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ON BEST SIMULTANEOUS APPROXIMATION IN QUOTIENT SPACES

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Abstract. We assume that X is a normed linear space, W and M are subspaces of X. We develop a theory of best simultaneous approximation in quotient spaces and introduce equivalent assertions between the subspaces W and $W+M$ and the quotient space W/M.

Key words: *quotient space, best simultaneous approximation, simultaneous pseudo-Chebyshev, simultaneous quasi-Chebyshev, simultaneous weakly-Chebyshev*

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1 Introduction

The theory of best simultaneous approximation has been studied by many authors, e.g., [2,4,6,7,15,16,18,19,20]. Most of these work have dealt with the characterization of best simultaneous approximation in the space of continuous functions with values in a Banach space. The results on best simultaneous approximation in general Banach space can be found in [7,15]. Some existence results for best simultaneous approximation in the Banach space of p-Bochner integrable functions have been obtained in [16]. The characterizations of best simultaneous approximation for closed convex sets can be found in [10,11].

In this paper, we use the concept simultaneous proximinality and simultaneous Chebyshevity to introduce a theory of best simultaneous approximation in quotient spaces. The structure of this paper is as follows. In section 2, we give some preliminary results on simultaneous proximinal sets. In section 3, we give characterizations of best simultaneous approximation in quotient spaces. In section 4, we investigate simultaneous pseudo-Chebychevity in quotient spaces. We present characterizations of simultaneous quasi-Chebyshevity and weakly-Chebyshevity in sections 5 and 6.

2 Preliminaries

Let X be a normed linear space. For a non-empty subset W of X and each $x \in X$, we define

$$d(x,W) = \inf_{w \in W} \|x - w\|.$$

We recall (see [1,17]) that a point $w_0 \in W$ is called a best approximation for $x \in X$, if

$$\|x - w_0\| = d(x,W).$$

If each $x \in X$ has at least one best approximation $w \in W$, then W is called a proximinal subset of X. If each $x \in X$ has a unique best approximation in W, then W is called a Chebyshev subset of X.

Let W be a subset of X. For each $x \in X$ we denote by $\mathbf{P}_W(x)$ the set of all best approximations of x in W and define as follows:

$$\mathbf{P}_W(x) := \{w \in W : \|x - w\| = d(x,W)\}.$$

Let X be a normed linear space, W be a subset of X and S be a bounded set in X. We define

$$d(S,W) := \inf_{w \in W} \sup_{s \in S} \|s - w\|. \tag{2.1}$$

An element $w_0 \in W$ is called a best simultaneous approximation to S from W if

$$d(S,W) = \sup_{s \in S} \|s - w_0\|.$$

The set of all best simultaneous approximations to S from W will be denoted by $\mathbf{S}_W(S)$, and we have

$$\mathbf{S}_W(S) := \left\{ w \in W : \sup_{s \in S} \|s - w\| = d(S,W) \right\}. \tag{2.2}$$

It is well-known that $\mathbf{S}_W(S)$ is a bounded subset of X and if W is a closed and convex subset of X, then $\mathbf{S}_W(S)$ is closed and convex.

If for each bounded set S in X there exists at least one best simultaneous approximation to S from W, then W is called a simultaneous proximinal subset of X.

If for each bounded set S in X there exists a unique best simultaneous approximation to S from W, then W is called a simultaneous Chebyshev subset of X.

From the above definitions the following assertions are obvious:

1) $d(S+x, W+x) = d(S,W)$ for every $x \in X$.

2) $d(\lambda S, \lambda W) = |\lambda| d(S,W)$ for every $\lambda \in \mathbf{R}$.

3) $\mathbf{S}_{W+x}(S+x) = \mathbf{S}_W(S) + x$ for every $x \in X$.

4) $\mathbf{S}_{\lambda W}(\lambda S) = \lambda \mathbf{S}_W(S)$ for every $\lambda \in \mathbf{R}$.

Definition 2.1[3]. Let M be a subspace of a normed linear space X . The quotient space X/M is the set of all cosets $x+M$ of M with the following operations:

$$(x+M)+(y+M) = (x+y)+M,$$
$$\lambda(x+M) = \lambda x+M,$$

for every $x, y \in X$ and arbitrary scalar λ. Then the quotient space X/M is a normed linear space with the norm

$$\|x+M\| = \inf_{m\in M} \|x-m\|.$$

We recall that the canonical map $\pi : X \longrightarrow X/M$ which is defined by $\pi(x) = x+M$, is linear, continuous and open mapping. Let $f \in M^{\perp}$ be arbitrary . Define the linear functional T_f on X/M by $T_f(x+M) = f(x)$ for all $x+M \in X/M$. Then $T_f \in (X/M)^{\star}$ (the dual space of X/M).

