Numerical Functional Analysis and Optimization, 31(11):1261–1271, 2010

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630563.2010.519135

I_m-QUASI UPWARD SETS, WITH THEIR BEST APPROXIMATIONS

M. Iranmanesh and F. Solimany

Department of Mathematics, Shahrood University of Technology, Shahrood, Iran

□ *The main purpose of this article is to introduce particular subsets of R^I, which are not necessarily convex, and we call them I_m-quasi upward, or I_m-quasi downward. We show that these sets can be translated to downward or upward sets. We introduce the connection of these sets with downward and upward subsets of R^I, and discuss the best approximation of these sets. Also we introduce embedded I_m-quasi upward and embedded I_m-quasi downward subsets of a normed space X.*

Keywords Banach Lattice Space; Best approximation; Downward set; I_m-quasi downward hull; I_m-quasi downward set; I_m-quasi upward hull; I_m-quasi upward set; Proximinal set; Upward set.

AMS Mathematics Subject Classification Primary 41A50; Secondary 46B40, 26B25, 52A40.

1. INTRODUCTION

The theory of best approximation by elements of convex sets in normed linear spaces, which has many important applications in mathematics and some other sciences, is well developed. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best approximation by not necessarily convex sets. In this direction, Rubinov and Singer [2, 3] developed a theory of best approximation by elements of so-called normal sets in the non-negative orient R^I_+, of a finite-dimensional coordinate space R^I endowed with the max- norm. Martinez-Legaz, Rubinov and Singer in [1] have developed a theory of best approximation of downward subsets of the space R^I. Downward sets play an important role in some parts of mathematical

Received 5 May 2010; Revised 26 July 2010; Accepted 23 August 2010.
Address correspondence to M. Iranmanesh, Department of Mathematics, Shahrood University of Technology, P.O. Box 3619995161-316, Shahrood, Iran; E-mail: m.iranmanesh2012@gmail.com

economics (see, e.g., [8]) and game theory. In this article we develop the theory of best approximation by subsets of R^I which are not necessary downward or upward, but this sets can translate to downward or upward sets. We use the results obtained in downward and upward sets as a tool for founding the distance and best approximation of a point to new sets. The structure of the article is as follows: In Section 2, we present some preliminary results. In Section 3, we define the new sets which is called I_m-quasi upward sets and discuss about the connection between I_m-quasi upward sets and upward or downward sets. In particular, we obtain a condition for uniqueness of best approximation by elements of closed I_m-quasi upward sets. Also, we show that every collection $(A_t)t \in T$ of I_m-quasi upward subsets of R^I, is linearly regular. In Section 4, we define positive I_m-quasi upward sets and the connection between positive I_m-quasi upward and I_m-quasi upward sets as subsets of R^I, based on the notion of I_m-quasi upward hull sets. In Section 5, we introduce embedded I_m-quasi upward or downward subsets of the space X. We discuss about the conditions that embedded I_m-quasi upward or downward sets are proximinal.

2. PRELIMINARIES

Let I be a finite set of indices. Consider the space R^I of all vectors $(x_i)_{i\in I}$. We shall use the following notations:

if $x \in R^I$ then x_i is the ith coordinate of x.
if $x, y \in R^I$, then $x \leq y \Leftrightarrow x_i \leq y_i$ for all $i \in I$.
$R^I_+ := \{x = (x_i)_{i\in I} \in R^I; x_i \geq 0 \text{ for all } i \in I\}$.
$1 = (1, \dots, 1)$.

For each $x = (x_i)_{i\in I} \in R^I$ let $x^+ = \max(x, 0)$ coordinatewise (i.e., $x_i^+ = \max(x_i, 0)$ for all $i \in I$). Now consider that $I_m = \{i_1, i_2, \dots, i_m\}$ be an arbitrarily subset of I and $x = (x_i)_{i\in I}$ be an arbitrary element of R^I we define the following sets:

$$R_x^{I_m} := \{y = (y_i)_{i\in I} \in R^I : x_i \leq y_i \text{ if } i \in I_m \text{ and } x_i \geq y_i \text{ if } i \in I\backslash I_m\},$$
$$\operatorname{co} R_x^{I_m} := \{y = (y_i)_{i\in I} \in R^I : x_i \geq y_i \text{ if } i \in I_m \text{ and } x_i \leq y_i \text{ if } i \in I\backslash I_m\},$$
$$(R_x^{I_m})_+ := R^I_+ \cap R_x^{I_m}.$$

