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MAPS PRESERVING STRONG SKEW LIE PRODUCT ON FACTOR VON NEUMANN ALGEBRAS*

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Abstract Let $\mathcal{A}$ be a factor von Neumann algebra and Φ be a nonlinear surjective map from $\mathcal{A}$ onto itself. We prove that, if Φ satisfies that $\Phi(A)\Phi(B)-\Phi(B)\Phi(A)^*=AB-BA^*$ for all $A,B\in\mathcal{A}$, then there exist a linear bijective map $\Psi:\mathcal{A}\to\mathcal{A}$ satisfying $\Psi(A)\Psi(B)-\Psi(B)\Psi(A)^*=AB-BA^*$ for $A,B\in\mathcal{A}$ and a real functional h on $\mathcal{A}$ with $h(0)=0$ such that $\Phi(A)=\Psi(A)+h(A)I$ for every $A\in\mathcal{A}$. In particular, if $\mathcal{A}$ is a type I factor, then, $\Phi(A)=cA+h(A)I$ for every $A\in\mathcal{A}$, where $c=\pm1$.

Key words Skew Lie product; factor von Neumann algebras; preserver problems

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1 Introduction

For a Hilbert space H, $\mathcal{B}(H)$ stands for the Banach algebra of all bounded linear operators on H. The first result concerning the relation between the subspaces of $\mathcal{B}(H)$ which are ideals with respect to different types of (possibly nonassociate) ring operations can be found in [4]. It was proved there that, if H is a complex infinite dimensional separable Hilbert space, then considering respectively the Lie and Jordan products on $\mathcal{B}(H)$

$$[T,S]=TS-ST,\qquad T\circ S=\frac{1}{2}(TS+ST),$$

every Lie ideal can be "approximated" by an associative ideal and every Jordan ideal is an associative ideal [4, Theorem 2 and 3]. An associative ideal means a two-sided ideal under the usual multiplication of operators.

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The Lie products $[T,S]$ are in a close connection with the derivations on $\mathcal{B}(H)$ (see, for example, [10]). Another derivationlike map also attains more and more importance. Let $\mathcal{A}$ be a $*$-ring. The additive map $\delta:\mathcal{A}\to\mathcal{A}$ is called a Jordan $*$-derivation if $\delta(A^2)=A\delta(A)+\delta(A)A^*$ for all $A\in\mathcal{A}$. These maps are extensively studied (see, for example, [2, 5–7, 9]) because, by the fundamental theorem of Šemrl in [8], their structure is intimately related to the problem of representability of quadratic functionals via sesquilinear forms (see [7]). Concerning operator algebras, it was also Šemrl [7] who proved that, for a real or complex Hilbert space H, every Jordan $*$-derivation $\delta:\mathcal{B}(H)\to\mathcal{B}(H)$ is of the form $\delta(T)=TA-AT^*$ $(\forall T\in\mathcal{B}(H))$ with $A\in\mathcal{B}(H)$ (see [9]). Motivated by the work of Šemrl and [4], Molnár [6] studied the relation between subspaces of $\mathcal{B}(H)$ which are ideals with respect to this product $TA-AT^*$. Where he showed that, if H is a real or complex Hilbert space of dimension greater than 1, then, a subspace $\mathcal{N}$ of $\mathcal{B}(H)$ is an ideal if and only if $AB-BA^*\in\mathcal{N}$ for $A\in\mathcal{B}(H)$ and $B\in\mathcal{N}$; and also, if the dimension of H is an odd natural number, then $\mathcal{N}=\mathcal{B}(H)$. In addition, it was also proved in [6] that, if $\mathcal{N}\subseteq\mathcal{B}(H)$ is an ideal, then, $\operatorname{span}\{AB-BA^*\mid A\in\mathcal{N},B\in\mathcal{B}(H)\}=\operatorname{span}\{AB-BA^*\mid A\in\mathcal{B}(H),B\in\mathcal{N}\}=\mathcal{N}$. In particular, every element of $\mathcal{B}(H)$ is a finite sum of $TS-ST^*$ type operators. In [1], Brešar and Fonšner generalized Molnár's results to rings with involution in different ways, and studied the relationship between (ordinary) ideals of a $*$-ring R and left and right ideals of R with respect to the product $AB-BA^*$. Their approach is entirely algebraic and is completely different from that used by Molnár, and it is based on discovering certain identities that connect the product $AB-BA^*$ with the initial associative product.