We recall (see [17]) that for an arbitrary non-empty convex set A in X the linear manifold spanned by A which is denoted by $\ell(A)$ is defined as follows:

$$\ell(A) := \{\alpha x+(1-\alpha)y : x, y \in A;\ \alpha \text{ is scalar}\}.$$

For every fixed $y \in A$ the set $\ell(A-y)$ is a linear subspace of X satisfying

$$\ell(A-y) = \ell(A)-y := \{x-y : x \in \ell(A)\}.$$

It is clear that for an arbitrary non-empty convex set A in X

$$\ell(\pi(A)) = \pi(\ell(A)).$$

The dimension of A is defined by

$$\dim A := \dim \ell(A).$$

Then for every $y \in A$ we have,

$$\dim A = \dim \ell(A) = \dim[\ell(A)-y] = \dim \ell(A-y) = \dim(A-y).$$

For more details see [17].

3 Simultaneous Proximinality and Simultaneous Chebyshevity in Quotient Spaces

In this section, we give characterizations of best simultaneous proximinality and simultaneous Chebyshevity in quotient spaces. The characterizations of best approximation in quotient spaces have been obtained in [5,14]. We start with the following lemma.

Lemma 3.1. *Let X be a normed linear space, and M a proximinal subspace of X, then for each non-empty bounded set S in X, we have*

$$d(S,M) = \sup_{s\in S} \inf_{m\in M} \|s-m\|.$$

Proof. Since M is proximinal, it follows that for each $s \in S$, there exists $m_s \in \mathbf{P}_M(S)$ such that

$$\|s-m_s\| = \inf_{m\in M} \|s-m\|. \tag{3.1}$$

Hence, in view of (3.1), we have

$$\begin{aligned} d(S,M) &= \inf_{m\in M} \sup_{s\in S} \|s-m\| \\ &\leq \sup_{s\in S} \|s-m_s\| \\ &= \sup_{s\in S} \inf_{m\in M} \|s-m\| \\ &\leq \inf_{m\in M} \sup_{s\in S} \|s-m\| \\ &= d(S,M). \end{aligned}$$

This implies that

$$d(S,M) = \sup_{s\in S} \inf_{m\in M} \|s-m\|,$$

which completes the proof.

Lemma 3.2. *Let X be a normed linear space, M a proximinal subspace of X, and let S be an arbitrary subset of X. Then the following assertions are equivalent:*

i) *S is a bounded subset of X,*

ii) *S/M is a bounded subset of X/M.*

Proof. (i) $\Rightarrow$ (ii). It is obvious that if S is a bounded subset of X, then S/M is a bounded subset of X/M, because for each $s \in S$ we have

$$\|s+M\| = \inf_{m\in M} \|s-m\| \leq \|s\|.$$

(ii)$\Rightarrow$ (i). Suppose that S/M is a bounded subset of X/M. Thus

$$\sup_{s\in S} \|s+M\| < \infty.$$

Since M is proximinal, then for each $s \in S$ there exists $m_s \in M$ such that $m_s \in \mathbf{P}_M(S)$.

This implies that for each $s \in S$, we get

$$\|s-m_s\| = \inf_{m\in M} \|s-m\|. \tag{3.2}$$

In view of Lemma 3.1, we conclude that

$$\sup_{s\in S} \|s-m_s\| = \sup_{s\in S} \inf_{m\in M} \|s-m\| = \inf_{m\in M} \sup_{s\in S} \|s-m\|.$$

Then for each $\varepsilon > 0$, there exists $m_\varepsilon \in M$ such that

$$\sup_{s\in S}\|s-m_\varepsilon\| \quad\leq\quad \sup_{s\in S}\|s-m_s\|+\varepsilon.$$

Hence, due to (3.2) we have

$$\begin{aligned}\sup_{s\in S}(\|s\|-\|m_\varepsilon\|) &\leq \sup_{s\in S}\|s-m_\varepsilon\| \\ &\leq \sup_{s\in S}\|s-m_s\|+\varepsilon \\ &= \sup_{s\in S}\|s+M\|+\varepsilon.\end{aligned}$$

Therefore

$$\sup_{s\in S}\|s\| \leq \sup_{s\in S}\|s+M\|+\|m_\varepsilon\|+\varepsilon,$$

for each $\varepsilon > 0$, which completes the proof.