We shall use the notation $\mathbf{1}^{I_m}$ for vector $y = (y_i)_{i\in I}$ which

$$y_i = \begin{cases} 1 & \text{if } i \in I_m \\ -1 & \text{if } i \in I\backslash I_m \end{cases}. \tag{2.1}$$

Also we define $Pr^{I_m}(x)$ as follows:

$$(Pr^{I_m}(x))_i = \begin{cases} x_i & \text{if } i \in I_m \\ 0 & \text{if } i \in I \setminus I_m \end{cases}. \tag{2.2}$$

In the sequel, we shall assume the space R^I is equipped with the coordinatewise order relation $\leq$ and with the max-norm $\|x\| = \|x\|_\infty = \max_{i\in I} |x_i|$. Note that the ball $B(x,r) = \{y : \|y - x\| \leq r\}$ can be represented as

$$B(x,r) = \{y \in R^I : x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}. \tag{2.3}$$

We shall denote by $\operatorname{int} A$, $\operatorname{bd} A$; the interior of A and the boundary of A, respectively. If there exists the least element of A, we denote it by $\min A$. In the following we introduce some definitions (see [1, 2]). Let A be a subset of R^I and $x \in R^I$, then we consider the following notations:

$$d_A(x) := dist(x, A) = \inf_{a\in A} \|x - a\|, \tag{2.4}$$

$$\mathbf{P}_A(x) := \{a : a \in A, \|x - a\| = d_A(x)\}, \tag{2.5}$$

where the function d_A is called distance function. Geometrically $\mathbf{P}_A(x)$ is the set of all elements of A which are nearest to x. It is well known that for each closed set $A \subset R^I$ and each $x \in R^I$ the set $\mathbf{p}_A(x)$ is not empty, and $\mathbf{p}_\mathbf{A}(x) \subset \operatorname{bd} A$ for each $A \subset R^I$ and $x \in R^I$.

Definition 2.1. A set $U \subset R^I$ is called downward, if $(u \in U, x \leq u) \Longrightarrow x \in U$.

Definition 2.2. *A set $U \subset R^I$ is called upward if $(u \in U, x \geq u) \Longrightarrow x \in U$.*

Definition 2.3. *A set $U \subset R^I$ is called* strictly downward if for each boundary point w_0 of U inequality $w > w_0$ implies $w \notin U$.

Proposition 2.1 (see [4]). *Let U be a downward subset of R^I and $x \in R^I$. Then the following assertions are true:*

(1) *best approximation of x is unique.*
(2) *U is strictly downward.*

Proposition 2.2 (see [1]). *Let A be a closed downward set and $x \in R^I$. Then there exists the least element $a_0 = \min \mathbf{P}_A(x)$ of the set $\mathbf{P}_A(x)$, and we have $a_0 = x - r\mathbf{1}$ where $r = d_A(x)$.*

Proposition 2.3 (see [1]). *Let A be a closed upward set and $x \in R^I$. Then there exists the least element $w_0 = \max \mathbf{P}_A(x)$ of the set $\mathbf{P}_A(x)$, and we have $w_0 = x + r\mathbf{1}$ where $r = d_A(x)$.*

Corollary 2.1 (see [1]). *The following is valid for a closed downward set A and any $x \in R^I$,*

$$d_A(x) = \min\{\lambda \geq 0 : x - \lambda\mathbf{1} \in A\}.$$

3. I_m-QUASI UPWARD SETS AND THEIR BEST APPROXIMATIONS

In this section, we introduce a new definitions for some special subsets of R^I which are not necessarily downward or upward, but this sets can translate to downward or upward sets. We use the results obtained in downward and upward sets as a tool for founding the distance and best approximation of a point to new sets.

As previous section we consider that $I_m = \{i_1, i_2, \ldots, i_m\}$ be an arbitrarily subset of I. It is clear that m is the cardinality of I_m. We introduce the following definitions.

Definition 3.1. *A* set $U \subset R^I$ is called I_m-quasi upward if $R_u^{I_m} \subseteq U$ for all $u \in U$.

