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On discretizations of cumulative distribution functions

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Stochastic processes in discrete time are considered which develop through the successive application of independent positive multipliers and also are martingales.

We construct optimal discretizations and derive properties of the Mellin-Stieltjes transforms of the cumulative distribution functions of the multipliers. Discretization means approximation by positive random variables with values in a given discrete set. It will be shown that the independence of the factors will be preserved in this procedure. The important case that discretization leads to multipliers with values in some fixed geometric progression allows one to write the Mellin-Stieltjes transforms as Laurent series.

The processes are then investigated by using the fact that the Mellin-Stieltjes transform of an independent product is the product of the transforms of its factors.

1 Introduction

We consider stochastic processes $(S_n)_{n=0,1,\dots}$ or $(S_n)_{n=0,1,\dots,n_0}$ which develop through successive application of independent positive multipliers and have constant expectations, so that the processes are martingales. If the process starts with $S_0 = 1$ then

$$S_n = X_1 \cdots X_n \qquad (n = 1, 2, \dots) \quad \text{with} \quad X_n = \frac{S_n}{S_{n-1}}\,.$$

The martingale property is expressed by $\mathbb{E}(X_n) = 1$. So our basic entity is the space $\mathcal{M}$ of the cumulative probability distributions

$$F(x) = \mathbb{P}(X \le x) \qquad (0 \le x < \infty)$$

of positive random variables X with $\mathbb{E}(X) = 1$.

Processes of this kind occur if the conditions that prevail cause the increase and decrease of quantities to be proportional to their present sizes, as is the case for financial assets. The prices S_n at the end of the n-th trading period are positive random variables on some probability space. Under the usual hypothesis that there is no arbitrage, the fundamental theorem of asset pricing asserts that there is a measure with respect to which $(S_n)_n$ is a martingale if the prices are discounted with a risk free interest rate, see for example [4, p. 232], [7, p. 114] or [9]. The well-known formula for the price $p(x)$ of a put option at strike price x,

$$p(x) = \int_0^x F(y)\,\mathrm{d}y\,, \tag{1.1}$$

shows that, in principle, F can be recovered from public market data. This example suggests that it is essential to allow for not identically distributed X_n.

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In Sections 2 and 3 we consider only a single random variable. In Section 2, we derive the basic properties of $\mathcal{M}$ and study discretizations $\hat{F}$ of the $F \in \mathcal{M}$ relative to a given increasing sequence $(x_k)_{k\in\mathbb{Z}}$, which we call canonical. They preserve the expectation and can be characterized by the property that they minimize

$$\|F - H\|^2 = \int_0^\infty \big(F(x) - H(x)\big)^2 \mathrm{d}x$$

among all cumulative probability distributions $H \in \mathcal{M}$ of positive random variables with values in $\{x_k : k \in \mathbb{Z}\}$. We show that we may assume that the discretized functions $\hat{F}$ are cumulative distribution functions of random variables $\hat{X}$ which are defined on the same probability space as X, and that the proximity of $\hat{X}$ to X matches that of $\hat{F}$ to F.

The main analytical tool in Section 3 will be the Mellin-Stieltjes transform

$$\Phi(s) = \mathbb{E}(X^s) = \int_0^\infty x^s \mathrm{d}F(x),$$

see the book of Galambos and Simonelli [5], and, for example, [10]. It would, of course, be possible to replace the Mellin transform of X by the more familiar Laplace transform of $Y = \log X$, thereby transforming products of independent positive variables into sums of independent variables. Nevertheless, the multiplicative version together with the Mellin transform are superior in our setting since the hypothesis $\mathbb{E}(X) = 1$ does not transform into $\mathbb{E}(Y) = 0$, so we do not arrive at the familiar case of centered random variables. Equivalently, the condition $\mathbb{E}(X) = 1$ means that $\Phi(1) = 1$, but there is no simple correspondent for the Laplace transform of Y.

In Section 4 we turn to products $Y_n = X_1 \cdots X_n$ of independent random variables $X_n > 0$ with $\mathbb{E}(X_n) = 1$. Our construction of the $\hat{X}_n$ preserves independence. The distribution function G_n of Y_n is the n-fold multiplicative convolution and its Mellin-Stieltjes transform is

$$\int_0^\infty x^s \mathrm{d}G_n(x) = \mathbb{E}\big(X_1^s \cdots X_n^s\big) = \Phi_1(s) \cdots \Phi_n(s).$$

We consider the analogous expressions for the discretizations $\hat{X}_1 \cdots \hat{X}_n$. Among all discrete random variables, those whose values lie in a geometric sequence are the only ones that uniformly preserve the discrete structure under multiplication and therefore lead to a comparably simple structure of the products. Their Mellin-Stieltjes transform can be written as a Laurent series [8]

$$\varphi(z) = \sum_{k\in\mathbb{Z}} a_k z^k \qquad \text{with} \quad z = u^s, \quad u > 1 \quad \text{fixed.}$$

2 The space $\mathcal{M}$ and the canonical discretization

We need the following well-known elementary facts. We write $F(+\infty) = \lim_{x\to\infty} F(x)$.

