BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **34**(1) (2011), 21–30

Certain Sufficient Conditions for Strongly Starlike Functions Associated with an Integral Operator

JIN-LIN LIU
Department of Mathematics, Yangzhou University,
Yangzhou, 225002, P. R. China
jlliu@yzu.edu.cn

Abstract. By using the method of differential subordinations, we derive certain sufficient conditions for strongly starlike functions associated with an integral operator. All these results presented here are sharp.

2010 Mathematics Subject Classification: Primary: 30C45; Secondary 30A10

Keywords and phrases: Analytic function, starlike function, strongly starlike function, subordination, Hadamard product (or convolution).

1. Introduction and preliminaries

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Also let the Hadamard product (or convolution) of two functions

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \quad (j = 1, 2),$$

be given by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k} = (f_2 * f_1)(z).$$

Given two functions $f(z)$ and $g(z)$, which are analytic in U, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g(z) \prec f(z) \quad (z \in U)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in U)$ such

Communicated by V. Ravichandran.
Received: June 18, 2009; *Revised:* October 13, 2009.

that $g(z) = f(w(z)) \quad (z \in U)$. In particular, if $f(z)$ is univalent in U, we have the following equivalence

$$g(z) \prec f(z) \quad (z \in U) \Longleftrightarrow g(0) = f(0) \quad \text{and} \quad g(U) \subset f(U).$$

A function $f(z) \in A_p$ is called p-valently starlike in U if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \quad (z \in U).$$

A function $f(z) \in A_p$ is called p-valent strongly starlike of order α $(0 < \alpha \leq 1)$ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{1.2}$$

For any integer n greater than $-p$, let $f_{n+p-1}(z) = z^p/(1-z)^{n+p}$ and let $f_{n+p-1}^{(-1)}(z)$ be defined such that

$$f_{n+p-1}(z) * f_{n+p-1}^{(-1)}(z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.3}$$

Then for $f(z) \in A_p$, we define an integral operator I_{n+p-1} as follows.

$$\begin{aligned} I_{n+p-1}f(z) &= f_{n+p-1}^{(-1)}(z) * f(z) \\ &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(p+k+1)\Gamma(p+n)}{\Gamma(p+k+n)\Gamma(p+1)} a_{p+k} z^{p+k}. \end{aligned} \tag{1.4}$$

It is obvious that $I_p f(z) = f(z)$. The operator I_{n+p-1} was introduced by Liu and Noor [3]. When $p = 1$, the operator I_n was first defined by Noor and Noor [6]. Many interesting subclasses of analytic functions, associated with the integral operator I_{n+p-1} and its many special cases, were investigated recently by (for example) Noor [5], Noor and Noor [6], Liu and Noor [3], Liu [1, 2] and others.

In order to prove our main results, we need the following lemma.

Lemma 1.1. *Let the function $g(z)$ be analytic and univalent in U and let the functions $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ $(w \in g(U))$. Set*

$$Q(z) = zg'(z)\varphi(g(z)) \quad \text{and} \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) *$Q(z)$ is univalently starlike in U and*

(ii)

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} = \mathrm{Re}\left(\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in U).$$

If $q(z)$ is analytic in U with $q(0) = g(0)$, $q(U) \subset D$ and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \quad (z \in U), \tag{1.5}$$

then $q(z) \prec g(z)$ $(z \in U)$ and $g(z)$ is the best dominant of (1.5).

The lemma is due to Miller and Mocanu [4, p.132].

2. Sufficient conditions for strongly starlike functions

In this section, we assume that $\alpha, \lambda_0, \lambda, a, b \in R$ and $\mu \in C$.