For A, B in a $*$-ring $\mathcal{A}$, denote by $[A,B]_*=AB-BA^*$ the skew Lie product of A and B. A map $\phi:\mathcal{A}\to\mathcal{A}$ is called a strong skew Lie product preserver if $[\phi(A),\phi(B)]_*=[A,B]_*$ for all A, $B\in\mathcal{A}$. In this article, we will characterize strong skew Lie product preserving nonlinear maps on general factor von Neumann algebras. Our main result is as follows.

Theorem 1 Let $\mathcal{A}$ be a factor von Neumann algebra and $\Phi:\mathcal{A}\to\mathcal{A}$ be a nonlinear surjective map. Assume that Φ preserves strong skew Lie product. Then, there exist a functional $h:\mathcal{A}\to\mathbb{R}$ with $h(0)=0$ and a strong skew Lie product preserving bijective linear map $\Psi:\mathcal{A}\to\mathcal{A}$, such that $\Phi(A)=\Psi(A)+h(A)I$ for every $A\in\mathcal{A}$.

Recently, in [3], we characterized the bijective linear maps preserving zero skew Lie product on $\mathcal{B}(H)$, where H is a complex Hilbert space, that is, the map ϕ satisfies that $\phi(A)\phi(B)=\phi(B)\phi(A)^*$ whenever $AB=BA^*$ for A, $B\in\mathcal{B}(H)$. Thus, as an application of Theorem 1, we can obtain the following result.

Corollary 2 Let H be a complex Hilbert space and $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ be a nonlinear surjective map. Assume that Φ preserves strong skew Lie product. Then, there exists a functional $h:\mathcal{B}(H)\to\mathbb{R}$ with $h(0)=0$, such that $\Phi(A)=cA+h(A)I$ for every $A\in\mathcal{B}(H)$, where $c=\pm1$.

2 The Proofs of the Results

Recall that an algebra $\mathcal{R}$ is called prime if $A\mathcal{R}B=\{0\}$ for $A,B\in\mathcal{R}$ implies that $A=0$ or $B=0$. Clearly, every factor von Neumann algebra is prime. In this section, we assume always that $\mathcal{A}$ is a factor von Neumann algebra. As usual, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the real field and complex field. To prove our results, we need to prove several lemmas.

Lemma 1 Let $A \in \mathcal{A}$, and let $P \in \mathcal{A}$ be a nontrivial projection. Then, for every $T \in \mathcal{A}$, $[P, [P, [A, T]_*]_*]_* = [A, T]_*$ if and only if there exist constants γ, $\beta \in \mathbb{R}$ such that $A = \gamma P + \beta I$.

Proof Clearly, we need only to prove the necessity. Assume that $[P, [P, [A, T]_*]_*]_* = [A, T]_*$ for every $T \in \mathcal{A}$. Then, a direct computation implies that

$$P[A, T]_* P = 0 \quad \text{and} \quad (I - P)[A, T]_*(I - P) = 0. \tag{1}$$

Replacing T by $PT(I - P)$ in the above expression, it follows that, for every $T \in \mathcal{A}$, $PT(I - P)A^*P = 0$ and $(I - P)APT(I - P) = 0$. That is,

$$P\mathcal{A}(I - P)A^*P = \{0\} \quad \text{and} \quad (I - P)AP\mathcal{A}(I - P) = \{0\}.$$