Proposition 2.1 *Let F be nondecreasing in $[0, \infty)$ and let $F(+\infty) = 1$. If $\alpha > 0$ and $0 \le a < \infty$ then*

$$\int_a^\infty (x - a)^\alpha \,\mathrm{d}F(x) = \alpha \int_a^\infty (x - a)^{\alpha-1} (1 - F(x))\,\mathrm{d}x. \tag{2.1}$$

If $\alpha > 0$ and $\int_0^\infty x^\alpha \,\mathrm{d}F(x) = b < \infty$ then

$$x^\alpha (1 - F(x)) \le b \quad (x > 0)\,, \quad x^\alpha (1 - F(x)) \longrightarrow 0 \quad (x \longrightarrow \infty). \tag{2.2}$$

Let $\mathcal{M}$ denote the space of cumulative distribution functions

$$F(x) = \mathbb{P}(X \le x) \qquad (0 \le x < \infty) \tag{2.3}$$

of random variables with values in $(0, \infty)$ and expectation $\mathbb{E}(X) = 1$. A nondecreasing function F belongs to $\mathcal{M}$ if and only if it is continuous from the right and satisfies

$$F(0) = 0\,, \quad F(+\infty) = \int_0^\infty \mathrm{d}F(x) = 1\,, \quad \int_0^\infty x\,\mathrm{d}F(x) = \int_0^\infty (1 - F(x))\,\mathrm{d}x = 1; \tag{2.4}$$

see (2.1) with $\alpha = 1$ and $a = 0$. The condition $F(0) = 0$ holds because X is strictly positive.

Let $(x_k)_{k\in\mathbb{Z}}$ be a fixed doubly infinite sequence with

$$0 < x_k < x_{k+1} < \infty \ \ (k \in \mathbb{Z}), \quad x_k \longrightarrow 0 \ \ (k \longrightarrow -\infty), \quad x_k \longrightarrow \infty \ \ (k \longrightarrow +\infty)\,. \tag{2.5}$$

For $F \in \mathcal{M}$ we define $\hat{F}(x)$ as the mean value in the interval $[x_k, x_{k+1})$ that contains x, that is

$$\hat{F}(x) = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} F(y)\,\mathrm{d}y = \int_0^1 F((1-t)x_k + t x_{k+1}))\,\mathrm{d}t \tag{2.6}$$

for $x_k \le x < x_{k+1}$. We call $\hat{F}$ the *canonical discretization* of F with respect to (x_k). By definition, $\hat{F}$ is continuous from the right in $(0,\infty)$. The second integral in (2.6) shows that $\hat{F}$ is nondecreasing, and since $F(x_k) \le \hat{F}(x) \le F(x_{k+1})$ for $x_k \le x_k < x_{k+1}$ it follows from (2.4) and (2.5) that $\hat{F}(0) = 0$ and $\hat{F}(+\infty) = 1$. The definition of $\hat{F}$ as the mean value of F in each interval $[x_k, x_{k+1})$ implies that (2.4) is satisfied for $\hat{F}$. Therefore we have $\hat{F} \in \mathcal{M}$.

If we use the function p defined by (1.1) then

$$\hat{F}(x) = \frac{p(x_{k+1}) - p(x_k)}{x_{k+1} - x_k} \qquad \text{for} \quad x_k \le x < x_{k+1}\,.$$

Therefore $\hat{p}(x) := \int_0^x \hat{F}(y)\,\mathrm{d}y$ turns out to be the linear interpolation of p with support points x_k. This is a simple and natural choice in two contexts:

(i) The empirical data are given in terms of the function p, for instance as known prices of put options in financial mathematics.

(ii) The empirical data and therefore F are not smooth. Then passing to the integral p has a smoothing effect prior to the discretization.

In other contexts different discretizations may be more suitable.

Next we show that F and $\hat{F}$ are the distribution functions of random variables on a common probability space which are close-by to each other, parallel to the closeness of their distribution functions. It is easily seen that

$$G(x,y) = \sum_{x_k \le x} \int_0^y \left(\int_0^1 \mathbf{1}_{((1-t)x_{k-1} + t x_k,\ (1-t)x_k + t x_{k+1}]}(z)\,\mathrm{d}t \right) \mathrm{d}F(z)$$

is the distribution function of a probability measure in $(0,\infty)^2$ with marginal distributions $G(\infty, y) = F(y)$ and $G(x,\infty) = \hat{F}(x)$. Therefore G is the joint cumulative distribution function of random variables X', $\hat{X}'$ with distribution functions F and $\hat{F}$. Since we are less interested in the random variables themselves but rather in their distributions and joint distributions, we replace the original X by X', rename X' as X and set $\hat{X} = \hat{X}'$. Then the joint distribution of X and X' satisfies

$$\hat{X} = x_k \implies x_{k-1} < X \le x_{k+1} \ \text{ or, in other words, } \ X \in (x_k, x_{k+1}] \implies \hat{X} \in \{x_k, x_{k+1}\}. \tag{2.7}$$

Now we introduce a metric in $\mathcal{M}$ such that $\hat{F}$ can be characterized as a best approximation to F with respect to this metric. It is convenient to define it for the larger space $\overline{\mathcal{M}}$ of nondecreasing functions F in $[0,\infty)$ that are continuous from the right and satisfy

$$F(0) \ge 0\,, \quad F(+\infty) = F(0) + \int_0^\infty \mathrm{d}F(x) = 1\,, \quad \int_0^\infty x\,\mathrm{d}F(x) = \int_0^\infty \big(1 - F(x)\big)\,\mathrm{d}x \le 1\,. \tag{2.8}$$

In view of (2.8), we can interpret $F \in \overline{\mathcal{M}}$ as the distribution function of a nonnegative random variable. The distance in $\overline{\mathcal{M}}$ is defined by

$$\|F - G\|^2 = \int_0^\infty \big(F(x) - G(x)\big)^2 \mathrm{d}x \quad \text{for} \quad F, G \in \overline{\mathcal{M}}. \tag{2.9}$$

Since $0 \le F \le 1$ and $0 \le G \le 1$ we have $(F-G)^2 \le (1-F)+(1-G)$ and it follows from (2.8) that $\|F-G\| \le 2$. This makes $\overline{\mathcal{M}}$ into a metric space. In Proposition 3.4 we shall show that $\overline{\mathcal{M}}$ is indeed the closure of $\mathcal{M}$.