In particular, an I_m-quasi upward set U is downward, if $I_m = \varnothing$ and is upward, if $I_m = I$.

Definition 3.2. *A* set $U \subset R^I$ is called I_m-quasi downward, if its complement be an I_m-quasi upward (i.e, $\forall u \in U, coR_u^{I_m} \subseteq U$).

Proposition 3.1. *Let U be an I_m-quasi upward subset of R^I and $x \in R^I$. Then the following assertions are true:*

(1) *If $x \in U$, then $x + \varepsilon\mathbf{1}^{I_m} \in \operatorname{int} U$ for all $\varepsilon > 0$.*
(2) *We have* $\operatorname{int} U = \{x \in R^I : x - \varepsilon\mathbf{1}^{I_m} \in U$ *for some* $\varepsilon > 0\}$.

Proof. (1) Let $\varepsilon > 0$ and $x \in U$ be given and $N := \{y \in R^I : \|y - (x + \varepsilon\mathbf{1}^{I_m})\| \leq \varepsilon\}$ be an open neighborhood of $x + \varepsilon\mathbf{1}^{I_m}$. Then, by (2.3) we have,

$$N = \{y \in R^I : x + \varepsilon\mathbf{1}^{I_m} - \varepsilon\mathbf{1} \leq y \leq x + \varepsilon\mathbf{1}^{I_m} + \varepsilon\mathbf{1}\}.$$

By definition of $R_x^{I_m}$ and that U is an I_m-quasi upward set, it follows that $N \subset U$, and so $x + \varepsilon\mathbf{1}^{I_m} \in \operatorname{int} U$.

(2) Let $x \in \operatorname{int} U$. Then there exists $\varepsilon_0 > 0$ such that the closed ball $B(x, \varepsilon_0) \subset U$. In view of (2.3), we get $x - \varepsilon_0 \mathbf{1}^{I_m} \in U$.

Conversely, suppose that there exists $\varepsilon > 0$ such that $x - \varepsilon \mathbf{1}^{I_m} \in U$. Then, by part (1) we have $x = (x - \varepsilon \mathbf{1}^{I_m}) + \varepsilon \mathbf{1}^{I_m} \in \operatorname{int} U$, which completes the proof. □

Assumption 3.1. If A be an I_m-quasi upward subset of R^I with respect to the subset $I_m = \{i_1, i_2, \dots, i_m\}$ of I; we denote A by A^{I_m}.

Definition 3.3. Consider two I_m-quasi upward subset A^{I_m} and B^{J_m} of R^I, where $I_m = \{i_1, i_2, \dots, i_m\}$ and $J_m = \{j_1, j_2, \dots, j_m\}$. We say that A^{I_m} and B^{J_m} are homogeneous if $I_m = J_m$.

Lemma 3.1. *The following assertions are true:*

(1) *if A is an I_m-quasi upward set in R^I, then so is the shift $x + A$ for $x \in R^I$.*
(2) *If A is an I_m-quasi upward set in R^I, then* $\operatorname{int} A$ *is so.*
(3) *Let $(A_t)_{t\in T}$ be a family of I_m-quasi upward sets where T is an arbitrary set of indices, then $\bigcap_{t\in T} A_t$ and $\bigcup_{t\in T} A_t$ are I_m-quasi upward.*

Proof. The proof is trivial. □

Consider that I_m is a subset of I. In the following we define two useful maps:

We define $T_m : R^I \to R^I$ by $T_m(x) = y = (y_i)_{i\in I}$ for each $x \in R^I$ where:

$$y_i = \mathbf{1}^{I_m} \cdot x_i \tag{3.1}$$

and $(coT)_m := R^I \to R^I$ by $(coT)_m(x) = y = (y_i)_{i\in I}$ such that:

$$y_i = -\mathbf{1}^{I_m} \cdot x_i \tag{3.2}$$

Lemma 3.2. *The maps T_m defined by* (3.1) *and $(coT)_m$ defined by* (3.2) *are difeomorphism. Moreover*

$$T_m(\mathbf{1}^{I_m}) = 1, \quad T_m(\mathbf{1}) = 1^{I_m} \tag{3.3}$$

and

$$(coT)_m(\mathbf{1}^{I_m}) = -1, \quad (coT)_m(\mathbf{1}) = -1^{I_m} \tag{3.4}$$

