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Eigenvalue asymptotics for the Schrödinger operators on the hyperbolic plane

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Abstract

In this paper we consider the Schrödinger operator $H_V = -\frac{1}{2}\triangle_{\mathbb{H}} + V$ on the hyperbolic plane $\mathbb{H} = \{z = (x, y) \mid x \in \mathbb{R}, y > 0\}$, where $\triangle_{\mathbb{H}}$ is the hyperbolic Laplacian and V is a scalar potential on $\mathbb{H}$. It is proven that, under an appropriate condition on V at 'infinity', the number of eigenvalues of H_V less than λ is asymptotically equal to the canonical volume of the quasi-classically allowed region $\{(x, y; \xi, \eta) \in \mathrm{T}^*\mathbb{H} \mid y^2(\xi^2 + \eta^2)/2 + V(x, y) < \lambda\}$ as $\lambda \to \infty$. Our proof is based on the probabilistic methods and the standard Tauberian argument as in the proof of Theorem 10.5 in Simon (Functional Integration and Quantum Physics, Academic Press, New York, 1979).
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1. Introduction and main result

In this paper we study the asymptotic distribution of large eigenvalues of the Schrödinger operator of the form

$$H_V = -\frac{1}{2}\triangle_{\mathbb{H}} + V \tag{1.1}$$

on the hyperbolic plane $\mathbb{H}$. (The precise formulation is given below.)

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The investigation of spectral asymptotics for the Schrödinger operator has been one of the central problem in spectral theory, mathematical physics or other fields of mathematics. In particular, there is a very big collection of deep results in case of the (asymptotically) Euclidean domains (see, e.g., [7,9] and references therein).

In what follows we restrict ourselves to the case where the underlying space is non-compact without boundary and the spectrum of the Schrödinger operator is discrete. In the Euclidean case, a typical result is stated as follows: if V diverges at infinity (in an appropriate sense), the Schrödinger operator has compact resolvent and the number of eigenvalues (counting multiplicities) less than λ behaves like (a constant multiple of) the Lebesgue measure of the semi-classically allowed region $\{(x,p)\in\mathbb{R}^n\times\mathbb{R}^n \mid a(x,p)<\lambda\}$ as $\lambda\to\infty$, where $a(x,p)$ is a 'principal symbol' of the Schrödinger operator (see, e.g., [16, Theorem 10.5; 14, Theorem 30.1]).

It seems that less has been done in the case of general non-compact Riemannian manifolds. For a class of non-compact Riemannian manifolds, Matsumoto [12] obtained the large eigenvalue asymptotics for the Schrödinger operator with electro-magnetic fields, which diverges at infinity, roughly speaking.

The purpose of this paper is to obtain similar eigenvalue asymptotics for the operator (1.1). Our proof is based on the probabilistic methods as in the proof of Theorem 10.5 in [16]; using the Feynman–Kac representation of the heat kernel of the Schrödinger operator H_V, we have the short time asymptotics of the trace of the heat semigroup e^{-tH_V}, from which the results follows by the Tauberian theorem.

Let $\mathbb{H}=\{z=x+y\sqrt{-1}\in\mathbb{C}\mid x\in\mathbb{R},\ y>0\}$ be the hyperbolic plane endowed with the metric $g(\partial_i,\partial_j)=\delta_{ij}y^{-2}$, where δ_{ij} is Kronecker's delta and ∂_1,∂_2 stand for ∂_x,∂_y, respectively. With this metric, $\mathbb{H}$ is a complete Riemannian manifold with the Riemannian measure $m(dz)=y^{-2}\,dx\,dy$ and the Laplace–Beltrami operator is given by $\triangle_{\mathbb{H}}=y^2(\partial_x^2+\partial_y^2)$. The Riemannian distance $d_{\mathbb{H}}$ is given by

$$\cosh d_{\mathbb{H}}(z_1,z_2)=\frac{(x_1-x_2)^2+y_1^2+y_2^2}{2y_1y_2}, \tag{1.2}$$

where $z_i=x_i+y_i\sqrt{-1}\in\mathbb{H}$ for $i=1,2$.

Remark 1.1. Matsumoto [12] assumed that, in the geodesic coordinate, the Riemannian metric g and its derivatives of all order are bounded and g is uniformly positive definite. In our case, the geodesic (polar) coordinate (t,θ) is given by $(z-\sqrt{-1})/(z+\sqrt{-1})=\tanh(t/2)e^{\theta\sqrt{-1}}$. Then, in the coordinate, the Riemannian metric g is expressed as

$$\begin{pmatrix}1 & 0\\ 0 & (\sinh t)^2\end{pmatrix},$$

which is not bounded at infinity, hence, does not satisfy the assumption as in [12].

In addition, in the geodesic coordinate, the coefficients of the hyperbolic Laplacian $\triangle_{\mathbb{H}}$ vanish at infinity and the typical example of the scalar potential V we keep in mind (see Section 7 below) has exponential growth. It seems (to the authors) that it is not so easy to invoke the standard pseudo-differential technique as in [9,14].

To formulate the precise conditions on the scalar potentials, we define an auxiliary potential V_ε by

$$V_\varepsilon(z) = \sup\{V(z') \mid z' \in \mathbb{H}, d_{\mathbb{H}}(z, z') \leqslant \varepsilon\} \tag{1.3}$$

for any real-valued function V and for any positive number ε. Then we introduce the conditions (A.0)–(A.2) on V as follows:

(A.0) The scalar potential V is a real-valued, continuous function on $\mathbb{H}$. Moreover, V is bounded from below.

(A.1) In addition to (A.0), the quantity

$$\int_{\mathbb{H}} \exp(-tV(z))m(dz)$$

is finite for each $t > 0$.