Theorem 2.1. *Let*

$$0<\alpha\le 1,\ \lambda_0 a\ge 0,\ |b+1|\le \frac{1}{\alpha} \quad \text{and} \quad |a-b-1|\le \frac{1}{\alpha}. \tag{2.1}$$

If $f(z)\in A_p$ *satisfies* $I_{n+p-1}f(z)(I_{n+p-1}f(z))'\neq 0$ $(z\in U\setminus\{0\})$ *and*

$$\lambda_0\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^a + z\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)'\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^b \prec h(z), \tag{2.2}$$

for $(z\in U)$ *where*

$$h(z)=\lambda_0\left(\frac{1+z}{1-z}\right)^{a\alpha}+\left(\frac{1+z}{1-z}\right)^{(b+1)\alpha}\cdot\frac{2\alpha z}{1-z^2}, \tag{2.3}$$

then the function $I_{n+p-1}f(z)$ *is p-valent strongly starlike of order* α *in* U. *The number* α *is sharp for the function* $f(z)$ *defined by*

$$\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}=\left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{2.4}$$

Proof. We choose

$$q(z)=\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)},\quad g(z)=\left(\frac{1+z}{1-z}\right)^{\alpha},\quad \theta(w)=\lambda_0 w^a \quad\text{and}\quad \varphi(w)=w^b$$

in lemma. Clearly, the function $g(z)$ is analytic and univalently convex in U and

$$|\arg g(z)|<\frac{\pi}{2}\alpha\le\frac{\pi}{2}\quad (z\in U). \tag{2.5}$$

The function $q(z)$ is analytic in U with $q(0)=g(0)=1$ and $q(z)\neq 0$ $(z\in U)$. The functions $\theta(w)$ and $\varphi(w)$ are analytic in a domain D containing $g(U)$ and $q(U)$, with $\varphi(w)\neq 0$ when $w\in g(U)$. For

$$-\frac{1}{\alpha}\le b+1\le\frac{1}{\alpha},$$

the function $Q(z)$ given by

$$Q(z)=zg'(z)\varphi(g(z))=\frac{2\alpha z}{(1-z)^{1+(b+1)\alpha}(1+z)^{1-(b+1)\alpha}}$$

is univalently starlike in U because

$$\begin{aligned}\mathrm{Re}\frac{zQ'(z)}{Q(z)}&=1+(1+(b+1)\alpha)\mathrm{Re}\frac{z}{1-z}-(1-(b+1)\alpha)\mathrm{Re}\frac{z}{1+z}\\ &>1-\frac{1}{2}(1+(b+1)\alpha)-\frac{1}{2}(1-(b+1)\alpha)=0\quad (z\in U).\end{aligned} \tag{2.6}$$

Further, we have

$$\begin{aligned}\theta(g(z))+Q(z)&=\lambda_0\left(\frac{1+z}{1-z}\right)^{a\alpha}+\frac{2\alpha z}{(1-z)^{1+(b+1)\alpha}(1+z)^{1-(b+1)\alpha}}\\ &=h(z),\end{aligned}$$

where $h(z)$ is given by (2.3), and so

$$\begin{aligned}\frac{zh'(z)}{Q(z)} &= \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} \\ &= \lambda_0 a(g(z))^{a-b-1} + \frac{zQ'(z)}{Q(z)}. \end{aligned} \tag{2.7}$$

Also, for

$$|a-b-1| \le \frac{1}{\alpha},$$

we find that

$$\left|\arg(g(z))^{a-b-1}\right| \le |a-b-1| \cdot \frac{\alpha\pi}{2} \le \frac{\pi}{2} \quad (z \in U). \tag{2.8}$$

Therefore, it follows from (2.1) and (2.5) to (2.8) that

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} > 0 \quad (z \in U).$$

The other conditions of lemma are also satisfied. Hence we conclude that

$$q(z) = \frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} = g(z) \quad (z \in U)$$

and $g(z)$ is the best dominant of (2.2). By (2.5) we see that the function $I_{n+p-1}f(z)$ is p-valent strongly starlike of order α in U.