Note that $\mathcal{A}$ is prime. We have

$$PA = PAP = AP \quad \text{and} \quad (I - P)A = (I - P)A(I - P) = A(I - P). \tag{2}$$

It follows from (1) and (2) that, for every $T \in \mathcal{A}$,

$$PAPTP = PTPA^*P \quad \text{and} \quad (I - P)A(I - P)T(I - P) = (I - P)T(I - P)A^*(I - P).$$

Taking respectively $T = P$ and $I - P$ in the above expression, then both PAP and $(I - P)A(I - P)$ are self-adjoint, and consequently, the above expression implies again that PAP and $(I - P)A(I - P)$ belong, respectively, to the center of $P\mathcal{A}P$ and $(I - P)\mathcal{A}(I - P)$, thus, there exist α, $\beta \in \mathbb{R}$, such that

$$PAP = \alpha P \quad \text{and} \quad (I - P)A(I - P) = \beta(I - P).$$

This, together with (2), ensures that

$$\begin{aligned} A &= PAP + (I - P)A(I - P) + PA(I - P) + (I - P)AP \\ &= \alpha P + \beta(I - P) = (\alpha - \beta)P + \beta I \\ &= \gamma P + \beta I \end{aligned}$$

with $\gamma = \alpha - \beta \in \mathbb{R}$.

In the sequel, we assume always that Φ satisfies the assumptions in Theorem 1.

Lemma 2 $\Phi(\mathbb{R}I) = \mathbb{R}I$ and $\Phi(0) = 0$.

Proof For any $A \in \mathcal{A}$ and any $\alpha \in \mathbb{R}$, we have $\Phi(\alpha I)\Phi(A) = \Phi(A)\Phi(\alpha I)^*$. As Φ is surjective,

$$\Phi(\alpha I)X = X\Phi(\alpha I)^* \quad \text{for every } X \in \mathcal{A}. \tag{3}$$

Take $X = I$ in (3), then $\Phi(\alpha I)$ is self-adjoint, and consequently, (3) implies again $\Phi(\alpha I) \in \mathbb{R}I$. Conversely, assume that $\Phi(A) \in \mathbb{R}I$, then, for every $B \in \mathcal{A}$, we have $AB - BA^* = [\Phi(A), \Phi(B)]_* = 0$, hence, $A \in \mathbb{R}I$.

Next, we prove that $\Phi(0) = 0$. Otherwise, assume that $\Phi(0) = bI$ for some nonzero real number b. Then, for every $T \in \mathcal{A}$, we have $\Phi(T)\Phi(0) = \Phi(0)\Phi(T)^*$, so $\Phi(T)$ is self-adjoint. This implies that every element in the range of Φ is self-adjoint, which contradicts to the surjectivity of Φ.

Lemma 3 Let $P \in \mathcal{A}$ be a nontrivial projection. Then, there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$, such that $\Phi(P) = \alpha P + \beta I$.

Proof For every $T \in \mathcal{A}$, we have $[P, [P, [P, T]_*]_*]_* = [P, T]_*$. So,

$$[P, [P, [\Phi(P), \Phi(T)]_*]_*]_* = [\Phi(P), \Phi(T)]_*.$$

As Φ is surjective, it follows from Lemma 1 that there exist $\alpha, \beta \in \mathbb{R}$ such that $\Phi(P) = \alpha P + \beta I$. Now, Lemma 2, together with P being nontrivial, ensures that $\alpha \neq 0$.

Lemma 4 Let $P \in \mathcal{A}$ be a nontrivial projection. Then, there exists a nonzero $a \in \mathbb{R}$ such that, for any $T \in \mathcal{A}$,

$$P\Phi(T)(I - P) = aPT(I - P) \quad \text{and} \quad (I - P)\Phi(T)P = a(I - P)TP.$$

Proof By Lemma 3, there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ such that $\Phi(P) = \alpha P + \beta I$. Thus, for every $T \in \mathcal{A}$, we have

$$PT - TP = \Phi(P)\Phi(T) - \Phi(T)\Phi(P) = \alpha(P\Phi(T) - \Phi(T)P).$$

In the above expression, multiplying both the left side and right side by $I - P$, we get

$$P\Phi(T)(I - P) = aPT(I - P) \quad \text{and} \quad (I - P)\Phi(T)P = a(I - P)TP$$

with $a = \frac{1}{\alpha}$. This completes the proof of Lemma 4.

