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WHEN IS $C(X)$ A *po*-COHERENT RING?

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ABSTRACT

Let X be a completely regular Hausdorff topological space. It is shown that the ring of real valued continuous functions $C(X)$ is *po*-coherent (in the sense of F. Wehrung) if and only if X is basically disconnected.

INTRODUCTION

In his article [8] F. Wehrung defined *po*-coherent modules and rings. We shall only need the definition of a *po*-coherent commutative ring.

Let R be a partially ordered directed commutative ring in which $0 \leq 1$ with positive cone R^+. Then R is said to be po-*coherent* if for any $r_1, \ldots, r_n \in R$ the solution set of the mixed system

$$\begin{aligned} &r_1 x_1 + \cdots + r_n x_n \geq 0 \\ &x_1, \ldots, x_n \geq 0 \end{aligned} \tag{1}$$

(with unknowns $x_1, \ldots, x_n \in R$) is a finitely generated R^+-subsemimodule of $\mathfrak{M}_{n,1}(r)$, where $\mathfrak{M}_{n,1}(r)$ denotes the set of all $n \times 1$ matrices with entries in R.

That is, each solution $[x_1, \ldots, x_n]^\top \in \mathfrak{M}_{n,1}(r^+)$ can be written as a non-negative linear combination of finitely many (prechosen) columns from $\mathfrak{M}_{n,1}(r)$.

F. Wehrung showed that each *po*-coherent ring R is also coherent (in the classical sense) [8, Corollary 3.7]. That is, every finitely generated ideal of R is finitely presented. For an excellent treatment of commutative coherent rings see [3].

Let X be a completely regular Hausdorff topological space and $C(X)$ the ring of continuous real valued functions on X. There is a topological characterization of those spaces X for which the ring $C(X)$ is coherent. C. Neville proved [7, Theorem 2.2] that $C(X)$ is coherent if and only if X is basically disconnected (see also [6]). A space X is *basically disconnected* if the (closed) support of every continuous function is open. Remember that a boolean algebra is σ-complete if and only if its maximal ideal space is basically disconnected. Also some classical spaces from functional analysis, for example ℓ^∞, can be represented as a space of continuous functions on a basically disconnected space.

So a natural question arises. What is the topological characterization of those spaces X for which the ring $C(X)$ is *po*-coherent? Obviously X must be basically disconnected. We shall show that this condition is also sufficient. Thus we shall extend the topological characterization of those spaces X for which the ring $C(X)$ is coherent, *po*-coherent, semihereditary, projectable and Dedekind σ-complete (see Theorem 2).

PRELIMINARIES

Let X be a completely regular Hausdorff topological space and $C(X)$ the ring of continuous real valued functions on X with positive cone $C^+(X)$. We shall use standard notation, introduced in [2]: for a continuous function $f \in C(X)$

$$\mathbf{Z}(f) := \{x \in X;\ f(x) = 0\}, \quad \operatorname{coz} f := X \setminus \mathbf{Z}(f),$$
$$\operatorname{supp} f := \overline{\operatorname{coz} f}, \quad \operatorname{pos} f := \{x \in X;\ f(x) > 0\},$$
$$f^+ := f \vee 0, \quad f^- := (-f) \vee 0.$$

If every finitely generated ideal in $C(X)$ is principal, we call X an *F-space*. This happens exactly when every cozero-set in X is C^*-embedded (see [2, Theorem 14.25]). That is, any bounded real function, which is continuous on the cozero-set of some $f \in C(X)$, can be extended to a continuous function on the whole X.

Let X be an F-space, $f, g \in C(X)$ and $0 \le f \le g$. Define $s : \operatorname{coz} g \to \mathbb{R}$ by $s(x) := f(x)/g(x)$. Then s is bounded continuous on the cozero-set of g, so it has an extension to a function $t \in C(X)$. Since $f(x) = g(x) = 0$ for $x \in \mathbf{Z}(g)$, we have $f(x) = t(x)g(x)$ for every $x \in X$ and $f = tg$. We may replace t by $(t \wedge 1) \vee 0$. Thus we showed that for all $f, g \in C(X)$, such that $0 \le f \le g$, we can find $t \in C(X)$, $0 \le t \le 1$, such that $f = tg$.

Every basically disconnected space is an F-space ([2, Exercise 14.N]).