Furthermore, for the function $f(z)$ defined by (2.4), we have

$$\lambda_0(q(z))^a + zq'(z)(q(z))^b = h(z),$$

which shows that the number α is sharp. The proof of Theorem 2.1 is completed. ∎

Theorem 2.2. *Let*

$$0 < \alpha \le 1,\ \lambda(b+2) \ge 0,\ (b+1)\mathrm{Re}\mu \ge 0 \quad \text{and} \quad |b+1| \le \frac{1}{\alpha}. \tag{2.9}$$

If $f(z) \in A_p$ satisfies $I_{n+p-1}f(z)(I_{n+p-1}f(z))' \neq 0 \quad (z \in U \setminus \{0\})$ and

$$\begin{aligned}&\lambda\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b+2} + \mu\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b+1} \\ &\quad + z\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)'\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b} \\ &\prec h(z) \quad (z \in U),\end{aligned} \tag{2.10}$$

where

$$h(z) = \left(\frac{1+z}{1-z}\right)^{(b+1)\alpha}\left(\mu + \lambda\left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2}\right), \tag{2.11}$$

then the function $I_{n+p-1}f(z)$ is p-valent strongly starlike of order α in U. The number α is sharp for the function $f(z)$ defined by (2.4).

Proof. Let

$$q(z) = \frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)},\ g(z) = \left(\frac{1+z}{1-z}\right)^{\alpha},\ \theta(w) = \lambda w^{b+2} + \mu w^{b+1} \quad \text{and} \quad \varphi(w) = w^b$$

in lemma. Clearly, the functions $q(z), g(z), \theta(w), \varphi(w)$ and $Q(z) = zg'(z)\varphi(g(z))$ satisfy the conditions of lemma respectively. Further, we have

$$\begin{aligned}\theta(g(z)) + Q(z) &= \lambda\left(\frac{1+z}{1-z}\right)^{(b+2)\alpha} + \mu\left(\frac{1+z}{1-z}\right)^{(b+1)\alpha} \\ &\quad + \frac{2\alpha z}{(1-z)^{1+(b+1)\alpha}(1+z)^{1-(b+1)\alpha}} \\ &= h(z),\end{aligned}$$

where $h(z)$ is given by (2.11), and so

$$\begin{aligned}\frac{zh'(z)}{Q(z)} &= \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} \\ &= \lambda(b+2)g(z) + \mu(b+1) + \frac{zQ'(z)}{Q(z)}.\end{aligned}$$

Now, for

$$\lambda(b+2) \geq 0 \quad \text{and} \quad (b+1)\text{Re}\ \mu \geq 0,$$

we have

$$\text{Re}\frac{zh'(z)}{Q(z)} > 0 \quad (z \in U).$$

The other conditions of lemma are also satisfied. Hence we obtain the desired result of the theorem.

Furthermore, for the function $f(z)$ defined by (2.4), we have

$$\lambda(q(z))^{b+2} + \mu(q(z))^{b+1} + zq'(z)(q(z))^b = h(z),$$

which shows that the number α is sharp. The proof of Theorem 2.2 is completed. ∎

Theorem 2.3. *Let*

(2.12) $$0 < \alpha \leq 1, \quad \mu > 0 \quad \textit{and} \quad 0 \leq b+1 \leq 1.$$

If $f(z) \in A_p$ *satisfies* $I_{n+p-1}f(z)(I_{n+p-1}f(z))' \neq 0$ $(z \in U \setminus \{0\})$ *and*

(2.13) $$\left|\arg\left\{\mu\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b+1} + z\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)'\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b}\right\}\right| < \frac{\pi}{2}\beta \quad (z \in U),$$

where

(2.14) $$\beta = (b+1)\alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{\mu}\right),$$

then

(2.15) $$\left|\arg\left(\frac{z(I_{n+p-1}f(z))'}{I_{n+p-1}f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U).$$

This shows that the function $I_{n+p-1}f(z)$ is p-valent strongly starlike of order α in U. The bound β in (2.13) is the largest number such that (2.15) holds true.