Now, we are in a position to prove our main result.

Proof of Theorem 1 Fix an arbitrary nontrivial projection $P \in \mathcal{A}$. Let

$$\begin{aligned} \mathcal{A}_{11} &= P\mathcal{A}P, & \mathcal{A}_{12} &= P\mathcal{A}(I - P), \\ \mathcal{A}_{21} &= (I - P)\mathcal{A}P, & \mathcal{A}_{22} &= (I - P)\mathcal{A}(I - P). \end{aligned}$$

Then, $\mathcal{A} = \sum\limits_{i,j=1}^{2} \mathcal{A}_{ij}$.

Claim 1 There exists a nonzero $a \in \mathbb{R}$ such that, for every $A \in \mathcal{A}_{ij}$ $(i \neq j)$, $\Phi(A) = aA$.

For any $T, S \in \mathcal{A}$, as $TS - ST^* = \Phi(T)\Phi(S) - \Phi(S)\Phi(T)^*$, it follows that

$$\begin{aligned} (I - P)(TS - ST^*)P &= (I - P)(\Phi(T)\Phi(S) - \Phi(S)\Phi(T)^*)P \\ &= (I - P)\Phi(T)P\Phi(S)P + (I - P)\Phi(T)(I - P)\Phi(S)P \\ &\quad -(I - P)\Phi(S)P\Phi(T)^*P - (I - P)\Phi(S)(I - P)\Phi(T)^*P. \end{aligned}$$

By applying Lemma 4, there exists a nonzero $a \in \mathbb{R}$, such that

$$\begin{aligned} (I - P)(TS - ST^*)P = a((I - P)&TP\Phi(S)P + (I - P)\Phi(T)(I - P)SP \\ &-(I - P)SP\Phi(T)^*P - (I - P)\Phi(S)(I - P)T^*P). \end{aligned}$$

Let $A \in \mathcal{A}_{12}$ and replace S by A in the above expression. Then, for every $T \in \mathcal{A}$,

$$(I - P)TP\Phi(A)P = (I - P)\Phi(A)(I - P)T^*P. \tag{4}$$

Let $V \in \mathcal{A}$ be arbitrary. Take $T = (I-P)VP$ in (4), then (4) ensures that $(I-P)VP\Phi(A)P = 0$, and consequently,

$$P\Phi(A)P = 0 \tag{5}$$

as $\mathcal{A}$ is prime. Thus, (4) implies that $(I-P)\Phi(A)(I-P)T^*P=0$ for every $T\in\mathcal{A}$. And hence, that $\mathcal{A}$ is prime implies again that

$$(I-P)\Phi(A)(I-P)=0. \tag{6}$$

For every $A\in\mathcal{A}_{12}$, as

$$P\Phi(A)(I-P)=aA \quad \text{and} \quad (I-P)\Phi(A)P=0, \tag{7}$$

it follows from (5)–(7) that, for every $A\in\mathcal{A}_{12}$,

$$\begin{aligned}\Phi(A)&=P\Phi(A)P+(I-P)\Phi(A)(I-P)+P\Phi(A)(I-P)+(I-P)\Phi(A)P\\&=aA.\end{aligned}$$

A similar discussion implies that $\Phi(A)=aA$ for every $A\in\mathcal{A}_{21}$.

Claim 2 For every $A\in\mathcal{A}_{ii}$ $(i=1,2)$, $\Phi(A)\in\mathcal{A}_{ii}$.