PREPARATIONS

In this section we shall suppose that X is a basically disconnected completely regular Hausdorff topological space.

Lemma 1. *Suppose that* $f_1, \dots, f_n \in C^+(X)$ *satisfy* $X = \operatorname{supp} f_i$ *for* $1 \le i \le n$. *Then the intersection* $\operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_n$ *is a dense subset of* X.

Proof. We shall use mathematical induction on the number of functions n. Obviously $\overline{\operatorname{pos} f_1} = \operatorname{supp} f_1 = X$.

Suppose that

$$\overline{\operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_{n-1}} = X.$$

We shall use the following characterization of open sets, found in [1, Problem III.4.4]: $G \subseteq X$ is open in X if and only if $\overline{G \cap \bar{A}} = \overline{G \cap A}$ for every $A \subseteq X$.

Remember that the set $\operatorname{pos} f_n$ is always open in X. Then

$$\begin{aligned}\overline{(\operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_{n-1}) \cap \operatorname{pos} f_n} &= \overline{\overline{\operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_{n-1}} \cap \operatorname{pos} f_n} \\ &= \overline{X \cap \operatorname{pos} f_n} = \operatorname{supp} f_n = X. \qquad \square\end{aligned}$$

Proposition 1. *Suppose that* $1 \le m < n$, $f_1, \dots, f_n \in C^+(X)$ *and* $X = \operatorname{supp} f_i$ *for* $1 \le i \le n$. *Then the solution set of the mixed system*

$$f_1 k_1 + \cdots + f_m k_m \ge f_{m+1} k_{m+1} + \cdots + f_n k_n \quad k_1, \dots, k_n \ge 0 \tag{2}$$

(with unknowns $k_1, \dots, k_n \in C(X)$*) is a finitely generated* $C^+(X)$*-subsemimodule of* $\mathfrak{M}_{n,1}(C(X))$.

Proof. Let $\mathcal{I}:=\{i \in \mathbb{N};\ 1 \le i \le m\}$ and $\mathcal{J}:=\{j \in \mathbb{N};\ m+1 \le j \le n\}$. Since X is an F-space and

$$0 \le f_i, \qquad f_j \le f_i \vee f_j \quad \text{for } i \in \mathcal{I}, j \in \mathcal{J},$$

we can find functions $s_{ij}, t_{ij} \in C(X)$, $0 \le s_{ij}, t_{ij} \le 1$, such that

$$f_j = s_{ij}(f_i \vee f_j) \quad \text{and} \quad f_i = t_{ij}(f_i \vee f_j) \quad \text{for } i \in \mathcal{I},\ j \in \mathcal{J}.$$

Note that $f_i s_{ij} = f_j t_{ij}$ for $i \in \mathcal{I}$ and $j \in \mathcal{J}$. For any functions $p_i, q_{ij} \in C^+(X)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, the functions

$$k_i = p_i + \sum_{j \in \mathcal{J}} q_{ij} s_{ij} \quad \text{for } i \in \mathcal{I} \quad k_j = \sum_{i \in \mathcal{I}} q_{ij} t_{ij} \quad \text{for } j \in \mathcal{J} \tag{3}$$

satisfy the mixed system (2).

Now let $k_1, \dots, k_n \in C(X)$ be the solutions of the mixed system (2). We shall show that they can be written in the form (3) for suitable $p_i, q_{ij} \in C^+(X)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$. We shall recursively define the functions q_{ij} and then show that

$$k_i - \sum_{j \in \mathcal{J}} q_{ij} s_{ij} \ge 0 \quad \text{for } i \in \mathcal{I} \quad \text{and} \tag{P_i}$$

$$k_j = \sum_{i \in \mathcal{I}} q_{ij} t_{ij} \quad \text{for } j \in \mathcal{J}. \tag{E_j}$$

It is easy to see that

$$0 \le (f_i \vee f_j)(k_i f_i \wedge k_j f_j) \le f_i f_j (k_i \vee k_j) \quad \text{for } i \in \mathcal{I},\ j \in \mathcal{J}.$$