Proof. By taking

$$\lambda = 0, \quad \mu > 0 \quad \text{and} \quad 0 \le b+1 \le \frac{1}{\alpha}$$

in Theorem 2.2, we see that if

$$\mu\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b+1} + z\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)'\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b} \prec h(z) \tag{2.16}$$

for $(z \in U)$, where

$$h(z) = \left(\frac{1+z}{1-z}\right)^{(b+1)\alpha}\left(\mu + \frac{2\alpha z}{1-z^2}\right), \tag{2.17}$$

then (2.15) is true.

For $z = e^{i\theta}$ $(\theta \in R)$, $z \neq 1$ and $z \neq -1$, we get

$$\frac{z}{1-z} = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}, \quad \frac{z}{1+z} = \frac{1}{2} + \frac{i}{2}\tan\frac{\theta}{2} \tag{2.18}$$

$$\frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \cot\frac{\theta}{2}e^{\frac{\pi}{2}i} \neq 0. \tag{2.19}$$

The following two cases arise.
(i) If

$$k(\theta) = \cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{1}{2}\sin\theta > 0,$$

then we deduce from (2.17) to (2.19) that

$$h(e^{i\theta}) = \left(\cot\frac{\theta}{2}\right)^{(b+1)\alpha} e^{\frac{1}{2}(b+1)\alpha\pi i}\left(\mu + i\frac{\alpha}{2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right),$$

which yields

$$\arg\, h(e^{i\theta}) = \frac{1}{2}(b+1)\alpha\pi + \tan^{-1}\left(\frac{\alpha}{2\mu k(\theta)}\right) \tag{2.20}$$

for $\mu > 0, e^{i\theta} \neq 1$ and $e^{i\theta} \neq -1$. Let $\theta_1 = \frac{\pi}{2}$, then

$$0 < k(\theta) \le k(\theta_1) = \frac{1}{2} \tag{2.21}$$

and it follows from (2.12),(2.20) and (2.21) that

$$\begin{aligned} \pi > \arg\, h(e^{i\theta}) \ge \arg\, h(e^{i\theta_1}) &= \frac{1}{2}(b+1)\alpha\pi + \tan^{-1}\left(\frac{\alpha}{\mu}\right) \\ &= \frac{\pi}{2}\beta > 0. \end{aligned} \tag{2.22}$$

(ii) If $k(\theta) < 0$, then it follows from (2.17) to (2.19) that

$$h(e^{i\theta}) = \left(-\cot\frac{\theta}{2}\right)^{(b+1)\alpha} e^{-\frac{1}{2}(b+1)\alpha\pi i}\left(\mu + i\frac{\alpha}{2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right),$$

and so

$$\arg h(e^{i\theta}) = -\frac{1}{2}(b+1)\alpha\pi + \tan^{-1}\left(\frac{\alpha}{2\mu k(\theta)}\right) \tag{2.23}$$

for $\mu > 0$, $e^{i\theta} \neq 1$ and $e^{i\theta} \neq -1$. Let $\theta_2 = -\frac{\pi}{2}$. Then

$$0 > k(\theta) \geq k(\theta_2) = -\frac{1}{2} \tag{2.24}$$

and from (2.12),(2.23) and (2.24) we have

$$\begin{aligned} -\pi < \arg h(e^{i\theta}) \leq \arg h(e^{i\theta_2}) &= -\frac{1}{2}(b+1)\alpha\pi - \tan^{-1}\left(\frac{\alpha}{\mu}\right) \\ &= -\frac{\pi}{2}\beta < 0. \end{aligned} \tag{2.25}$$

Noting that $h(0) = \mu > 0$, we find from (2.22) and (2.25) that $h(U)$ properly contains the angular region $-\frac{\pi}{2}\beta < \arg w < \frac{\pi}{2}\beta$ in the complex w-plane. Consequently, if $f(z) \in A_p$ satisfies (2.13), then the subordination relation (2.16) holds true, and so we have the assertion (2.15) of Theorem 2.3.