Lemma 3.3. *Let M be a proximinal subspace of a normed linear space X, and $W \supseteq M$ a subspace of X. Let S be a bounded set in X. If $\omega_0 \in \mathbf{S}_W(S)$, then $\omega_0 + M \in \mathbf{S}_{W/M}(S/M)$.*

Proof. Since S is a bounded set in X, it follows from Lemma 3.2 that S/M is a bounded set in X/M. Assume that $\omega_0 \in \mathbf{S}_W(S)$ and that $\omega_0 + M \notin \mathbf{S}_{W/M}(S/M)$. Thus there exists $\omega' \in W$ such that

$$\sup_{s\in S}\|s-\omega'+M\| < \sup_{s\in S}\|s-\omega_0+M\| \leq \sup_{s\in S}\|s-\omega_0\| = d(S,W). \tag{3.3}$$

On the other hand, for each $s \in S$ we have

$$\|s-\omega'+M\| = \inf_{m\in M}\|s-\omega'-m\|.$$

It follows that for each $\varepsilon > 0$ and each $s \in S$ there exists $m_s \in M$ such that

$$\|s-\omega'-m_s\| \leq \|s-\omega'+M\|+\varepsilon.$$

Since $\omega' + m_s \in W$, we conclude that

$$d(S,W) \leq \sup_{s\in S}\|s-(\omega'+m_s)\| \leq \sup_{s\in S}\|s-\omega'+M\|+\varepsilon. \tag{3.4}$$

Thus

$$d(S,W) \leq \sup_{s\in S}\|s-\omega'+M\|. \tag{3.5}$$

By (3.3) and (3.5) we have

$$d(S,W) \leq \sup_{s\in S}\|s-\omega'+M\| < d(S,W),$$

which is impossible.

Therefore, we have

$$\omega_0 + M \in \mathbf{S}_{W/M}(S/M),$$

this completes the proof.

Corollary 3.1. *Let M be a proximinal subspace of a normed linear space X, and $W \supseteq M$ a subspace of X. If W is simultaneous proximinal, then W/M is a simultaneous proximinal subspace of X/M.*

Proof. This is an immediate consequence of Lemma 3.3.

Corollary 3.2. *Let M be a proximinal subspace of a normed linear space X, and $W \supseteq M$ a subspace of X. If W is simultaneous proximinal, then for each bounded set S in X, we have*

$$\pi(\mathbf{S}_W(S)) \subseteq \mathbf{S}_{W/M}(S/M).$$

Proof. This is an immediate consequence of Lemma 3.3.

Proposition 3.1. *Let M be a proximinal subspace of a normed linear space X, and $W \supset M$ a subspace of X. If S is a bounded set in X such that*

$$\omega_0 + M \in \boldsymbol{S}_{W/M}(S/M) \text{ and } m_0 \in \boldsymbol{S}_M(S-\omega_0), \tag{3.6}$$

then

$$\omega_0 + m_0 \in \mathbf{S}_W(S).$$

Proof. In view of Lemma 3.1 and (3.6) we have

$$\begin{aligned}
\sup_{s\in S}\|s-\omega_0-m_0\| &= \inf_{m\in M}\sup_{s\in S}\|s-\omega_0-m\| \\
&= \sup_{s\in S}\inf_{m\in M}\|s-\omega_0-m\| \\
&= \sup_{s\in S}\|s-\omega_0+M\| \\
&\leq \sup_{s\in S}\|s-\omega+M\| \quad \forall\, \omega\in W \\
&\leq \sup_{s\in S}\|s-\omega\|, \qquad \forall\, \omega\in W.
\end{aligned}$$

Hence

$$\sup_{s\in S}\|s-(\omega_0+m_0)\| \leq \sup_{s\in S}\|s-\omega\|, \qquad \forall\, \omega\in W.$$

Since $\omega_0 + m_0 \in W$, we conclude that

$$\omega_0 + m_0 \in \mathbf{S}_W(S).$$

This completes the proof.