So we can find $u_{1,m+1} \in C(X)$, $0 \le u_{1,m+1} \le 1$, such that

$$(f_1 \vee f_{m+1})(k_1 f_1 \wedge k_{m+1} f_{m+1}) = u_{1,m+1} f_1 f_{m+1}(k_1 \vee k_{m+1})$$

and define

$$q_{1,m+1} := (k_1 \vee k_{m+1}) u_{1,m+1}. \tag{5}$$

To get q_{ij}, where $i \in \mathcal{I} \backslash \{1\}$ or $j \in \mathcal{J} \backslash \{m+1\}$, we shall have to define all $q_{i\ell}$, $\ell < j$, and all $q_{\ell j}$, $\ell < i$, first. Let

$$q_{ij} := (k_i \vee k_j)u_{ij}, \tag{6}$$

where $u_{ij} \in C(X)$, $0 \leq u_{ij} \leq 1$, satisfies the equation

$$\begin{aligned}(f_i \vee f_j)\left(\left(f_i\left(k_i - \sum_{\ell=m+1}^{j-1} q_{i\ell}s_{i\ell}\right)\right) \wedge \left(f_j\left(k_j - \sum_{\ell=1}^{i-1} q_{\ell j}t_{\ell j}\right)\right)\right)\\ = u_{ij}f_if_j(k_i \vee k_j).\end{aligned} \tag{7}$$

An empty sum is defined to be 0. Note that both differences above are nonnegative by the definition of $q_{i-1,j}$ and $q_{i,j-1}$. Then the existence of u_{ij} follows from the inequality (4).

By Lemma 1 the set $\operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_n$ is dense in X. Thus it is enough to prove that the conditions (P_i) and (E_j) hold pointwise on $\operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_n$.

Each of the conditions (P_i), $i \in \mathcal{I}$, is satisfied by the definition of q_{in}, $i \in \mathcal{I}$. We shall develop a schematic description of the definition of q_{ij} to see that also the conditions (E_j), $j \in \mathcal{J}$, hold.

For each $x \in \operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_n$ we shall recursively define a $m \times (n-m)$ matrix $Q(x) = [q_{ij}(x)]_{i \in \mathcal{I}, j \in \mathcal{J}}$. First we get the upper left element $q_{1,m+1}(x)$ by (5). If $f_1(x)k_1(x) \leq f_{m+1}(x)k_{m+1}(x)$, we have

$$\begin{aligned}f_1(x)(k_1 - q_{1,m+1}s_{1,m+1})(x) &= f_1(x)\left(k_1 - \left(\frac{f_1 \vee f_{m+1}}{f_1 f_{m+1}} f_1 k_1\right)\frac{f_{m+1}}{f_1 \vee f_{m+1}}\right)(x)\\ &= 0\end{aligned}$$

and $q_{1,m+2}(x) = 0$. Then $q_{1,j}(x) = 0$ for all $j > m+1$. So all the elements in the first row but the first one are zero. Then we can move downwards to define $q_{2,m+1}(x)$.

Likewise, if $f_1(x)k_1(x) \geq f_{m+1}(x)k_{m+1}(x)$, the remaining elements in the first column are zero. Now we can move rightwards to define $q_{1,m+2}(x)$. Note that both cases can happen simultaneously. Then we move two steps forward at a time instead of one.

To define $q_{ij}(x)$, we must compare $\lambda_{ij}(x)$ and $\rho_{ij}(x)$, where

$$\begin{aligned}\lambda_{ij}(x) &:= f_i(x)\left(k_i(x) - \sum_{\ell=m+1}^{j-1} q_{i\ell}(x)s_{i\ell}(x)\right),\\ \rho_{ij}(x) &:= f_j(x)\left(k_j(x) - \sum_{\ell=1}^{i-1} q_{\ell j}(x)t_{\ell j}(x)\right).\end{aligned}$$

As before: if $\lambda_{ij}(x) \le \rho_{ij}(x)$,

$$\begin{aligned} f_i(x)\left(k_i - \sum_{\ell=m+1}^{j} q_{i\ell} s_{i\ell}\right)(x) &= f_i(x)\left(k_i - \sum_{\ell=m+1}^{j-1} q_{i\ell} s_{i\ell}\right)(x) - (f_i q_{ij} s_{ij})(x) \\ &= \left(\frac{f_i f_j}{f_i \vee f_j} q_{ij}\right)(x) - (f_i q_{ij} s_{ij})(x) = 0 \end{aligned}$$

and all the remaining elements in the ith row are zero. Symmetrically, if $\lambda_{ij}(x) \ge \rho_{ij}(x)$, the remaining elements in the jth column are zero. In the first case we can move downwards and in the second case we can move rightwards.