Proof. The proof is trivial. □

Theorem 3.1. *Let $U \subset R^I$ be an I_m-quasi upward set, T_m be the map defined by (3.1) and $(coT)_m$ be the map defined by (3.2), then $T_m(U)$ is upward, and $(coT)_m(U)$ is downward.*

Proof. By Definition 2.2, $T_m(U)$ is upward if and only if for $x \in R^I$, and $h \in T_m(U)$, $x \geq h$ implies that $x \in T_m(U)$. Let $h \in T_m(U)$, By Lemma 3.2 there exists $u \in U$ such that $T_m(u) = h$, as $x \geq h$ then for each $i \in I$, $x_i \geq h_i$. Then by (3.1) we have

$$\begin{cases} x_i \geq u_i & \text{if } i \in I_m \\ -x_i \leq u_i & \text{if } i \in I \backslash I_m \end{cases}.$$

Since $u \in U$ and U is I_m-quasi upward, we conclude that $w = (w_i)_{i \in I} \in U$, where $(w_i)_{i \in I}$ is defined by

$$w_i = \mathbf{1}^{I_m} \cdot x_i.$$

This shows that $x = T_m(w) \in T_m(U)$. Hence, $T_m(U)$ s upward. Similarly it can be shown that $(coT)_m(U)$ is downward. This completes the proof. □

Theorem 3.2. *Let $U \subset R^I$ be an I_m-quasi downward then then $T_m(U)$ is a downward set and $(coT)_m(U)$ is an upward set.*

Proof. It is similar to the proof of Theorem 3.1. □

By (3.1), (3.2) we conclude the following proposition.

Proposition 3.2. *Let $U \subset R^I$ be a closed I_m-quasi upward set and $x \in R^I$, if $r := d_U(x)$, $r' := d_{T_m(U)}(T_m(x))$, and $r'' = d_{(coT)_m(U)}((coT)_m(x))$, then $r = r' = r''$.*

Proof. Let $u \in U$, we have

$$\begin{aligned} \|T_m(x) - T_m(u)\| &= \max_{i \in I} \left| \big(T_m(x)\big)_i - \big(T_m(u)\big)_i \right| \\ &= \max\left\{ \max_{i \in I_m} |x_i - u_i|, \max_{i \in I \backslash I_m} |x_i - u_i| \right\} \\ &= \max_{i \in I} |x_i - u_i| \\ &= \|x - u\|. \end{aligned}$$

By taking inf we get $r = r'$. In other hand, we can similarly show that $r = r''$. This completes the proof. □

Remark 3.1. Let $U \subset R^I$ be a closed I_m-quasi downward, $x \in R^I$ then

$$\mathbf{P}_U(x) = \{u \in U : T_m(u) \in \mathbf{P}_{T_m(U)}(T_m(x))\}.$$

Proposition 3.3. *Let $U \subset R^I$ be a closed I_m-quasi upward set, $x \in R^I$ and T_m be as in* (3.1)*, then the following assertions are true:*

$$\begin{aligned} \text{i)} \quad \mathbf{P}_U(x) &= \{u \in U : T_m(u) \in \mathbf{P}_{T_m(U)}(T_m(x))\} \\ &= \{u \in U : (coT)_m(u) \in \mathbf{P}_{(coT)_m(U)}((coT)_m(x))\}, \\ \text{ii)} \quad d_U(x) &= \min\{\lambda \geq 0 : T_m(x) - \lambda \mathbf{1} \in T_m(U)\} \\ &= \min\{\lambda \geq 0 : (coT)_m(x) - \lambda \mathbf{1} \in (coT)_m(U)\}. \end{aligned}$$

Proof. i) This follows by Lemma 3.2 and Proposition 3.2.

ii) This follows by Corollary 2.1 and Proposition 3.2. □

Proposition 3.4. *Let $U \subset R^I$, $x \in R^I$ and $r := d_U(x)$, then*

(i) *If U is a closed m-quasi upward, then $u_0 = x + r\mathbf{1}^{I_m} \in \mathbf{P}_U(x)$,*
(ii) *If U is a closed m-quasi downward, then $d_0 = x - r\mathbf{1}^{I_m} \in \mathbf{P}_U(x)$.*