Furthermore, for the function $f(z) \in A_p$ defined by (2.4), we have (2.15) and

$$\mu\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b+1} + z\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)'\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b} = h(z).$$

Hence, by using (2.22) and (2.25), we conclude that the bound β in (2.13) is the best possible. This completes our proof. ∎

Theorem 2.4. *Let*

$$0 < \alpha < 1, \quad \lambda > 0 \quad \text{and} \quad 0 \leq b+2 \leq 1. \tag{2.26}$$

If $f(z) \in A_p$ satisfies $I_{n+p-1}f(z)(I_{n+p-1}f(z))' \neq 0$ $(z \in U \setminus \{0\})$ and

$$\begin{aligned} &\left|\arg\left\{\lambda\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b+2} + z\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)'\left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)}\right)^{b}\right\}\right| \\ &< \frac{\pi}{2}\gamma \quad (z \in U), \end{aligned} \tag{2.27}$$

where

$$\gamma = (b+2)\alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\left(\frac{\pi}{2}\alpha\right)}{2\lambda\delta(\alpha) + \alpha\sin\left(\frac{\pi}{2}\alpha\right)}\right), \tag{2.28}$$

and

$$\delta(\alpha) = \frac{1}{2}(1-\alpha)^{\frac{1-\alpha}{2}} \cdot (1+\alpha)^{\frac{1+\alpha}{2}}, \tag{2.29}$$

then

$$\left|\arg\left(\frac{z(I_{n+p-1}f(z))'}{I_{n+p-1}f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.30}$$

This shows that the function $I_{n+p-1}f(z)$ is p-valent strongly starlike of order α in U. The bound γ is the largest number such that (2.30) holds true.

Proof. Putting
$$\mu = 0, \quad \lambda > 0 \quad \text{and} \quad 0 \le b+2 \le \frac{1}{\alpha},$$
we easily have (2.9) and it follows from Theorem 2.2 that if

$$\lambda \left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \right)^{b+2} + z \left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \right)' \left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \right)^{b} \prec h(z) \tag{2.31}$$

for $(z \in U)$, where

$$h(z) = \left(\frac{1+z}{1-z} \right)^{(b+1)\alpha} \left(\lambda \left(\frac{1+z}{1-z} \right)^{\alpha} + \frac{2\alpha z}{1-z^2} \right), \tag{2.32}$$

then (2.30) holds true.

Proceeding as in the proof of Theorem 2.3, we consider the following two cases.
(i) If
$$k(\theta) = \cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{1}{2}\sin\theta > 0,$$
then from (2.18) and (2.19) (used in the proof of Theorem 2.3) and (2.32) we get
$$h(e^{i\theta}) = \left(\cot\frac{\theta}{2} \right)^{(b+1)\alpha} e^{\frac{1}{2}(b+1)\alpha\pi i} \left(\lambda \left(\cot\frac{\theta}{2} \right)^{\alpha} e^{\frac{\alpha\pi i}{2}} + i\frac{\alpha}{2k(\theta)} \right)$$
and so
$$\arg h(e^{i\theta}) = \frac{1}{2}(b+2)\alpha\pi + \tan^{-1} \left(\frac{\alpha\cos\left(\frac{\pi}{2}\alpha\right)}{2\lambda k_1(\theta) + \alpha\sin\left(\frac{\pi}{2}\alpha\right)} \right), \tag{2.33}$$
where $\lambda > 0,\ 0 < \alpha < 1,\ e^{i\theta} \neq 1,\ e^{i\theta} \neq -1$ and
$$k_1(\theta) = \left(\cot\frac{\theta}{2} \right)^{\alpha} k(\theta) > 0. \tag{2.34}$$