This movement through the matrix $Q(x)$ will be called *a non-zero path* through the matrix $Q(x)$. Figure 1 shows an example for the case $m = 2$, $n = 5$.

The conditions (E_j), $j \in \mathcal{J}$, are equivalent to the inequalities

$$\rho_{mj}(x) \le \lambda_{mj}(x) \quad \text{for } j \in \mathcal{J}.$$

This inequalities can be reformulated in the language of our schematic definition of the matrix $Q(x)$:

Claim 1. If we get to the bottom of the matrix $Q(x)$ (i.e., after the definition of some element from the mth row), we shall only move rightwards.

To prove this, we shall need the following claim.

Claim 2. Let $x \in \operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_n$, $i \in \mathcal{I}$ and $j \in \mathcal{J}$. If $q_{ij}(x)$ lies on the nonzero path through the matrix $Q(x)$,

$$\begin{aligned} \lambda_{ij}(x) - \rho_{ij}(x) = {} & (f_1(x)k_1(x) + \cdots + f_i(x)k_i(x)) \\ & - (f_{m+1}(x)k_{m+1}(x) + \cdots + f_j(x)k_j(x)). \end{aligned}$$

	3	4	5
1	↓	0	0
2	→	→	→

	3	4	5
1	→	↓	0
2	0	→	→

	3	4	5
1	→	→	↓
2	0	0	→

	3	4	5
1	→	→	→
2	0	0	0

Figure 1. The four cases that can appear in solving the inequality $f_1k_1 + f_2k_2 \ge f_3k_3 + f_4k_4 + f_5k_5$ at certain $x \in \operatorname{pos} f_1 \cap \cdots \cap \operatorname{pos} f_5$. For example, in the second case: $(f_1k_1)(x) \ge (f_3k_3)(x)$, $(f_1k_1)(x) \le (f_3k_3 + f_4k_4)(x)$, $(f_1k_1 + f_2k_2)(x) \ge (f_3k_3 + f_4k_4)(x)$, $(f_1k_1 + f_2k_2)(x) \ge (f_3k_3 + f_4k_4 + f_5k_5)(x)$.

Proof. The claim is obviously true for $i = 1$ and $j = m + 1$. We shall use mathematical induction on the length of the nonzero path through the matrix $Q(x)$.

Suppose that $q_{ij}(x)$, where $i > 1$ or $j > m + 1$, lies on the non-zero path through $Q(x)$. One of the following two cases must have happened:

First case: We got to the element $q_{ij}(x)$ from the left. That is, $\lambda_{i,j-1}(x) \geq \rho_{i,j-1}(x)$. In this case we moved rightwards and

$$q_{i+1,j-1}(x) = \cdots = q_{m,j-1}(x) = 0.$$

Also, while moving downwards, we filled the upper rows with zeros and

$$q_{1,j}(x) = \cdots = q_{i-1,j}(x) = 0.$$

Now

$$\begin{aligned}
\lambda_{ij}(x) - \rho_{ij}(x) &= f_i(x)k_i(x) - \sum_{\ell=m+1}^{j-1} f_i(x)q_{i\ell}(x)s_{i\ell}(x) - f_j(x)k_j(x) \\
&= f_i(x)k_i(x) - \sum_{\ell=m+1}^{j-2} f_i(x)q_{i\ell}(x)s_{i\ell}(x) \\
&\quad - f_i(x)q_{i,j-1}(x)s_{i,j-1}(x) - f_j(x)k_j(x) \\
&= f_i(x)k_i(x) - \sum_{\ell=m+1}^{j-2} f_i(x)q_{i\ell}(x)s_{i\ell}(x) - f_{j-1}(x)k_{j-1}(x) \\
&\quad + \sum_{\ell=1}^{i-1} f_{j-1}(x)q_{\ell,j-1}(x)t_{\ell,j-1}(x) - f_j(x)k_j(x) \\
&= \lambda_{i,j-1}(x) - \rho_{i,j-1}(x) - f_j(x)k_j(x)
\end{aligned}$$

Note that we used $x \in \operatorname{pos} f_i \cap \operatorname{pos} f_{j-1}$ in the evaluation of $f_i(x)q_{i,j-1}(x)$ $s_{i,j-1}(x)$ in the second line of the above calculation.