The metric (2.9) defines the same topology as the weak convergence of (F_n), see Proposition 3.3. For smooth functions F, other metrics may be preferable, see for instance [3, p. 286, no. 12]. But this metric assigns the distance 2 to the difference of the functions $F_1(x)=1$ for $x \ge 0$ and $F_2(x)=1$ for $x \ge \varepsilon$ and $=0$ otherwise, whereas $\|F_1 - F_2\| = \sqrt{\varepsilon}$. The Levy metric [3, p. 285, no. 11] avoids this problem but is more difficult to handle. The next theorem reflects the familiar connection between mean-values and the L^2-norm.

Theorem 2.2 *Let $F \in \mathcal{M}$ and let $\hat{F}$ be its canonical discretization. Then*

$$\|F-\hat{F}\| \le \|F-H\| \tag{2.10}$$

for all $H \in \mathcal{M}$ that are constant in each interval $[x_k, x_{k+1})$. Equality only holds if $H=\hat{F}$.

Proof. By assumption H has the constant value $H(x_k)$ in $[x_k, x_{k+1})$. Hence

$$\|F-H\|^2 = \sum_k \int_{x_k}^{x_{k+1}} (F(x)-H(x_k))^2 \,\mathrm{d}x\,.$$

Since $\int_{x_k}^{x_{k+1}} (F(x)-\hat{F}(x_k))\,\mathrm{d}x = 0$ by (2.6), we conclude that

$$\begin{aligned}\|F-H\|^2 &= \sum_k \int_{x_k}^{x_{k+1}} \big(F(x)-\hat{F}(x_k)\big)^2 \,\mathrm{d}x + \sum_k \int_{x_k}^{x_{k+1}} \big(\hat{F}(x_k)-H(x_k)\big)^2 \,\mathrm{d}x \\ &= \|F-\hat{F}\|^2 + \|\hat{F}-H\|^2\,.\end{aligned}$$

□

If G is an approximation to F then $\hat{G}$ is an even better approximation to $\hat{F}$, as the following theorem shows.

Theorem 2.3 *Let $F, G \in \mathcal{M}$ and let $\hat{F}$ and $\hat{G}$ be their canonical discretizations. Then*

$$\big\|\hat{F}-\hat{G}\big\| \le \|F-G\|\,. \tag{2.11}$$

Proof. Let $x_k \le x < x_{k+1}$. By (2.6) and the Schwarz inequality we have

$$\big(\hat{F}(x)-\hat{G}(x)\big)^2 \le \frac{1}{x_{k+1}-x_k} \int_{x_k}^{x_{k+1}} (F(y)-G(y))^2 \mathrm{d}y$$

and therefore

$$\int_{x_k}^{x_{k+1}} \big(\hat{F}(y)-\hat{G}(y)\big)^2 \mathrm{d}y \le \int_{x_k}^{x_{k+1}} (F(y)-G(y))^2\,\mathrm{d}y\,.$$

Summing over $k \in \mathbb{Z}$ we obtain (2.11). □

Theorem 2.4 *If $\hat{F}$ is the canonical discretization of $F \in \mathcal{M}$ then*

$$\big\|F-\hat{F}\big\|^2 \le \frac{1}{4}\sum_{k\in\mathbb{Z}} (F(x_{k+1})-F(x_k))^2 (x_{k+1}-x_k)\,. \tag{2.12}$$

Proof. Let $H(x) = (F(x_k)+F(x_{k+1}))/2$ for $x_k \le x < x_{k+1}$. Since F is nondecreasing we have $|F(x)-H(x)| \le (F(x_{k+1})-F(x_k))/2$ for $x_k \le x < x_{k+1}$ and thus, by Theorem 2.2,

$$\big\|F-\hat{F}\big\|^2 \le \|F-H\|^2 \le \frac{1}{4}\sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} (F(x_{k+1})-F(x_k))^2 \mathrm{d}x\,.$$

□

Example 2.5 Let $F(x) = 0$ for $0 \le x < 1$ and $= 1$ for $1 \le x < \infty$, furthermore $x_0 < 1$ and $x_1 = 2 - x_0$. Then $\hat{F}(x) = 0$ for $0 \le x < x_0$, $= \frac{x_1 - 1}{x_1 - x_0}$ for $x_0 \le x < x_1$ and $= 1$ for $x_1 \le x < \infty$. Hence we have $\|F - \hat{F}\|^2 = (x_1 - x_0)/4$. It follows that (2.12) is best possible.