Theorem 3.1. *Let M be a proximinal subspace of a normed linear space X, and $W \supseteq M$ a simultaneous proximinal subspace of X. Then for each bounded set S in X, we have*

$$\pi(\mathbf{S}_W(S)) = \mathbf{S}_{W/M}(S/M).$$

Proof. By Corollary 3.2, we obtain

$$\pi(\mathbf{S}_W(S)) \subset \mathbf{S}_{W/M}(S/M).$$

It follows from Corollary 3.1 that W/M is simultaneous proximinal. Now, let

$$\omega_0 + M \in \mathbf{S}_{W/M}(S/M), \tag{3.7}$$

where $w_0 \in W$. By simultaneous proximinality of M, there exists $m_0 \in M$ such that

$$m_0 \in \mathbf{S}_M(S - \omega_0).$$

Then, in view of Proposition 3.1, we conclude that $\omega_0 + m_0 \in \mathbf{S}_W(S)$. Therefore

$$\omega_0 + M \in \pi(\mathbf{S}_W(S)),$$

and the proof is complete.

Corollary 3.3. *Let W and M be subspaces of a normed linear space X. If M is simultaneous proximinal, then the following assertions are equivalent:*

i) *W/M is simultaneous proximinal in X/M,*

ii) *$W+M$ is simultaneous proximinal in X.*

Proof. (i) $\Rightarrow$ (ii). Let S be an arbitrary bounded set in X. Then by lemma 3.2, we have S/M is a bounded set in X/M. Since $(W+M)/M = W/M$ and M are simultaneous proximinal, it follows that there exists $w_0 + M \in (W+M)/M$ and $m_0 \in M$ such that

$$w_0 + M \in \mathbf{S}_{(W+M)/M}(S/M) \quad \text{and} \quad m_0 \in \mathbf{S}_M(S - w_0).$$

Now, in view of Proposition 3.1, we get $w_0 + m_0 \in \mathbf{S}_{W+M}(S)$. This shows that $W+M$ is simultaneous proximinal in X.

(ii) $\Rightarrow$ (i). Since $W+M$ is simultaneous proximinal and $W+M \supseteq M$, then due to Corollary 3.1, we have $(W+M)/M = W/M$ is simultaneous proximinal.

Theorem 3.2. *Let W and M be subspaces of a normed linear space X. If M is simultaneous Chebyshev , then the following assertions are equivalent:*

i) *W/M is simultaneous Chebyshev in X/M,*

ii) *$W+M$ is simultaneous Chebyshev in X.*

Proof. (i) $\Rightarrow$ (ii). By hypothesis $(W+M)/M = W/M$ is simultaneous Chebyshev. Assume that (ii) is false , then some bounded subset S of X has two distinct simultaneous best approximations , say ℓ_0 and ℓ_1 in $W+M$. Thus, we have

$$\ell_0 \ \textit{and} \ \ell_1 \in \mathbf{S}_{W+M}(S). \tag{3.8}$$

Since $W+M \supseteq M$, it follows from Lemma 3.3 that

$$\ell_0 + M \text{ and } \ell_1 + M \in \mathbf{S}_{(W+M)/M}(S/M) = \mathbf{S}_{W/M}(S/M).$$

By hypothesis W/M is simultaneous Chebyshev, and so $\ell_0 + M = \ell_1 + M$. Then there exists $m_0 \in M \setminus \{0\}$ such that $\ell_1 = \ell_0 + m_0$.

Thus, in view of (3.8), we conclude that

$$\begin{aligned} \sup_{s\in S} \|(s-\ell_0)-m_0\| &= \sup_{s\in S} \|s-\ell_1\| \\ &= \sup_{s\in S} \|s-\ell_0\| \\ &= d(S, W+M) \\ &= d(S-\ell_0, W+M) \\ &\leq d(S-\ell_0, M). \end{aligned}$$

This shows that both m_0 and 0 are simultaneous best approximations to $S-\ell_0$ from M.

Hence M is not simultaneous Chebyshev. This is a contradiction.

(ii)$\Rightarrow (i)$. Assume that (i) does not hold. Then for some bounded subset S of X, S/M has two distinct simultaneous best approximations , say , $\omega + M$ and $\omega' + M$ belong to W/M. Thus, $\omega - \omega' \notin M$.

Since M is simultaneous proximinal, there exist simultaneous best approximations m and m' to $S-\omega$ and $S-\omega'$ from M, respectively.