Second case: We could also get to the element $q_{ij}(x)$ from the top. That is, $\lambda_{i-1,j}(x) \leq \rho_{i-1,j}(x)$ and

$$q_{i-1,j+1}(x) = \cdots = q_{i-1,n}(x) = 0 \quad \text{and} \\ q_{i,m+1}(x) = \cdots = q_{i,j-1}(x) = 0.$$

Similar calculation as before gives us

$$\lambda_{ij}(x) - \rho_{ij}(x) = \lambda_{i-1,j}(x) - \rho_{i-1,j}(x) + f_i(x)k_i(x).$$

In both cases the path from $q_{1,m+1}(x)$ to either $q_{i,j-1}(x)$ or $q_{i-1,j}(x)$ is one step shorter than the path from $q_{1,m+1}(x)$ to $q_{ij}(x)$. Thus we can apply the induction hypothesis and Claim 2 is proved. □

Now it is easy to prove Claim 1. The functions $k_1, \dots, k_n$ solve the mixed system (2) thus

$$f_1k_1 + \cdots + f_mk_m \geq f_{m+1}k_{m+1} + \cdots + f_nk_n \geq f_{m+1}k_{m+1} + \cdots + f_jk_j$$

for any $j \in \mathcal{J}$. This is by Claim 2 the same as saying that we shall always move rightwards if we get to the bottom of the matrix $Q(x)$.

Proposition 1 is proved. The solution $[k_1, \dots, k_n]^\top \in \mathfrak{M}_{n,1}(C(X))$ of the mixed system (2) is (nonnegatively) generated by the following set

$$\{[\delta_{1i}, \dots, \delta_{mi}, 0, \dots, 0]^\top; i \in \mathcal{I}\} \\ \cup \{[s_{ij}\delta_{1i}, \dots, s_{ij}\delta_{mi}, t_{ij}\delta_{m+1,j}, \dots, t_{ij}\delta_{nj}]^\top;\ i \in \mathcal{I}, j \in \mathcal{J}\}$$

with at most $m(n - m + 1)$ elements. Here δ denotes Kronecker's delta. □

THE MAIN RESULT

Now we can prove the analogue of Grillet, Effros, Handelman and Shen theorem (see [4, Proposition 3.15] or [8, Proposition 6.2]) for the case $C(X)$.

Theorem 1. *Let X be a completely regular Hausdorff topological space. Then $C(X)$ is* po-*coherent if and only if X is basically disconnected.*

Proof. (⇒): F. Wehrung proved [8, Corollary 3.7] that every *po*-coherent ring is coherent. C. Neville [7, Theorem 2.2] on the other hand proved that $C(X)$ is coherent if and only if X is basically disconnected.

(⇐): Suppose that X is basically disconnected. We have to prove that for any $f_1, \dots, f_n \in C(X)$ the solution set of the mixed system

$$f_1k_1 + \cdots + f_nk_n \geq 0 \quad k_1, \dots, k_n \geq 0 \tag{8}$$

(with unknowns $k_1, \dots, k_n \in C(X)$) is a finitely generated $C^+(X)$-subsemimodule of $\mathfrak{M}_{n,1}(C(X))$.

Let $\mathcal{N} := \{1, \dots, n\}$. For all $i \in \mathcal{N}$ define

$$X_i^+ := \operatorname{supp} f_i^+, \quad X_i^- := \operatorname{supp} f_i^-, \quad X_i^0 := X \backslash \operatorname{supp} f_i.$$

Each of the above sets is clopen since X is basically disconnected.

For any subsets $\mathcal{I} \subseteq \mathcal{N}$, $\mathcal{J} \subseteq \mathcal{N} \backslash \mathcal{I}$, $\mathcal{L} := \mathcal{N} \backslash (\mathcal{I} \cup \mathcal{J})$ let

$$X_{\mathcal{I},\mathcal{J}} := \bigcap_{i \in \mathcal{I}} X_i^+ \cap \bigcap_{j \in \mathcal{J}} X_j^- \cap \bigcap_{\ell \in \mathcal{L}} X_\ell^0,$$

where an empty intersection is defined to be X. Thus we decomposed X into 3^n (not necessarily disjoint) pieces,

$$X = \bigcup_{\mathcal{I},\mathcal{J}} X_{\mathcal{I},\mathcal{J}}.$$

Note that each piece is clopen.