Example 2.6 Suppose that $F \in \mathcal{M}$ is linear in $[x_k, x_{k+1}]$. A short calculation shows that

$$\|F - \hat{F}\|^2 = \frac{1}{12} \sum_{k \in \mathbb{Z}} (F(x_{k+1}) - F(x_k))^2 (x_{k+1} - x_k)\,, \tag{2.13}$$

which is by a factor 3 smaller than the estimate (2.12). If $G \in \mathcal{M}$ then, by Theorem 2.3,

$$\|G - \hat{G}\| \le \|G - F\| + \|F - \hat{F}\| + \|\hat{F} - \hat{G}\| \le 2\|G - F\| + \|F - \hat{F}\|\,,$$

where $\|F - \hat{F}\|$ is given by (2.13). If G is very smooth then $\|G - F\|$ is small for suitable piecewise linear F.

Example 2.7 Suppose that $F \in \mathcal{M}$ is continuous and $F(b) = 1$ for some $b < \infty$. We determine x_k $(k = 0, 1, \ldots, m)$ such that $F(x_k) = k/m$ for $k < m$ and $x_m = b$. Then it follows from Theorem 2.4 that $\|F - \hat{F}\| \le \sqrt{b}/(2m)$.

Now we show that $\hat{F}$ is close to F if the $x_{k+1} - x_k$ are small; for $x_k > 1$ we only need that the $\log(x_{k+1}/x_k)$ are small.

Theorem 2.8 *Let $F \in \mathcal{M}$. If $x_{k+1} - x_k \le \delta \max(1, x_k)$ $(k \in \mathbb{Z})$ then*

$$\|F - \hat{F}\| \le \sqrt{\delta/2}\,. \tag{2.14}$$

If $\hat{F}_n$ are the canonical discretizations of F with respect to $(x_{n,k})_{k \in \mathbb{Z}}$ for $n \in \mathbb{N}$ and if

$$\sup_k [(x_{n,k+1} - x_{n,k})/\max(1, x_{n,k})] \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty, \tag{2.15}$$

then $\hat{F}_n$ converges weakly to F.

P r o o f. Since $F(x_{k+1}) - F(x_k) \le 1$ we see from Theorem 2.4 that

$$\begin{aligned}
\|F - \hat{F}\|^2 &\le \frac{1}{4} \sum_k (x_{k+1} - x_k)(F(x_{k+1}) - F(x_k)) \\
&\le \frac{\delta}{4} \sum_{x_k \le 1} (F(x_{k+1}) - F(x_k)) + \frac{\delta}{4} \sum_{x_k > 1} x_k (F(x_{k+1}) - F(x_k)) \\
&\le \frac{\delta}{4} + \frac{\delta}{4} \sum_k \int_{x_k}^{x_{k+1}} x \, \mathrm{d}F(x) \le \frac{\delta}{2}
\end{aligned}$$

by (2.4). Furthermore, it follows from (2.15) that $\|F - \hat{F}_n\| \to 0$ as $n \to \infty$ and we deduce from Proposition 3.3 that $\hat{F}_n$ converges weakly to F. □

3 The Mellin-Stieltjes transform

The Mellin-Stieltjes transform of $F \in \overline{\mathcal{M}}$ is defined by

$$\Phi(s) = \int_0^\infty x^s \, \mathrm{d}F(x)\,. \tag{3.1}$$

It follows from (2.8) that $\Phi(0) = 1 - F(0) \le 1$ and $\Phi(1) \le 1$. If $F \in \mathcal{M}$ is given by (2.3) then

$$\Phi(s) = \mathbb{E}(X^s), \quad \Phi(0) = \Phi(1) = 1\,. \tag{3.2}$$

Proposition 3.1 *Let $F \in \overline{\mathcal{M}}$. The Mellin-Stieltjes transform Φ is analytic and bounded by 1 in $\{s \in \mathbb{C}\colon 0 < \operatorname{Re} s < 1\}$ and continuous in $\{0 \le \operatorname{Re} s \le 1\}$. If $G \in \overline{\mathcal{M}}$ has the same Mellin-Stieltjes transform Φ as F, then $F = G$.*

Proof. By (2.8), we have for $0 \le \sigma \le 1$

$$|\Phi(\sigma + \mathrm{i}t)| \le \int_0^\infty x^\sigma F(x)\,\mathrm{d}x \le \int_0^\infty ((1-\sigma) + \sigma x)\,\mathrm{d}F(x) \le 1\,. \tag{3.3}$$

Furthermore x^s is continuous in $0 \le x < \infty$ and analytic in $\{0 < \operatorname{Re} s < 1\}$. Hence it follows from the estimate in (3.3) that Φ is analytic in $\{0 < \operatorname{Re} s < 1\}$. Now let $0 \le \operatorname{Re} s \le 1$ and let $\varepsilon > 0$ be given. By (2.8) we can choose $0 < a < 1 < b < \infty$ such that

$$\left|\int_0^a x^s\,\mathrm{d}F(x)\right| \le \int_0^a \mathrm{d}F(x) = F(a) - F(0) < \varepsilon\,,$$
$$\left|\int_b^\infty x^s\,\mathrm{d}F(x)\right| \le \int_b^\infty \mathrm{d}F(x) < \varepsilon\,,$$

note that F is right-continuous. If $0 \le \operatorname{Re} s_0 \le 1$ it follows that

$$|\Phi(s) - \Phi(s_0)| < 4\varepsilon + \int_a^b |x^s - x^{s_0}|\,\mathrm{d}F(x) < 5\varepsilon$$

if $|s - s_0|$ is sufficiently small. Hence Φ is continuous in $\{0 \le \operatorname{Re} s \le 1\}$.