Let us now calculate the maximum value of $k_1(\theta)$. It is easy to verify that
$$\lim_{\theta \to 0} k_1(\theta) = \lim_{e^{i\theta} \to -1} k_1(\theta) = 0 \tag{2.35}$$
and that
$$\begin{aligned} k_1'(\theta) &= -\frac{\alpha}{2} \left(\cot\frac{\theta}{2} \right)^{\alpha-1} \cdot \frac{k(\theta)}{\left(\sin\frac{\theta}{2}\right)^2} + \frac{1}{2} \left(\cot\frac{\theta}{2} \right)^{\alpha} \cos\theta \\ &= \frac{1}{2} \left(\cot\frac{\theta}{2} \right)^{\alpha} (\cos\theta - \alpha). \end{aligned} \tag{2.36}$$
Set
$$\theta_1 = \cos^{-1}\alpha. \tag{2.37}$$
Then $k_1'(\theta_1) = 0$. Noting that $0 < \alpha < 1$, we easily have
$$0 < \theta_1 < \frac{\pi}{2}. \tag{2.38}$$
Hence, $k(\theta_1) > 0$ and it follows from (2.34) to (2.38) that
$$0 < k_1(\theta) \le k_1(\theta_1) = \left(\sin\frac{\theta_1}{2} \right)^{-2\alpha} \left(\cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2} \right)^{1+\alpha}$$

$$= \left(\frac{1-\cos\theta_1}{2}\right)^{-\alpha}\left(\frac{1}{2}\sin\theta_1\right)^{1+\alpha}$$

$$= \frac{(1-\alpha^2)^{\frac{1+\alpha}{2}}}{2(1-\alpha)^{\alpha}} = \delta(\alpha). \tag{2.39}$$

Thus, by using (2.26),(2.33) and (2.39), we arrive at

$$\pi > \arg\, h(e^{i\theta}) \geq \arg\, h(e^{i\theta_1}) = \frac{1}{2}(b+2)\alpha\pi + \tan^{-1}\left(\frac{\alpha\cos\left(\frac{\pi}{2}\alpha\right)}{2\lambda\delta(\alpha)+\alpha\sin\left(\frac{\pi}{2}\alpha\right)}\right)$$

$$= \frac{\pi}{2}\gamma > 0. \tag{2.40}$$

(ii) If $k(\theta) < 0$, then we obtain

$$h(e^{i\theta}) = \left(-\cot\frac{\theta}{2}\right)^{(b+1)\alpha} e^{-\frac{1}{2}(b+1)\alpha\pi i}\left(\lambda\left(-\cot\frac{\theta}{2}\right)^{\alpha} e^{-\frac{\alpha\pi i}{2}} + i\frac{\alpha}{2k(\theta)}\right),$$

which leads to

$$\arg\, h(e^{i\theta}) = -\frac{1}{2}(b+2)\alpha\pi - \tan^{-1}\left(\frac{\alpha\cos\left(\frac{\pi}{2}\alpha\right)}{2\lambda k_2(\theta)+\alpha\sin\left(\frac{\pi}{2}\alpha\right)}\right), \tag{2.41}$$

where $\lambda > 0,\ 0 < \alpha < 1,\ e^{i\theta} \neq 1$ and $e^{i\theta} \neq -1$ and

$$k_2(\theta) = \left(-\cot\frac{\theta}{2}\right)^{\alpha}(-k(\theta)) > 0.$$

Now we have

$$\lim_{\theta\to 0} k_2(\theta) = \lim_{e^{i\theta}\to -1} k_2(\theta) = 0$$

and

$$k_2'(\theta) = \frac{1}{2}\left(-\cot\frac{\theta}{2}\right)^{\alpha}(\alpha - \cos(-\theta)).$$