Proof. i) By Theorem 3.1, $T_m(U)$ is an upward set. Therefore, by Proposition 2.3, $w_0 = \max \mathbf{P}_{T_m(U)}(T_m(x))$ exists and $w_0 = T_m(x) + r\mathbf{1}$ then by Proposition 3.3 and Lemma 3.2, we get $u_0 = x + r\mathbf{1}^{I_m} = T_m^{-1}(T_m(x) + r\mathbf{1}) = T_m^{-1}(w_0) \in \mathbf{P}_U(x)$.

ii) By Theorem 3.2, $T_m(U)$ is a downward set, hence, by Proposition 2.2, $w_0 = \min \mathbf{P}_{T_m(U)}(T_m(x))$ exists and $w_0 = T_m(x) - r\mathbf{1}$, then by Remark 3.1, we get $d_0 = T_m^{-1}(w_0) \in \mathbf{P}_U(x)$. □

Corollary 3.1. *Let $U \subset R^I$ be a closed I_m-quasi upward set and $x \in R^I$, then best approximation of x with respect to U is unique.*

Proof. Since $(coT)_m(U)$ is a closed strictly downward set, by Proposition 2.1, best approximation of $T_m(x)$ with respect to $(coT)_m(U)$ is unique. This with Proposition 3.3, completes the proof. □

Definition 3.4. A family $(A_t)_{t \in T}$ of subsets of a normed linear space X is called linearly regular, if there exists a constant $c > 0$ such that for all $x \in X$

$$d_{\bigcap_{t \in T} A_t}(x) \leq c \sup_{t \in T} d_{A_t}(x)$$

Theorem 3.3. *Let $(A_t)_{t\in T}$ be a family of closed I_m-quasi upward subsets of R^I, where T is an arbitrary indices set such that $A = \bigcap_{t\in T} A_t$ is not empty. Then for each $x \in R^I$, we have*

$$d_A(x) = \sup_{t\in T} d_{A_t}(x). \tag{3.5}$$

Proof. Let $x \in R^I$ and $r_t := d_{A_t}(x)$. Since for each $t \in T, A \subseteq A_t$ and $r_t \leq d_A(x)$, hence $s := \sup_{t\in T} r_t \leq d_A(x)$. Thus, if $s = \infty$ then $d_A(x) = \infty$, and we have (3.4). Assume now that $s < \infty$, by Proposition 3.4 we have $u_t = x + r_t\mathbf{1}^{I_m} \in A_t$ for each $t \in T$. Let $w = x + s\mathbf{1}^{I_m}$. Since $r_t \leq s$, and A_t is closed I_m-quasi upward, we conclude that $w \in A_t$ for each $t \in T$, thus $w \in \bigcap_{t\in T} A_t = A$. In other hand we have $\|x - w\| = s$. Thus $d_A(x) \leq s$. Consequently $d_A(x) = s$, which proves (3.5). □

Remark 3.2. A family $(A_t)_{t\in T}$ of homogeneous and closed I_m-quasi upward subsets of R^I is linearly regular.

4. POSITIVE I_m-QUASI UPWARD AND POSITIVE I_m-QUASI DOWNWARD SETS

Definition 4.1. A set $V \subset R^I_+$ is called positive I_m-quasi upward if $(R^{I_m}_v)_+ \subset V$ for each $v \in V$.

Definition 4.2. Let $V \subset R^I_+$ be positive I_m-quasi upward set. The intersection of all homogeneous I_m-quasi upward sets which contains V is again I_m-quasi upward set, and is called I_m-quasi upward hull of V. We shall denote it by V^*.

Let us indicate some properties of the I_m-quasi upward hull of a positive I_m-quasi upward set in the following.

Proposition 4.1. *Let V^* be the I_m-quasi upward hull of $V \subset R^I_+$, then*

(1) $V^* = \{x \in R^I : Pr^{I_m}(x) \text{ and } x^+ \in V\}$,
(2) $V = V^* \bigcap R^I_+$.