We obtain from (2.1) with $a = 0$ that

$$\Phi(s) = s\int_0^\infty x^{s-1}(1 - F(x))\,\mathrm{d}x \tag{3.4}$$

holds for $s \in (0,1)$ and it follows from the identity theorem that (3.4) holds for $0 < \operatorname{Re} s < 1$. If F and G have the same Mellin-Stieltjes transform Φ then it follows from (3.4) and the uniqueness theorem [2, p. 87] of the standard Mellin transformation that $F = G$. □

Theorem 3.2 *Let Φ and Ψ be the Mellin-Stieltjes transforms of $F \in \overline{\mathcal{M}}$ and $G \in \overline{\mathcal{M}}$, respectively. Then*

$$\|F - G\|^2 = \frac{1}{\pi}\int_{-\infty}^\infty \left|\Phi\left(\frac{1+\mathrm{i}t}{2}\right) - \Psi\left(\frac{1+\mathrm{i}t}{2}\right)\right|^2 \frac{\mathrm{d}t}{1+t^2}\,. \tag{3.5}$$

Proof. We set $f = F - G$ and $\varphi = \Phi - \Psi$. Then we have

$$\varphi\left(\frac{1+\mathrm{i}t}{2}\right) = \int_0^\infty x^{\frac{1+\mathrm{i}t}{2}}\,\mathrm{d}f(x) \quad \text{for} \quad t \in \mathbb{R}$$

and therefore

$$\int_{-\infty}^\infty \left|\varphi\left(\frac{1+\mathrm{i}t}{2}\right)\right|^2 \frac{\mathrm{d}t}{1+t^2} = \int_0^\infty \int_0^\infty \left(\int_{-\infty}^\infty x^{\frac{1+\mathrm{i}t}{2}} y^{\frac{1-\mathrm{i}t}{2}} \frac{\mathrm{d}t}{1+t^2}\right) \mathrm{d}f(x)\,\mathrm{d}f(y)\,; \tag{3.6}$$

note that all integrals converge absolutely in view of (3.3). The innermost integral is equal to

$$2(xy)^{\frac{1}{2}} \int_0^\infty \cos\left(\frac{t}{2}\log\frac{y}{x}\right)\frac{\mathrm{d}t}{1+t^2} = \min(x,y)\,.$$

To prove this for $0 < x < y$, we substitute $\tau = (t/2)\log(y/x)$ and use that

$$\int_0^\infty \frac{a\cos\tau}{a^2+\tau^2}\,\mathrm{d}\tau = \frac{\pi}{2}\,\mathrm{e}^{-a} \quad \text{for} \quad a > 0.$$

Hence the right-hand side of (3.6) is equal to

$$\int_0^\infty \left(\int_0^\infty \min(x,y)\,\mathrm{d}f(y)\right)\mathrm{d}f(x) = \int_0^\infty \left(\int_0^x y\,\mathrm{d}f(y) + x\int_x^\infty \mathrm{d}f(y)\right)\mathrm{d}f(x)\,.$$

We integrate the first inner integral by parts and use that $f(+\infty) = F(+\infty) - G(+\infty) = 0$. We obtain

$$-\int_0^\infty \left(\int_0^x f(y)\,\mathrm{d}y\right)\mathrm{d}f(x) = \int_0^\infty f(x)^2\,\mathrm{d}x\,;$$

the last equality follows by another integration by parts using that $f(+\infty) = 0$ and that

$$\left|\int_0^x f(y)\,\mathrm{d}y\right| = \left|\int_0^x \big(1 - G(y)\big)\,\mathrm{d}y - \int_0^x \big(1 - F(y)\big)\,\mathrm{d}y\right| \le 1$$

because of (2.8). □

We assume that F is continuous from the right. The non-increasing functions F_n *converge weakly* to F on $(0,\infty)$ if

$$F_n(x) \longrightarrow F(x) \quad (n \longrightarrow \infty) \quad \text{on a dense subset of } (0,\infty)\,. \tag{3.7}$$

It follows that $F_n(x)$ converges to $F(x)$ whenever F is continuous at x. If $F_n \in \overline{\mathcal{M}}$ then $F \in \overline{\mathcal{M}}$; by (2.2) the condition $F(+\infty) = 1$ follows from $\int_0^\infty x\,\mathrm{d}F(x) \le 1$. However $F_n \in \mathcal{M}$ does not imply $F \in \mathcal{M}$.

The following proposition is related to many well-known theorems, see, for example, [5, Theorem 1.2]. One difference is our moment condition in (2.8).

Proposition 3.3 *Let $F_n \in \overline{\mathcal{M}}$ and $F \in \overline{\mathcal{M}}$ with Mellin-Stieltjes transforms Φ_n and Φ, respectively. Then the following three conditions are equivalent*:

(i) $F_n \longrightarrow F \ (n \longrightarrow \infty)$ *weakly,*

(ii) $\Phi_n \longrightarrow \Phi \ (n \longrightarrow \infty)$ *locally uniformly in* $\{0 < \operatorname{Re} s < 1\}$,

(iii) $\|F_n - F\| \longrightarrow 0 \ (n \longrightarrow \infty)$.