Let T, $S\in\mathcal{A}$ be arbitrary. Then,

$$TS-ST^*=\Phi(T)\Phi(S)-\Phi(S)\Phi(T)^*.$$

Multiplying both sides of the above expression by $I-P$, and applying Lemma 4, one gets that there exists a nonzero $a\in\mathbb{R}$ such that

$$\begin{aligned}&(I-P)(TS-ST^*)(I-P)\\&=a^2(I-P)TPS(I-P)+(I-P)\Phi(T)(I-P)\Phi(S)(I-P)\\&\quad -a^2(I-P)SPT^*(I-P)-(I-P)\Phi(S)(I-P)\Phi(T)^*(I-P).\end{aligned}$$

Let $A\in\mathcal{A}_{11}$ and replace S by A in the above expression, then, for any $T\in\mathcal{A}$,

$$(I-P)\Phi(T)(I-P)\Phi(A)(I-P)=(I-P)\Phi(A)(I-P)\Phi(T)^*(I-P). \tag{8}$$

As Φ is surjective, there exists $W\in\mathcal{A}$, such that $\Phi(W)=\mathrm{i}I$ (here i is the imaginary unit). Replacing T by W in (8), we have

$$(I-P)\Phi(A)(I-P)=0. \tag{9}$$

Let $A\in\mathcal{A}_{11}$ be arbitrary. Applying Lemma 3, there exists a nonzero $\alpha\in\mathbb{R}$, such that

$$\alpha(P\Phi(A)-\Phi(A)P)=\Phi(P)\Phi(A)-\Phi(A)\Phi(P)=PA-AP=0,$$

so,

$$P\Phi(A)(I-P)=(I-P)\Phi(A)P=0. \tag{10}$$

Hence, (9) and (10) imply that

$$\begin{aligned}\Phi(A)&=P\Phi(A)P+P\Phi(A)(I-P)+(I-P)\Phi(A)P+(I-P)\Phi(A)(I-P)\\&=P\Phi(A)P\in\mathcal{A}_{11}.\end{aligned}$$

Similarly, for every $A\in\mathcal{A}_{22}$, $\Phi(A)=(I-P)\Phi(A)(I-P)\in\mathcal{A}_{22}$.

Claim 3 For all A, $B \in \mathcal{A}$, $\Phi(A+B)-\Phi(A)-\Phi(B) \in \mathbb{R}I$.

Let A, $B \in \mathcal{A}$ be arbitrary. For any $T \in \mathcal{A}$, we have

$$\begin{aligned} & [\Phi(A+B)-\Phi(A)-\Phi(B), \Phi(T)]_* \\ = & [\Phi(A+B), \Phi(T)]_* - [\Phi(A), \Phi(T)]_* - [\Phi(B), \Phi(T)]_* \\ = & [A+B, T]_* - [A, T]_* - [B, T]_* = 0. \end{aligned}$$

The above expression, together with the surjectivity of Φ, implies that

$$(\Phi(A+B)-\Phi(A)-\Phi(B))X = X(\Phi(A+B)-\Phi(A)-\Phi(B))^*, \quad \forall X \in \mathcal{A}.$$

So, $\Phi(A+B)-\Phi(A)-\Phi(B)$ is self-adjoint, and therefore, the above expression implies again that $\Phi(A+B)-\Phi(A)-\Phi(B) \in \mathbb{R}I$. The proof of Claim 3 is completed.