Therefore, we have

$$m \in \mathbf{S}_M(S-\omega) \text{ and } m' \in \mathbf{S}_M(S-\omega').$$

Since $W+M \supseteq M$ and

$$\omega + M,\ \omega' + M \in \mathbf{S}_{W/M}(S/M) = \mathbf{S}_{(W+M)/M}(S/M),$$

it follows from Proposition 3.1 that

$$\omega + m \text{ and } \omega' + m' \in \mathbf{S}_{W+M}(S).$$

But, $W+M$ is simultaneous Chebyshev . Thus, we get $\omega + m = \omega' + m'$, and therefore $\omega - \omega' \in M$. This is a contradiction.

Theorem 2 is proved.

Corollary 3.4. *Let M be a simultaneous Chebyshev subspace of a normed linear space X. If $W \supset M$ is a subspace of X, then the following assertions are equivalent:*

i) *W is simultaneous Chebyshev in X,*

ii) *W/M is simultaneous Chebyshev in X/M.*

Proof. This is an immediate consequence of Theorem 3.2.

4 Simultaneous Pseudo-Chebyshevity in Quotient Spaces

The characterizations of pseudo-Chebyshevity in normed linear spaces are given in [8]. In this section, we present characterizations of simultaneous pseudo-Chebshevity in quotient spaces.

Definition 4.1. A closed subset W of a normed linear space X is called simultaneous pseudo-Chebyshev, if the set $\mathbf{S}_W(S)$ is non-empty and finite dimensional in X for every bounded set S in X.

Theorem 4.1. *Let M and W be subspaces of a normed linear space X such that $W+M$ is simultaneous proximinal . If M is finite dimensional , then the following assertions are equivalent:*

i) *W/M is simultaneous pseudo-Chebyshev in X/M,*

ii) *$W+M$ is simultaneous pseudo-Chebyshev in X.*

Proof. (i) $\Rightarrow$ (ii). Let S be an arbitrary bounded set in X and $\kappa_0 \in \mathbf{S}_{W+M}(S)$ be arbitrary. Since $W+M \supset M$ and $W/M = (W+M)/M$, it follows from Theorem 3.1 that

$$\begin{aligned}\pi[\ell(\mathbf{S}_{W+M}(S)-\kappa_0)] &= \ell[\pi(\mathbf{S}_{W+M}(S)-\kappa_0)]\\ &= \ell[\mathbf{S}_{(W+M)/M}(S/M)-(\kappa_0+M)]\\ &= \ell[\mathbf{S}_{W/M}(S/M)-(\kappa_0+M)].\end{aligned}$$

But, we have W/M is simultaneous pseudo-Chebyshev in X/M. Thus

$$\dim \ell[\mathbf{S}_{W/M}(S/M)-(\kappa_0+M)] < \infty,$$

and hence

$$\dim \pi[\ell(\mathbf{S}_{W+M}(S)-\kappa_0)] < \infty.$$

Since M is finite dimensional, we conclude that

$$\dim(\ell(\mathbf{S}_{W+M}(S)-\kappa_0) < \infty.$$

Therefore, $W+M$ is simultaneous pseudo-Chebyshev in X.

(ii) $\Rightarrow$ (i). Let S be an arbitrary bounded set in X. Since $W+M$ is simultaneous pseudo-Chebyshev, $\mathbf{S}_{W+M}(S)$ is a non-empty and finite dimensional set in X. In view of Corollary 3.3 and that $W+M$ and M are simultaneous proximinal, we get $(W+M)/M = W/M$ is simultaneous proximinal in X/M.

Thus, we have

$$\begin{aligned}\dim(\mathbf{S}_{W/M}(S/M)) &= \dim[\ell(\mathbf{S}_{W/M}(S/M))]\\ &= \dim[\ell(\mathbf{S}_{(W+M)/M}(S/M))]\\ &= \dim[\ell(\pi(\mathbf{S}_{W+M}(S))]\\ &= \dim[\ell(\mathbf{S}_{W+M}(S))]\\ &= \dim[\mathbf{S}_{W+M}(S)].\end{aligned}$$

Therefore, W/M is simultaneous pseudo-Chebyshev in X/M, which completes the proof.

Corollary 4.1. *Let M and $W \supset M$ be simultaneous proximinal subspaces of a normed linear space X. If M is finite dimensional, then the following assertions are equivalent:*

i) *W is simultaneous pseudo-Chebyshev in X,*

ii) *W/M is simultaneous pseudo-Chebyshev in X/M.*

Proof. This is an immediate consequence of Theorem 4.1.