Proof. (i) implies (ii). Let $0 < s < 1$ and $\varepsilon > 0$. We choose $b > 1$ such that $b^{s-1} < \varepsilon$. Then, by (2.8),

$$\left|\int_b^\infty x^s\,\mathrm{d}F_n(x)\right| \le b^{s-1}\int_b^\infty x\,\mathrm{d}F_n(x) \le b^{s-1} < \varepsilon\,,$$

similarly to F. Hence, by (i), we have

$$|\Phi_n(s) \longrightarrow \Phi(s)| \le \left|\int_0^b x^s\,\mathrm{d}F_n(x) - \int_0^b x^s\,\mathrm{d}F(x)\right| + 2\varepsilon < 3\varepsilon$$

for $n > n_0(\varepsilon, s)$. It follows that $\Phi_n(s) \longrightarrow \Phi(s)$ for $0 < s < 1$, and since $|\Phi_n(s) - \Phi(s)| \le 2$ this implies (ii) by Vitali's theorem [6, p. 324].

(ii) implies (iii). Let $a > 0$. Theorem 3.2 shows that

$$\|F_n - F\|^2 \le \frac{1}{\pi}\int_{-a}^a \left|\Phi_n\left(\frac{1+\mathrm{i}t}{2}\right) - \Phi\left(\frac{1+\mathrm{i}t}{2}\right)\right|^2 \frac{\mathrm{d}t}{1+t^2} + \frac{8}{\pi}\int_a^\infty \frac{\mathrm{d}t}{1+t^2}\,.$$

We choose a so large that the last term is $< \varepsilon$. By (ii) we can find n_1 such that

$$\big|\Phi_n(s) - \Phi(s)\big| < \varepsilon \quad \text{for} \quad n > n_1, \quad \operatorname{Re} s = \frac{1}{2}, \quad \operatorname{Im} s \le \frac{a}{2}.$$

We conclude that $\|F_n - F\|^2 < 2\varepsilon$ for $n > n_1$, so that (iii) holds.

(iii) implies (i). It follows from (iii) and (2.9) that $F_n \to F$ in measure, and since F_n and F are nondecreasing this implies (i). □

Proposition 3.4 *The space $\overline{\mathcal{M}}$ is compact and is the closure of $\mathcal{M}$.*

Proof. Let $F_n \in \overline{\mathcal{M}}$. By Helly's selection theorem there is a sequence (n_ν) such that F_{n_ν} converges weakly. The limit function F belongs again to $\overline{\mathcal{M}}$. Since $\overline{\mathcal{M}}$ is a metric space it follows that $\overline{\mathcal{M}}$ is compact.

Now let $F \in \overline{\mathcal{M}}$. We define F_n^* by $F_n^*(x) = 0$ for $0 \le x < 1/n$ and $F_n^*(x) = F(x)$ otherwise. We have

$$1 + \delta_n := \int_0^\infty \big(1 - F_n^*(x)\big)\mathrm{d}x = \frac{1}{n} + \int_{\frac{1}{n}}^\infty (1 - F(x))\mathrm{d}x \le 1 + \frac{1}{n}$$

and thus $\delta_n \le 1/n$. Now we set $c_n = (2/\pi)\big(1 - (n-1)\delta_n\big)$ and define

$$F_n(x) = \left(1 - \frac{1}{n}\right) F_n^*(x) + \frac{1}{n}\,\frac{x^2}{x^2 + c_n^2} \quad \text{for} \quad 0 \le x < \infty\,.$$

Then $c_n > 0$ and we check that $\int_0^\infty \big(1 - F_n(x)\big)\,\mathrm{d}x = 1$. Hence $F_n \in \mathcal{M}$ and F_n converges weakly to F. □

We see from (3.1) and (2.6) that the Mellin-Stieltjes transform of a discretization $\hat{F}$ is

$$\hat{\Phi}(s) = \sum_{k\in\mathbb{Z}} a_k x_k^s \quad \text{with} \quad a_k = \hat{F}(x_k) - \hat{F}(x_{k-1}) \ge 0 \tag{3.8}$$

for $0 \le \operatorname{Re} s \le 1$. This is an almost periodic analytic function [1].

The *geometric discretization* is the special case

$$x_k = u^k \quad (k \in \mathbb{Z}), \quad u > 1, \tag{3.9}$$

of the canonical discretization. We can apply Theorem 2.8 with $\delta = u - 1$. Since $x_k^s = u^{ks}$ we see from (3.8) that $\hat{\Phi}$ is now a periodic function. If we write $z = u^s$ then (3.8) becomes

$$\varphi(z) := \hat{\Phi}(s) = \sum_{k\in\mathbb{Z}} a_k z^k \quad \text{for} \quad 1 \le |z| \le u, \tag{3.10}$$

where $a_k = \hat{F}\left(u^{k+1}\right) - \hat{F}\left(u^k\right) \ge 0$. The series converges for $z = 1$ because $\varphi(z) \to 1$ as $z \to 1-$ by Proposition 3.1. Hence we have

$$\sum_{k\in\mathbb{Z}} a_k = \varphi(1) = 1\,, \qquad \sum_{k\in\mathbb{Z}} a_k u^k = \varphi(u) = 1\,. \tag{3.11}$$

In the next section we show that the random variables with values in a fixed geometric progression have a property that singles them out among other discrete random variables.