We shall solve the mixed system (8) on each piece $X_{\mathcal{I},\mathcal{J}}$ separately and then glue all the solutions together to get the solution of (8) on the whole space X.

First suppose that $X = X_{\mathcal{I},\emptyset}$. We are looking for the solution of the inequality

$$f_1 k_1 + \cdots + f_n k_n \geq 0$$

where $f_1, \dots, f_n \geq 0$ holds on X. Obviously any choice of nonnegative $k_1, \dots, k_n$ will do. We can say that in this case the solution of (8) is a n-generated $C^+(X)$-subsemimodule.

In the case $X = X_{\emptyset,\mathcal{J}}$ we may renumber the indices so that $\mathcal{J} = \{1, \dots, j\}$. We have to solve the inequality

$$f_1 k_1 + \cdots + f_j k_j \geq 0$$

where $f_1, \dots, f_j < 0$ holds on a dense subset of X. Then $k_1 = \cdots = k_j = 0$, and if $j < n$, $k_{j+1}, \dots, k_n$ can be arbitrary nonnegative functions. The solution is a $(n-j)$-generated $C^+(X)$-subsemimodule.

Now suppose that $X = X_{\mathcal{I},\mathcal{J}}$ where $\mathcal{I} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$. We may renumber the indices so that $\mathcal{I} = \{1, \dots, m\}$ and $\mathcal{J} = \{m+1, \dots, j\}$ where $m < j \leq n$. We have to solve the inequality

$$f_1 k_1 + \cdots + f_m k_m \geq f_{m+1} k_{m+1} + \cdots + f_j k_j \tag{9}$$

where $f_1, \dots, f_j > 0$ holds on a dense subset of $\operatorname{supp} f_1 \cap \cdots \cap \operatorname{supp} f_j$, and if $j < n$, $f_{j+1} = \cdots = f_n = 0$. If $j < n$, $k_{j+1}, \dots, k_n$ can be arbitrary nonnegative functions. Proposition 1 gives us $k_1, \dots, k_j$. In this case the solution of (8) is a $(n-j) + m(j-m+1)$-generated $C^+(X)$-subsemimodule.

Let $[k_1^{\mathcal{I},\mathcal{J}}, \dots, k_n^{\mathcal{I},\mathcal{J}}]^\top \in \mathfrak{M}_{n,1}(C^+(X_{\mathcal{I},\mathcal{J}}))$ be the solution of the mixed system (8) on $X_{\mathcal{I},\mathcal{J}}$. Since $X_{\mathcal{I},\mathcal{J}}$ is clopen, each of the functions $\hat{k}_i^{\mathcal{I},\mathcal{J}}$, where

$$\hat{k}_i^{\mathcal{I},\mathcal{J}}(x) := \begin{cases} k_i^{\mathcal{I},\mathcal{J}}(x) & \text{if } x \in X_{\mathcal{I},\mathcal{J}}, \\ 0 & \text{otherwise.} \end{cases},$$

is nonnegative continuous on X. Then

$$\sum_{\mathcal{I},\mathcal{J}} [\hat{k}_1^{\mathcal{I},\mathcal{J}}, \dots, \hat{k}_n^{\mathcal{I},\mathcal{J}}]^\top \tag{10}$$

is the solution of (8) on X. On the other hand if $[k_1, \dots, k_n]^\top$ is the solution of (8) on X, its restriction to $X_{\mathcal{I},\mathcal{J}}$ must agree with $[k_1^{\mathcal{I},\mathcal{J}}, \dots, k_n^{\mathcal{I},\mathcal{J}}]^\top$. So all the solutions can be written in the form (10).

Thus we proved that the solution of the mixed system (8) is a finitely generated $C^+(X)$-subsemimodule of $\mathfrak{M}_{n,1}(C(X))$. □

This extends Brookshear-DeMarco's and Neville's result:

Theorem 2. *Let X be a completely regular Hausdorff topological space. The following statements are equivalent:*

(1) X is basically disconnected.
(2) C(X) is semihereditary.
(3) C(X) is coherent.
(4) C(X) is projectable.
(5) C(X) is Dedekind σ-complete.
(6) C(X) is po-*coherent.*

Proof. For the equivalence of (1),(2) and (3) see [7]. The equivalence of (1), (2), (3), (4) and (5) was proved (in a more general context) by B. Lavrič in [5]. You can also find the equivalence of (1) and (5) in [4]. □

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