Thus, for every $A \in \mathcal{A}$, we have

$$\Phi(A)-\Phi(PAP)-\Phi(PA(I-P))-\Phi((I-P)AP)-\Phi((I-P)A(I-P)) \in \mathbb{R}I.$$

Define a functional $h : \mathcal{A} \to \mathbb{R}$ by

$$h(A)I = \Phi(A)-\Phi(PAP)-\Phi(PA(I-P))-\Phi((I-P)AP)-\Phi((I-P)A(I-P)).$$

It follows from $\Phi(0)=0$ that $h(0)=0$. Let $\Psi(A)=\Phi(A)-h(A)I$ for every $A \in \mathcal{A}$. Then, $\Psi : \mathcal{A} \to \mathcal{A}$ is a map satisfying, for every $A \in \mathcal{A}$,

$$\Psi(A) = \Phi(PAP)+\Phi(PA(I-P))+\Phi((I-P)AP)+\Phi((I-P)A(I-P)). \tag{11}$$

Claim 4 Ψ is a bijective linear map satisfying $[\Psi(A), \Psi(B)]_* = [A, B]_*$ for all A, $B \in \mathcal{A}$.

We prove first that Ψ is linear. Write $P_1 = P$ and $P_2 = I-P$. For every $A_{ij} \in \mathcal{A}_{ij}$ $(i, j = 1, 2)$, (11) and $\Phi(0)=0$ imply that

$$\Psi(A_{ij}) = \Phi(A_{ij}). \tag{12}$$

A direct computation implies that $\Phi|_{\mathcal{A}_{ij}}$ is linear. In fact, let A_{ij}, $B_{ij} \in \mathcal{A}_{ij}$ $(i \neq j)$ and $\theta \in \mathbb{C}$ be arbitrary. By Lemma 3, there exist α_i, $\beta_i \in \mathbb{R}$ with $\alpha_i \neq 0$, such that $\Phi(P_i) = \alpha_i P_i + \beta_i I$, so,

$$\begin{aligned} \alpha_i [P_i, \Phi(\theta A_{ij} + B_{ij})]_* &= [\Phi(P_i), \Phi(\theta A_{ij} + B_{ij})]_* = [P_i, \theta A_{ij} + B_{ij}]_* \\ &= \theta [P_i, A_{ij}]_* + [P_i, B_{ij}]_* \\ &= \theta [\Phi(P_i), \Phi(A_{ij})]_* + [\Phi(P_i), \Phi(B_{ij})]_* \\ &= \theta \alpha_i [P_i, \Phi(A_{ij})]_* + \alpha_i [P_i, \Phi(B_{ij})]_* \\ &= \alpha_i [P_i, \theta \Phi(A_{ij}) + \Phi(B_{ij})]_*. \end{aligned}$$

Note that $\Phi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$. Hence,

$$\Phi(\theta A_{ij} + B_{ij}) = \theta \Phi(A_{ij}) + \Phi(B_{ij}). \tag{13}$$

Let A_{ii}, $B_{ii} \in \mathcal{A}_{ii}$ and $\theta \in \mathbb{C}$ be arbitrary. By Claim 1, there exists a nonzero $a \in \mathbb{R}$ such that, for every $X_{ji} \in \mathcal{A}_{ji}$ $(j \neq i)$, $\Phi(X_{ji}) = aX_{ji}$. A similar discussion just as (13) implies that

$$[X_{ji}, \Phi(\theta A_{ii} + B_{ii}) - \theta \Phi(A_{ii}) - \Phi(B_{ii})]_* = 0.$$

This, together with $\Phi(\mathcal{A}_{ii}) \subseteq \mathcal{A}_{ii}$, infers that

$$P_j\mathcal{A}[\Phi(\theta A_{ii} + B_{ii}) - \theta\Phi(A_{ii}) - \Phi(B_{ii})] = \{0\},$$

so,

$$\Phi(\theta A_{ii} + B_{ii}) = \theta\Phi(A_{ii}) + \Phi(B_{ii}). \tag{14}$$

Now, it follows from (11)–(14) that $\Psi(\theta A + B) = \theta\Psi(A) + \Psi(B)$ for all A, $B \in \mathcal{A}$ and any $\theta \in \mathbb{C}$, that is, Ψ is linear.