5 Simultaneous Quasi-Chebyshevity in Quotient Spaces

The characterizations of quasi-Chebyshevity in Banach spaces can be found in [8,12]. In this section, we give characterizations of simultaneous quasi-Chebyshevity in quotient spaces. First, we start with the following definition.

Definition 5.1. A closed subset W of a normed linear space X is called simultaneous quasi-Chebishev, if the set $\mathbf{S}_W(S)$ is non-empty and compact in X for every bounded set S in X.

Theorem 5.1. *Let M and W be subspaces of a normed linear space X. If M is finite dimensional, then the following assertions are equivalent:*

i) *W/M is simultaneous quasi-Chebyshev in X/M,*

ii) *$W+M$ is simultaneous quasi-Chebyshev in X.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $(W+M)/M$ is simultaneous proximinal in X/M. Since M is also simultaneous proximinal, it follows from Corollary 3.3 that $W+M$ is simultaneous proximinal in X. Let S be an arbitrary bounded set in X, then $\mathbf{S}_{W+M}(S) \neq \emptyset$. We show that $\mathbf{S}_{W+M}(S)$ is compact . Let $\ell_n \in \mathbf{S}_{W+M}(S)(n=1,2,\cdots)$ be an arbitrary sequence. Then, for each $n \geq 1$ we have

$$\sup_{s\in S}\|s-\ell_n+M\| \leq \sup_{s\in S}\|s-\ell_n\| = d(S,W+M).$$

Therefore, for each $n \geq 1$ we obtain

$$\sup_{s\in S}\|s-\ell_n+M\| \quad \leq \quad \sup_{s\in S}\|s-\ell-m\|, \qquad \forall\, \ell \in W+M, \forall\, m \in M.$$

Since M is proximinal, in view of Lemma 3.1, we have

$$\begin{aligned}\sup_{s\in S}\|s-\ell_n+M\| &\leq \inf_{m\in M}\sup_{s\in S}\|s-\ell-m\| \\ &= \sup_{s\in S}\inf_{m\in M}\|s-\ell-m\|, \qquad \forall\, \ell \in W+M.\end{aligned}$$

It follows that

$$\sup_{s\in S}\|s-\ell_n+M\| \leq \sup_{s\in S}\|s-\ell+M\|, \qquad \forall\, \ell \in W+M,\ \forall\, n \geq 1.$$

Thus

$$\sup_{s\in S}\|s-\ell_n+M\| = d(S/M,(W+M)/M), \qquad \forall\, n \geq 1.$$

Hence

$$\ell_n + M \in \mathbf{S}_{(W+M)/M}(S/M), \qquad \forall\, n \geq 1.$$

Since $\mathbf{S}_{(W+M)/M}(S/M)$ is compact by hypothesis, one may choose $\ell_0 \in W+M$ with $\ell_0 + M \in \mathbf{S}_{(W+M)/M}(S/M)$ and $\{\ell_{n_k} + M\}_{k\geq 1}$ converges to $\ell_0 + M$ for some subsequence $\{\ell_{n_k} + M\}_{k\geq 1}$ of $\{\ell_n + M\}_{n\geq 1}$.

Now, for all $k \geq 1$ we have

$$\begin{aligned}\|\ell_0 - \ell_{n_k} + M\| &= \inf_{m\in M} \|\ell_0 - \ell_{n_k} - m\| \\ &= d(\ell_0 - \ell_{n_k}, M), \quad \forall\; k \geq 1.\end{aligned}$$

Since M is proximinal in X, there exists $m_{n_k} \in M$ such that

$$m_{n_k} \in \mathbf{P}_M(\ell_0 - \ell_{n_k}), \qquad \forall\, k \geq 1,$$

and hence

$$\|\ell_0 - \ell_{n_k} - m_{n_k}\| = d(\ell_0 - \ell_{n_k}, M), \qquad \forall\, k \geq 1.$$

