||\cdot|||$. Now (69) follows by an application of (68) to $\sqrt{f'(A)}X\sqrt{f'(B)}$ instead of X. $\square$

Example 3. The functions $t\to t^\alpha$ for $0\leqslant\alpha\leqslant 1$ and log are operator monotone on $[0,+\infty)$ and $(0,+\infty)$ respectively, so

$$\left|\left|\left|A^{\frac{1+\alpha}{2}}XB^{\frac{1-\alpha}{2}} - A^{\frac{1-\alpha}{2}}XB^{\frac{1+\alpha}{2}}\right|\right|\right| \leqslant \alpha|||AX - XB|||, \tag{73}$$

(for a different proof see [11]), and

$$|||\sqrt{A}(\log(A)X - X\log(B))\sqrt{B}||| \leqslant |||AX - XB|||. \tag{74}$$

The last one is known as the geometric–logarithmic means inequality for operators (see [15]).

It is known ([5] or [22]) that $t/f(t)$ is operator monotone as long as $f(t)$ is. For a class of them we have the similar

Theorem 4.5. *Let $A,B>0$ be bounded operators and let f be a positive, operator monotone function on $(0,+\infty)$ such that both $g(x)=\frac{x}{f(x)}$ and $f(x)$ have integral representation* (70) *with $\alpha=\beta=0$, then*

$$|||f(A)Xg(B) - g(A)Xf(B)||| \leqslant |||AX - XB||| \tag{75}$$

for all bounded X and all unitarily invariant norms.

Proof. Let $f(A)=\int_0^\infty A(s+A)^{-1}\,d\mu(s)$ and $g(B)=\int_0^\infty B(t+B)^{-1}\,dv(t)$ be integral representations for f and g, respectively, with μ and v being appropriate positive measures (whose derivatives include factor s from the genuine representation (70)).

Therefore

$$
\begin{aligned}
&|||f(A)Xg(B) - g(A)Xf(B)|||\\
&\quad = \left|\left|\left|\int_0^\infty \int_0^\infty A(s+A)^{-1}XB(t+B)^{-1} - A(t+A)^{-1}XB(s+B)^{-1}\, d\mu(s)\, d\nu(t)\right|\right|\right|\\
&\quad = \left|\left|\left|\int_0^\infty \int_0^\infty (s-t)A(s+A)^{-1}(t+A)^{-1}(AX - XB)B(s+B)^{-1}\right.\right.\right.\\
&\qquad \left.\left.\left.\times (t+B)^{-1}\, d\mu(s)\, d\nu(t)\right|\right|\right|\\
&\quad \leqslant |||C(AX - XB)D|||,
\end{aligned}
$$

according to Theorem 3.2, where

$$C = \sqrt{\int_0^\infty \int_0^\infty |s-t|A^2(s+A)^{-2}(t+A)^{-2}\, d\mu(s)\, d\nu(t)}$$

and

$$D = \sqrt{\int_0^\infty \int_0^\infty |s-t|B^2(s+B)^{-2}(t+B)^{-2}\, d\mu(s)\, d\nu(t)}.$$

So, the proof of theorem will be completed by showing that C and D are contractions. Indeed, a straightforward computation shows that

$$|||s-t|A(s+A)^{-1}(t+A)^{-1}|| \leqslant ||(s+t)A(st + (s+t)A + A^2)^{-1}|| \leqslant 1$$

so

$$C^2 \leqslant \int_0^\infty \int_0^\infty A(s+A)^{-1}(t+A)^{-1}\, d\mu(s)\, d\nu(t) = \int_0^\infty f(A)(t+A)^{-1}\, d\nu(t) = I.$$

Similarly $D^2 \leqslant I$, and this concludes the proof. □

Remark 5. Under requirements of Theorem 4.5 we actually have $f(0+) = 0$, $f'(0+) = +\infty$ and $g(+\infty) = +\infty$.

Remark 6. As $t^\alpha = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t(t+s)^{-1}s^{\alpha-1}\, ds$ for all $0 < \alpha < 1$ and $t > 0$, so an application of Theorem 4.5 gives another inequality, which differs from (73) by the constant 1 instead of α on its right-hand side.

Acknowledgments

The author is thankful to the referee for drawing its attention to the norm convergence related to (53), which enabled the former separability assumption in Theorem 4.1 to be dropped.

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