Next, we prove that Ψ is bijective. The surjectivity of Ψ follows from the surjectivity of Φ. To prove that Ψ is injective, assume that $\Psi(A) = \Psi(B)$ for A, $B \in \mathcal{A}$. For every $T \in \mathcal{A}$, write $A = \sum\limits_{i=1}^{2} A_{ij}$, $B = \sum\limits_{i=1}^{2} B_{ij}$, and $T = \sum\limits_{i=1}^{2} T_{ij}$. Then, by (11) and (12),

$$\begin{aligned}
[T, A]_* = \sum_{i,j,k,l=1}^{2} [T_{ij}, A_{kl}]_* &= \sum_{i,j,k,l=1}^{2} [\Phi(T_{ij}), \Phi(A_{kl})]_* \\
&= [\Psi(T), \Psi(A)]_* = [\Psi(T), \Psi(B)]_* \\
&= \sum_{i,j,k,l=1}^{2} [\Phi(T_{ij}), \Phi(B_{kl})]_* \\
&= \sum_{i,j,k,l=1}^{2} [T_{ij}, B_{kl}]_* = [T, B]_*,
\end{aligned}$$

that is, for every $T \in \mathcal{A}$,

$$T(A - B) = (A - B)T^*.$$

Take $T = iI$ in the above expression, then, $A = B$. So, Ψ is injective.

Lastly, we prove that Ψ satisfies $[\Psi(A), \Psi(B)]_* = [A, B]_*$ for A, $B \in \mathcal{A}$. For any $A, B \in \mathcal{A}$, we write $A = \sum\limits_{i=1}^{2} A_{ij}$ and $B = \sum\limits_{i=1}^{2} B_{ij}$, then, it follows from (11) and $\Psi(\mathcal{A}_{\rangle|}) \subseteq \mathcal{A}_{\rangle|}$ that

$$\begin{aligned}
[\Psi(A), \Psi(B)]_* &= \Big[\sum_{i=1}^{2} \Phi(A_{ij}), \sum_{i=1}^{2} \Phi(B_{ij})\Big]_* \\
&= \sum_{i,j,k,l=1}^{2} [\Phi(A_{ij}), \Phi(B_{kl})]_* \\
&= \sum_{i,j,k,l=1}^{2} [A_{ij}, B_{kl}]_* = [A, B]_*.
\end{aligned}$$

So, Claim 4 holds, and the proof is completed.

To prove Corollary 2, we need the following result, which was proved in [3].

Lemma 5 Let H and K be complex Hilbert spaces. Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a linear bijective map. Then, Φ preserves zero skew Lie product if and only if there exist a nonzero scalar $c \in \mathbb{R}$ and a unitary operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = cUAU^*$ for all $A \in \mathcal{B}(H)$.

Proof of Corollary 2 As $\mathcal{B}(H)$ is a factor of type I, Theorem 1 implies that there exist a linear bijective map $\Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfying

$$[\Psi(A), \Psi(B)]_* = [A, B]_*, \ \forall A, B \in \mathcal{B}(H), \tag{15}$$

and a function $h:\mathcal{B}(H)\to\mathbb{R}$ with $h(0)=0$, such that $\Phi(A)=\Psi(A)+h(A)I$ for every $A\in\mathcal{B}(H)$. (15) implies that $\Psi(A)\Psi(B)=\Psi(B)\Psi(A)^*$ if and only if $AB=BA^*$ for $A,B\in\mathcal{B}(H)$. By Lemma 5, there exist a nonzero real number c and a unitary operator $U\in\mathcal{B}(H)$, such that $\Psi(A)=cUAU^*$ for every $A\in\mathcal{B}(H)$. Take $A=iI$ in (15), then,

$$c^2UBU^*=B \quad \text{for every } B\in\mathcal{B}(H).$$

Picking $B=I$ in the above expression, one has $c=\pm1$. Therefore, the above expression implies again $UB=BU$ for every $B\in\mathcal{B}(H)$, and hence, $U=\lambda I$ with $|\lambda|=1$. So, $\Phi(A)=cA+h(A)I$ for every $A\in\mathcal{B}(H)$. The proof is completed.

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