Therefore

$$\lim_{k\to\infty} \|\ell_0 - \ell_{n_k} - m_{n_k}\| = 0. \tag{5.1}$$

On the other hand, $\{\ell_{n_k}\}_{k\geq 1}$ is a bounded sequence because $\ell_n \in \mathbf{S}_{W+M}(S)$. Hence, by (5.1), we have $\{m_{n_k}\}$ is a bounded sequence in M. Moreover, M is a finite dimensional subspace of X. Without loss of generality we may assume that $\{m_{n_k}\}_{k=1}^{\infty}$ converges to an element $m_0 \in M$ (otherwise, we may consider a suitable subsequence of $\{m_{n_k}\}_{k=1}^{\infty}$). Let $\ell' = \ell_0 - m_0$. Thus , $\ell' \in W+M$ and we have

$$\|\ell' - \ell_{n_k}\| = \|\ell_0 - m_0 - \ell_{n_k}\| \leq \|\ell_0 - \ell_{n_k} - m_{n_k}\| + \|m_{n_k} - m_0\|, \qquad \forall\, k \geq 1.$$

Thus

$$\lim_{k\to\infty} \|\ell' - \ell_{n_k}\| = 0.$$

Since $\ell_{n_k} \in \mathbf{S}_{W+M}(S)$ for all $k \geq 1$, and $\mathbf{S}_{W+M}(S)$ is closed, we conclude that $\ell' \in \mathbf{S}_{W+M}(S)$. Therefore, $\mathbf{S}_{W+M}(S)$ is compact .

(ii) $\Rightarrow$ (i). Since M *and* $W+M$ are simultaneous proximinal and $W+M \supseteq M$, it follows from Corollary 3.3 that $(W+M)/M = W/M$ is simultaneous proximinal in X/M.

Now, let S be an arbitrary bounded set in X, then $\mathbf{S}_{W/M}(S/M)$ is non-empty. By hypothesis, we have $W+M$ is simultaneous quasi-Chebyshev in X, and hence $\mathbf{S}_{W+M}(S)$ is compact in X. But , by Theorem 3.1 we have

$$\mathbf{S}_{(W+M)/M}(S/M) = \pi(\mathbf{S}_{W+M}(S)).$$

It follows that $\mathbf{S}_{W/M}(S/M)$ is compact , and therefore W/M is simultaneous quasi-Chebyshev in X.

Corollary 5.1. *Let M and $W \supseteq M$ be subspaces of a normed linear space X. If M is finite dimensional, then the following assertions are equivalent:*

i) *W is simultaneous quasi-Chebyshev in X,*

ii) *W/M is simultaneous quasi-Chebyshev in X/M.*

Proof. This is an immediate consequence of Theorem 5.1.

6 Simultaneous Weakly-Chebyshevity in Quotient Spaces

In this section, we give characterizations of simultaneous weakly-Chebyshevity in quotient spaces. The characterizations of weakly-Chebyshevity in Banach spaces have been obtained in [13].

Definition 6.1. A closed subset W of a normed linear space X is called simultaneous weakly-Chebyshev, if the set $\mathbf{S}_W(S)$ is non-empty and weakly compact in X for every bounded set S in X.

Theorem 6.1. *Let M and W be subspaces of a normed linear space X, and M simultaneous proximinal, then the following assertions are equivalent:*

i) *W/M is simultaneous weakly-Chebyshev in X/M,*

ii) *$W+M$ is simultaneous weakly-Chebyshev in X.*

Proof. (i) $\Rightarrow$ (ii). Since $(W+M)/M$ and M are simultaneous proximinal in X/M and X, respectively, hence by Corollary 3.3, we have $W+M$ is simultaneous proximinal in X. Let S be any bounded set in X. We show that $\mathbf{S}_{W+M}(S)$ is weakly compact . Let $\ell_n \in \mathbf{S}_{W+M}(S)$ $(n = 1,2,\cdots)$ be arbitrary. Then $\ell_n + M \in \mathbf{S}_{(W+M)/M}(S/M)$. Therefore

$$\sup_{s\in S} \|s-\ell_n\| = d(S,W+M), \qquad \forall\, n \geq 1,$$

and

$$\sup_{s\in S} \|s-\ell_n+M\| \leq \sup_{s\in S} \|s-\ell_n\| = d(S,W+M).$$

Then, we have

$$\sup_{s\in S} \|s-\ell_n+M\| \leq \inf_{\ell\in W+M} \sup_{s\in S} \|s-\ell+M\| = d(S/M,W/M), \qquad \forall\, n \geq 1.$$