Proof. Let $A = \{x \in R^I : Pr^{I_m}(x), x^+ \in V\}$. First we Prove that A is I_m-quasi upward. Let $a \in A$, We show that $R^{I_m}_a \subseteq A$. Let $x \in R^{I_m}_a$, therefore:

$$\begin{cases} x_i \geq a_i & \text{if } i \in I_m \\ x_i \leq a_i & \text{if } i \in I\backslash I_m \end{cases}, \tag{4.1}$$

As $a \in A$, thus, $Pr^{I_m}(a), a^+ \in V$. Let $y = Pr^{I_m}(x)$, by (2.2) we get

$$y_i = \begin{cases} x_i & \text{if } i \in I_m \\ 0 & \text{if } i \in I \backslash I_m \end{cases}.$$

On the other hand by (4.1) we have

$$\begin{cases} x_i^+ \geq a_i^+ & \text{if } i \in I_m \\ x_i^+ \leq a_i^+ & \text{if } i \in I \backslash I_m \end{cases}.$$

We have $y, a^+ \in V$, since V is positive I_m-quasi upward set. Thus, $x \in A$. This proves that A is I_m-quasi upward. Because $V \subseteq A$, hence $V^* \subseteq A$. To prove $A \subset V^*$, let $x \in A$ thus $x^+ \in V \subseteq V^*$ and $Pr^{I_m}(x) = Pr^{I_m}(x^+)$. Also for $i \in I_m$, $x_i^+ \geq x_i$. Therefore, V^* is I_m-quasi upward then $x \in V^*$, thus $A \subset V^*$ which completes the proof.

(2) It is immediately a consequence of the first part. □

Proposition 4.2. *Let V^* be a closed I_m-quasi upward hull of $V \subseteq R_+^I$ and $x \in R_+^I$, then $d_V(x) = d_{V^*}(x)$.*

Proof. $V \subset V^*$, then for $d_{V^*}(x) \leq d_V(x)$, since $\|.\| = \|.\|_\infty$ we have

$$\begin{aligned} \|x - v\| &= \max_{i \in I} |x_i - v_i| \\ &= \max\left\{ \max_{i \in I_m} |x_i - v_i|, \max_{i \in I \backslash I_m} |x_i - v_i| \right\} \\ &\geq \max\left\{ \max_{i \in I_m} |x_i - v_i^+|, \max_{i \in I \backslash I_m} |x_i - v_i^+| \right\} \\ &= \max_{i \in I} |x_i - v_i^+| = \|x - v^+\| \geq d_V(x). \end{aligned}$$

Then $\inf_{v \in V^*} \|x - v\| = d_{V^*}(x) \geq d_V(x)$, which completes the proof. □

5. EMBEDDED I_m-QUASI UPWARD

Definition 5.1. Let X be a normed space and W be a subset of X. Then we say that $W \subseteq X$ is an embedded I_m-quasi upward (embedded I_m-quasi downward) if there exists a map $\varphi : X \to R^I$ such that $\varphi(W)$ is I_m-quasi upward (I_m-quasi downward).

Proposition 5.1. *Let X, Y are two normed space, $W \subset X$ and $\psi : X \longrightarrow Y$, is bijective. Then W is an embedded I_m-quasi upward, if and only if $\psi(W)$ is an embedded I_m-quasi upward.*

Proof. Suppose that W is an embedded I_m-quasi upward set, then there exists a map φ, such that $\varphi : X \longrightarrow R^I$, and $\varphi(W)$ is I_m-quasi upward. Now let $\theta : Y \longrightarrow R^I$ define by $\theta = \varphi o \psi^{-1}$. We have $\theta(\psi(W))$ is I_m-quasi upward, then by Definition 5.1 $\psi(W)$ is an embedded I_m-quasi upward. Now suppose $\psi(W)$ be an embedded I_m-quasi upward, then there exists a map τ, such that $\tau : Y \longrightarrow R^I$, and $\tau(\psi(W)$ is I_m-quasi upward. Now $\rho = \psi o \tau$, proves (similarity) that W is an embedded I_m-quasi upward. □

Let X be a vector lattice (see [7]). Recall (see [6]) that an element $\mathbf{1}_X \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in R$ such that $x \leq \mathbf{1}_X$: Using a strong unit $\mathbf{1}_X$ we can define a norm on X by

$$\|x\| := \inf\{\lambda \geq 0 : |x| \leq \mathbf{1}_X\} \quad (x \in X).$$

Definition 5.2. A linear mapping from a Banach lattice to a Banach lattice is positive if it carries positive vectors to positive vectors. This is equivalent to saying that the mapping is order preserving.