4 Products of independent random variables

Now we consider a sequence of processes. In the framework of asset pricing, this means that we consider a multi-period market. For $n \in \mathbb{N}$ let X_n be random variables with values in $(0, \infty)$ and expectation 1. Then their distribution functions belong to $\mathcal{M}$. The random variables

$$Y_n = X_1 \cdots X_n \quad (n \in \mathbb{N}) \tag{4.1}$$

are the factors by which the original values S_0 have changed at time n. Their distribution functions are

$$G_n = \mathbb{P}(Y_n \le x) \quad (0 \le x < \infty) \tag{4.2}$$

and their Mellin-Stieltjes transforms are

$$\Psi_n(s) = \mathbb{E}\big(Y_n^s\big) = \int_0^\infty x^s \mathrm{d}G_n(x) \quad (0 \le \operatorname{Re} s \le 1). \tag{4.3}$$

Now we make the key assumption that the random variables X_n are *independent*. Then we see from (4.1) and (4.3) that

$$\Psi_n(s) = \prod_{\nu=1}^{n} \mathbb{E}\big(X_\nu^s\big) = \prod_{\nu=1}^{n} \Phi_\nu(s) \quad \text{for} \quad n \in \mathbb{N}; \tag{4.4}$$

in particular we have $\Psi_n(0) = \Psi_n(1) = 1$ and thus $G_n \in \mathcal{M}$.

Let $(x_k)_{k\in\mathbb{Z}}$ be a fixed doubly infinite sequence satisfying (2.5) and let $\hat{F}_n$ be the canonical discretization of F_n. We suppose that the same discretization is applied to all X_n. As pointed out in Section 2, we may suppose that F_n and $\hat{F}_n$ are the distribution functions of random variables X_n and $\hat{X}_n$ on a common probability space $(\Omega_n, \mathscr{F}_n, \mathbb{P}_n)$ and the analogue to (2.7) holds for each pair X_n, $\hat{X}_n$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the product of the $(\Omega_n, \mathscr{F}_n, \mathbb{P}_n)$, further pr_n the projection maps and $X'_n(\omega) := X_n(pr_n(\omega))$, $\hat{X}'_n(\omega) := \hat{X}'_n(pr_n(\omega))$. Then X'_n and $\hat{X}'_n$ are random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ with distribution functions F_n and $\hat{F}_n$, satisfying the analogue to (2.7). Moreover, both $\{X'_1, X'_2, \ldots\}$ and $\{\hat{X}'_1, \hat{X}'_2, \ldots\}$ are independent sets of random variables. By an argument like that in the discussion preceding (2.7) we finally rename X'_n as X_n and $\hat{X}'_n$ as $\hat{X}_n$. Altogether, the X_n and $\hat{X}_n$ are all defined on a common probability space and the $\hat{X}_n$ are independent as are the X_n.

Let $\hat{Y}_n := \hat{X}_1 \cdots \hat{X}_n$ for $n \in \mathbb{N}$; this notation is somewhat misleading, it is not supposed to mean that $\hat{Y}_n$ is the canonical discretization of Y_n defined in (4.1). Since the $\hat{X}_n$ are independent, the Mellin-Stieltjes transforms of the distributions of $\hat{Y}_n$ satisfy

$$\hat{\Psi}_n(s) = \prod_{\nu=1}^{n} \hat{\Phi}_\nu(s) = \prod_{\nu=1}^{n} \left(\sum_{k\in\mathbb{Z}} a_{\nu,k}\, x_k^s \right) \tag{4.5}$$

where $a_{\nu,k} = \hat{F}_\nu(x_k) - \hat{F}_\nu(x_{k-1}) \geq 0$ as in (3.8). Multiplying (4.5) out we see that $\hat{\Psi}_n(s)$ is the sum of terms of the form

$$a_{1,k_1} \cdots a_{n,k_n} \big(x_{k_1} \cdots x_{k_n}\big)^s, \quad k_\nu \in \mathbb{Z} \quad (\nu = 1, \ldots, n). \tag{4.6}$$

There is no cancellation because $a_{\nu,k_\nu} \geq 0$.

The logarithms of the products $x_{k_1} \cdots x_{k_n}$ form an additive semigroup $\mathscr{H}$. By (2.5) it contains positive and negative numbers. If there are arbitrary small positive or negative numbers in $\mathscr{H}$ then $\mathscr{H}$ is clearly dense in $\mathbb{R}$ and the numbers $x_{k_1} \cdots x_{k_n}$ approach any given point in $(0, \infty)$; we conclude that the distributions of the $\hat{Y}_n$ tend to become less and less discrete as $n \to \infty$ and the series obtained from (4.5) become worthless for practical purposes.

In the opposite case it is obvious that $0 < d := \inf_{0<h\in\mathscr{H}} h = -\sup_{0>h\in\mathscr{H}} h$ and neither d nor $-d$ can be a limit point of $\mathscr{H}$. Hence $d, -d \in \mathscr{H}$, therefore $d\,\mathbb{Z} \subset \mathscr{H}$, and finally $d\,\mathbb{Z} = \mathscr{H}$ because of the minimality of d, i.e., the x_k take only values in a fixed geometric progression. We use the notation of (3.10) where $z = u^s$ and put $a^*_{\nu,k} = u^{k/2} a_{\nu,k}$. Then we obtain from (4.5) and (4.6) that

$$\varphi_1(z) \cdots \varphi_n(z) = \sum_{k\in\mathbb{Z}} \mathbb{P}\big(\hat{X}_1 \cdots \hat{X}_n = k\big)\, z^k = \sum_{k\in\mathbb{Z}} \left(\sum_{k_1+\cdots+k_n=k} a^*_{1,k_1} \cdots a^*_{n,k_n} \right) u^{-\frac{k}{2}} z^k\,. \tag{4.7}$$

It follows from $\varphi(1) = \varphi(u) = 1$ that $a^*_{\nu,k_\nu} \leq u^{-\frac{|k|}{2}}$. Hence we see that the last inner sum converges well.