Hence

$$\ell_n + M \in \mathbf{S}_{W/M}(S/M), \qquad \forall\, n \geq 1.$$

Since $\mathbf{S}_{(W+M)/M}(S/M) = \mathbf{S}_{W/M}(S/M)$ is weakly compact , there exists a subsequence $\{\ell_{n_k} + M\}_{k=1}^{\infty}$ of $\{\ell_n + M\}_{n=1}^{\infty}$ such that $\{\ell_{n_k} + M\}_{k=1}^{\infty}$ converges weakly to an element

$$\ell_0 + M \in \mathbf{S}_{(W+M)/M}(S/M).$$

Then, $\ell_0 + m_0 \in \mathbf{S}_{W+M}(S)$ for some $m_0 \in M$. But , for every $f \in X^\star$ we have $T_f \in (X/M)^\star$. Therefore

$$f(\ell_{n_k}) = T_f(\ell_{n_k} + M) \longrightarrow T_f(\ell_0 + M) = T_f(\ell_0 + m_0 + M) = f(\ell_0 + m_0).$$

Hence, $\{\ell_{n_k}\}_{k\geq 1}$ converges weakly to $\ell_0 + m_0 \in \mathbf{S}_{W+M}(S)$. Thus, $\mathbf{S}_{W+M}(S)$ is weakly compact, and hence $W + M$ is simultaneous weakly-Chebyshev in X.

(ii) $\Rightarrow$ (i). Since M and $W + M$ are simultaneous proximinal, it follows from Corollary 3.1 that W/M is simultaneous proximinal . Let S be an arbitrary bounded set in X. Then S/M is a bounded set in X/M. We show that $\mathbf{S}_{W/M}(S/M)$ is simultaneous weakly-Chebyshev. Let $\omega_n + M \in \mathbf{S}_{W/M}(S/M)$ $(n = 1, 2, \cdots)$ be arbitrary. Since M is simultaneous proximinal and for each $n \geq 1$, $S - \omega_n$ is a bounded set in X, it follows that there exists $m_n \in M$ such that

$$m_n \in \mathbf{S}_M(S - \omega_n).$$

Thus, in view of Lemma 3.1, we have

$$\begin{aligned}
\sup_{s\in S} \|s - \omega_n - m_n\| &= \inf_{m\in M} \sup_{s\in S} \|s - \omega_n - m\| \\
&= \sup_{s\in S} \inf_{m\in M} \|s - \omega_n - m\| \\
&= \sup_{s\in S} \|s - \omega_n + M\| \\
&= d(S/M, (W + M)/M) \\
&\leq d(S, W + M).
\end{aligned}$$

Therefore

$$\sup_{s\in S} \|s - (\omega_n + m_n)\| \leq d(S, W + M), \qquad \forall\, n \geq 1.$$

Since $\omega_n + m_n \in W + M$ $(n \geq 1)$, then

$$\omega_n + m_n \in \mathbf{S}_{W+M}(S), \quad \forall\, n \geq 1.$$

Since $\mathbf{S}_{W+M}(S)$ is non-empty and weakly compact, therefore there exists a subsequence $\{\omega_{n_k} + m_{n_k}\}_{k=1}^{\infty}$ of $\{\omega_n + m_n\}_{n=1}^{\infty}$ such that $\{\omega_{n_k} + m_{n_k}\}_{k=1}^{\infty}$ converges weakly to an element $\ell_0 \in \mathbf{S}_{W+M}(S)$. Then, by Corollary 3.1, $\ell_0 + M \in \mathbf{S}_{W/M}(S/M)$. Now, for every $f \in (X/M)^\star$, we get

$$f(\omega_{n_k} + M) = f(\omega_{n_k} + m_{n_k} + M) = f \circ \pi(\omega_{n_k} + m_{n_k}) \longrightarrow f \circ \pi(\ell_0) = f(\ell_0 + M).$$

It follows that $\{\omega_{n_k} + M\}_{k=1}^{\infty}$ converges weakly to $\omega_0 + M \in W/M$. Hence, $\mathbf{S}_{W/M}(S/M)$ is weakly compact for every bounded set S/M in X/M, and therefore W/M is simultaneous weakly-Chebyshev in X/M.

Corollary 6.1. *Let M be a simultaneous proximinal subspace of a normed linear space X, and $W \supset M$ a subspace of X. Then the following assertions are equivalent:*

i) *W is simultaneous weakly-Chebyshev in X,*

ii) *W/M is simultaneous weakly-Chebyshev in X/M.*

Proof. This is an immediate consequence of Theorem 6.1.

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