Corollary 5.1. *Let X be a Banach lattices with strong units $\mathbf{1}_X$. Let $T : X \longrightarrow R^I$ be an injective positive operator which satisfies the following statements:*

(1) *T^{-1} is positive operator,*
(2) $T(\mathbf{1}_X) = \mathbf{1}$
Then T is a norm isometry, that is $\|T(x)\| = \|x\|$ for all $x \in X$.

Proof. (see [5, Proposition 2.3]). □

Proposition 5.2. *Let X be a lattice space, W be a closed subset of X and $\varphi : X \to R^I$, be an injective positive operator such that T^{-1} is a positive operator. If $\varphi(W)$ be an I_m-quasi upward subset of R^I, then the following assertions are true.*

(1) *W is an embedded I_m-quasi upward and proximinal subset of X.*
(2) *if $x \in X$, $r = dist(x, W)$, then $\varphi^{-1}(\varphi(x) + r\mathbf{1}^{I_m}) \in P_W(x)$.*

Proof. (1) By Definition 5.1 W is an embedded I_m-quasi upward. Also it is clear that $\varphi(W) \subseteq R^I$, is a proximinal set. By Corollary 5.1, φ is a norm isometry thus we have $\|\varphi(x)\| = \|x\|$ for all $x \in X$. Then for all $x \in X$, and $w, w_0 \in W$,

$$\|x - w_0\| \leq \|x - w\| \Leftrightarrow \|\varphi(x) - \varphi(w_0)\| \leq \|\varphi(x) - \varphi(w)\|.$$

Therefore, $\varphi(\mathbf{P}_W(x)) = \mathbf{P}_{\varphi(W)}(\varphi(x))$, which proves that W is a proximinal set in X.

(2) As $\varphi(W)$ is an I_m-quasi upward, by Proposition 3.4, $\varphi(x) + r\mathbf{1}^{I_m} \in \mathbf{P}_{\varphi(W)}(\varphi(x))$. This proves that $\varphi^{-1}(\varphi(x) - r\mathbf{1}^{I_m}) \in P_W(x)$. □

Corollary 5.2. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively, and ϕ be a positive operator between X and Y, which is also an isomorphism. Then for each subset W of X, we have,*

(1) *W is an embedded I_m-quasi upward and proximinal subset of X, if and only if $\phi(W)$ is an embedded I_m-quasi upward and proximinal subset of Y.*
(2) *W is an embedded I_m-quasi upward and chebyshev subset of X, if and only if $\phi(W)$ is an embedded I_m-quasi upward and chebyshev subset of Y.*

Proof. The proof is a consequence of Proposition 5.1 and [5, Corollary 2.6]. □

REFERENCES

1. J.-E. Martinez-Legaz, A.M. Rubinov, and I. Singer (2002). Downward sets and their separation and approximation properties. *J. Global Optimization* 23:111–137.
2. A.M. Rubinov and I. Singer (2002). Best approximation by normal and conormal sets. *J. Approximation Theory* 107:212–243.
3. A.M. Rubinov (2000). *Abstract Convexity and Global Optimization*. Kluwer Academic Publishers, Boston.
4. A.M. Rubinov and A.J. Zaslavski (2002). Two porosity results in monotonic analysis. *Numerical Functional Analysis and Optimization* 23:651–668.
5. S.M.S. Modarres and M. Dehghani (2009). New results for best approximation on Banach lattices. *Nonlinear Analysis* 70:3342–3347.
6. B.Z. Vulikh (1967). *Introduction to the Theory of Partially Ordered Vector Spaces*. Wolters-Noordhoff, Groningen.
7. G. Birkhoff (1967). *Lattice Theory, Colloquium Publications*. Vol. 25. American Mathematical Society, Providence, RI.
8. V.L. Makarov and A.M. Rubinov (1977). *Mathematical Theory of Economic Dynamics and Equilibria*. Springer-Verlag, New York.