Let

$$\theta_2 = -\cos^{-1}\alpha.$$

Then $k_2'(\theta_2) = 0$, $\theta_1 + \theta_2 = 0$ and $-\frac{\pi}{2} < \theta_2 < 0$. Thus, we deduce that $k(\theta_2) < 0$ and

$$0 < k_2(\theta) \leq k_2(\theta_2) = \left(-\sin\frac{\theta_2}{2}\right)^{-2\alpha}\left(-\cos\frac{\theta_2}{2}\sin\frac{\theta_2}{2}\right)^{1+\alpha}$$

$$= \left(\frac{1-\cos\theta_2}{2}\right)^{-\alpha}\left(-\frac{1}{2}\sin\theta_2\right)^{1+\alpha}$$

$$= \frac{(1-\alpha^2)^{\frac{1+\alpha}{2}}}{2(1-\alpha)^{\alpha}} = \delta(\alpha). \tag{2.42}$$

Further, by (2.26), (2.41) and (2.42), we find that

$$-\pi < \arg\, h(e^{i\theta}) \leq \arg\, h(e^{i\theta_2}) = -\frac{1}{2}(b+2)\alpha\pi - \tan^{-1}\left(\frac{\alpha\cos\left(\frac{\pi}{2}\alpha\right)}{2\lambda\delta(\alpha)+\alpha\sin\left(\frac{\pi}{2}\alpha\right)}\right)$$

$$= -\frac{\pi}{2}\gamma < 0. \tag{2.43}$$

In view of $h(0) = \lambda > 0$, we conclude from (2.40) and (2.43) that $h(U)$ properly contains the angular region $-\frac{\pi}{2}\gamma < \arg w < \frac{\pi}{2}\gamma$ in the complex w-plane. Therefore, if $f(z) \in A_p$ satisfies (2.27), then the subordination relation (2.31) holds true, and thus we arrive at (2.30).

Furthermore, for the function $f(z) \in A_p$ defined by (2.4), we have (2.30) and

$$\lambda \left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \right)^{b+2} + z \left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \right)' \left(\frac{z(I_{n+p-1}f(z))'}{pI_{n+p-1}f(z)} \right)^{b} = h(z).$$

Hence, by using (2.40) and (2.43), we see that the bound γ in (2.27) is sharp. The proof is now completed. ∎

Remark 2.1. If we let $\lambda = p = n = 1$ and $b = -1$, Theorem 2.4 reduces to the result obtained earlier by Nunokawa [7] (see also Nunokawa and Thomas [9]) by using another method.

Acknowledgement. I would like to express sincere thanks to the referees for careful reading and suggestions which helped me to improve the paper.

References

[1] J.-L. Liu, The Noor integral and strongly starlike functions, *J. Math. Anal. Appl.* **261** (2001), no. 2, 441–447.

[2] J.-L. Liu, Properties of certain subclass of multivalent functions defined by an integral operator, *Complex Var. Elliptic Equ.* **54** (2009), no. 5, 471–483.

[3] J.-L. Liu and K. I. Noor, Some properties of Noor integral operator, *J. Nat. Geom.* **21** (2002), no. 1-2, 81–90.

[4] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[5] K. I. Noor, On new classes of integral operators, *J. Nat. Geom.* **16** (1999), no. 1-2, 71–80.

[6] K. I. Noor and M. A. Noor, On integral operators, *J. Math. Anal. Appl.* **238** (1999), no. 2, 341–352.

[7] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, *Proc. Japan Acad. Ser. A Math. Sci.* **69** (1993), no. 7, 234–237.

[8] M. Nunokawa, S. Owa, H. Saitoh, A. Ikeda and N. Koike, Some results for strongly starlike functions, *J. Math. Anal. Appl.* **212** (1997), no. 1, 98–106.

[9] M. Nunokawa and D. K. Thomas, On convex and starlike functions in a sector, *J. Austral. Math. Soc. Ser. A* **60** (1996), no. 3, 363–368.