For processes with independent factors, it therefore seems that the only discretizations that preserve the discrete structure are those which lead to variables X with values in a fixed geometric progression, represented by a Laurent series

$$\varphi(z) = \sum_{k\in\mathbb{Z}} a_k z^k\,, \quad a_k = \mathbb{P}(X = u^k)\,, \quad u > 1\,. \tag{4.8}$$

Proposition 4.1 *Let $F \in \overline{\mathcal{M}}$ and let $u > 1$. Suppose that F is constant in $(1/u, u)$ except for a jump of height $a \in [0,1]$ at 1. Then*

$$\Phi\left(\frac{1}{2}\right) \leq 1 - (1-a)\frac{(\sqrt{u}-1)^2}{u+1} \tag{4.9}$$

with equality exactly for the symmetric trinomial distribution

$$\Phi(s) = \frac{1-a}{u+1}\, u^{1-s} + a + \frac{1-a}{u+1}\, u^{s}\,, \quad 0 \le a \le 1\,. \tag{4.10}$$

The assumptions are satisfied for a variable with values in a fixed geometric progression as in (4.8), in which case we have $a = a_0$.

P r o o f. We have $(x+1)/\sqrt{x} \ge (u+1)/\sqrt{u}$ for $0 < x \le 1/u$ and for $x \ge u$. Hence we obtain from the definition (3.1) that

$$\begin{aligned}\Phi\left(\frac{1}{2}\right) &= a + \left(\int_0^{1/u} + \int_u^{\infty}\right) \sqrt{x}\,\mathrm{d}F(x) \le a + \left(\int_0^{1/u} + \int_u^{\infty}\right) \sqrt{u}\,\frac{x+1}{u+1}\,\mathrm{d}F(x)\\ &= a + \frac{\sqrt{u}}{u+1}(\Phi(1) - a + \Phi(0) - a) \le a + \frac{2\sqrt{u}}{u+1}\,(1-a)\end{aligned}$$

because $\Phi(0) \le 1$ and $\Phi(1) \le 1$. This implies (4.9).

Now suppose that equality holds in (4.9). Then F must be constant except for jumps in $\{u^{-1}, 1, u\}$ and furthermore $\Phi(0) = \Phi(1) = 1$. Hence $\Phi(s) = a_{-1}u^{-s} + a + a_1 u^s$ where $a_{-1} + a + a_1 = 1$ and $a_{-1}u^{-1} + a + a_1 u = 1$ which implies (4.10). It is clear that equality in (4.9) holds for this function. □

We denote the n-fold multiplicative convolution of F by $F^{[n]}$. Hence

$$F^{[n]}(x) = \mathbb{P}(X_1 \cdots X_n \le x)\,, \quad X_\nu \text{ independent and identically distributed.} \tag{4.11}$$

Theorem 4.2 *Let $F, G \in \overline{\mathcal{M}}$. Suppose that the Mellin-Stieltjes transforms satisfy $\Phi\big(\frac{1}{2}\big) \le \alpha$, $\Psi\big(\frac{1}{2}\big) \le \beta$ and $\alpha \le \beta$. Then*

$$\big\|F^{[n]} - G^{[n]}\big\| \le \left(1 + \frac{\alpha}{\beta} + \cdots + \left(\frac{\alpha}{\beta}\right)^{n-1}\right)\beta^{n-1}\|F - G\| \le n\beta^{n-1}\|F - G\|\,. \tag{4.12}$$

P r o o f. We obtain from Theorem 3.2 and the independence hypothesis that

$$\big\|F^{[n]} - G^{[n]}\big\|^2 = \frac{1}{\pi}\int_{-\infty}^{\infty} \left|\Phi\left(\frac{1+\mathrm{i}t}{2}\right)^n - \Psi\left(\frac{1+\mathrm{i}t}{2}\right)^n\right|^2 \frac{\mathrm{d}t}{1+t^2}\,.$$

Since $|\Phi\big(\frac{1+\mathrm{i}t}{2}\big)| \le \Phi\big(\frac{1}{2}\big) \le \alpha$ and $|\Psi\big(\frac{1+\mathrm{i}t}{2}\big)| \le \beta$, we can estimate the expression within the last square by $\big|\Phi\big(\frac{1+\mathrm{i}t}{2}\big) - \Psi\big(\frac{1+\mathrm{i}t}{2}\big)\big|\big(\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}\big)$ and (4.12) follows using again Theorem 3.2. □

We can sometimes estimate $\Phi\left(\frac{1}{2}\right)$ and $\Psi\left(\frac{1}{2}\right)$ by (4.9). Since $\Phi\left(\frac{1}{2}\right) < 1$ except if X is deterministic, we conclude that $\big\|F^{[n]} - G^{[n]}\big\|$ is not much greater than $\|F - G\|$ and finally tends to 0.

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