(A.2) There exist positive constants γ, C_V such that

$$\lim_{\lambda \nearrow \infty} (2\pi)^{-2}\lambda^{-\gamma}\left|\left\{(x, y; \xi, \eta) \in \mathbb{H} \times \mathbb{R}^2 \,\middle|\, \frac{y^2}{2}(\xi^2 + \eta^2) + V(x, y) < \lambda\right\}\right| = C_V, \tag{1.4}$$

holds. Moreover, for any small $\varepsilon > 0$, there exists positive constant $C_{V,\varepsilon}$ such that

$$\lim_{\lambda \nearrow \infty} (2\pi)^{-2}\lambda^{-\gamma}\left|\left\{(x, y; \xi, \eta) \in \mathbb{H} \times \mathbb{R}^2 \,\middle|\, \frac{y^2}{2}(\xi^2 + \eta^2) + V_\varepsilon(x, y) < \lambda\right\}\right| = C_{V,\varepsilon}$$

and $\lim_{\varepsilon \searrow 0} C_{V,\varepsilon} = C_V$ hold. Here, $|\cdot|$ denotes the four-dimensional Lebesgue measure and the function V_ε is as in (1.3).

For any positive constants α, δ, the function $\alpha(\cosh(d_{\mathbb{H}}(z, \sqrt{-1})/2))^\delta$ is a typical example of V which satisfies (A.1) and (A.2) (we show this in Section 7). Nevertheless, (A.1) and (A.2) does not necessarily assure that V diverges in all directions at infinity.

We start with the Schrödinger operator (1.1) with domain $C_0^\infty(\mathbb{H})$, the space of all complex-valued smooth functions with compact support. First of all, we establish the essential self-adjointness of H_V.

Lemma 1.2. *Assume* (A.0). *Then H_V is essentially self-adjoint* (*i.e.*, *the operator closure of H_V is self-adjoint*). *Assume further* (A.1). *Then H_V has discrete spectrum* (*i.e.*, *the spectrum consists of isolated eigenvalues of finite multiplicity*).

Proof. The first assertion is a special case of Theorem 1.1 in [15]. The condition (A.1) implies that, for any $r>0$ and $A>0$, the Lebesgue measure of the set $\{z' \in \mathbb{H} \mid d_{\mathbb{H}}(z,z') < r, V(z) > A\}$ in the geodesic coordinate converges to 0 as $d_{\mathbb{H}}(z, \sqrt{-1}) \to \infty$. Then the second assertion follows from Kondrat'ev and Shubin [8, Corollary 6.2]. $\square$

We note that our proof below also shows that H_V has compact resolvent. In the sequel we use the same symbol for any essentially self-adjoint operator and the self-adjoint realization.

We denote by $N(H_V < \lambda)$ the number of eigenvalues (counting multiplicity) of H_V less than λ. The quantity $N(H_V < \lambda)$ is finite for each fixed λ by Lemma 1.2 under the condition (A.1). Note that, under condition (A.0), the operator H_V is semibounded since $-\triangle_{\mathbb{H}} \geqslant 1/4$.

The main result of this paper is the following:

Theorem 1.3. *Suppose that V satisfies the assumptions* (A.1) *and* (A.2). *Let γ and C_V be the constants as in* (A.2). *Then we have the asymptotic relation*

$$\lim_{\lambda \nearrow \infty} \lambda^{-\gamma} N(H_V < \lambda) = C_V.$$

Remark 1.4. The use of the standard Tauberian and Abelian theorems makes us impose the condition (A.2), which seems to be too restrictive. It is expected that the similar asymptotic relation still holds for wider class of potentials. Unfortunately, the authors have not obtain the result in that case.

The organization of the paper is as follows: In Section 2, we give some elementary results concerning the trace of operators and some function spaces. Section 3 contains some basic notions from stochastic analysis (the Brownian motion on $\mathbb{H}$, the generalized expectation and the pinned Wiener measure $P_t^{z,z'}$ on $\mathbb{H}$, etc.). In Section 4, we show a Fubini-like lemma with respect to the generalized expectation (see Lemma 4.4) and the continuity of the measure $P_t^{z,z'}$ with respect to the initial and end points (z,z') (Lemma 4.6). In Section 5, we establish the Feynman–Kac representation of the integral kernel of the semigroup e^{-tH_V} generated by H_V (Proposition 5.1) and give upper and lower estimates for the trace of the semi-group (Lemmas 5.3 and 5.6). In Section 6, we give a proof of Theorem 1.3. In Section 7, we give an example of the scalar potential which satisfies the conditions (A.1) and (A.2).

2. Results from functional analysis

Throughout this paper, we use the symbols c and C (possibly with some suffix) for various positive constants, which may vary from line to line.

First, we give two lemmas on the trace of non-negative self-adjoint operators. We say that a bounded operator A acting on a separable Hilbert space K is *of trace class* (*of Hilbert–Schmidt class*) if $\mathrm{Tr}(\sqrt{A^*A})$ $(\mathrm{Tr}(A^*A))$ is finite. Here the trace is defined by $\mathrm{Tr}(A) = \sum_{i=1}^{\infty} \langle e_i, Ae_i \rangle_K$ for an(y) orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for K. For further details, we refer to Reed and Simon [13].

Lemma 2.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of bounded, non-negative, self-adjoint operators of trace class on a separable Hilbert space K satisfying $\sup_n \mathrm{Tr}(A_n) \leqslant M$ for some positive constant M. Moreover, assume that A_n converges strongly to a bounded operator A. Then A is a non-negative, self-adjoint operator of trace class and, moreover, the inequality $\mathrm{Tr}(A) \leqslant M$ holds.*

Proof. This is a special case of Lemma 2 in [2], however we give a proof for our simple case.

Obviously, A is self-adjoint and non-negative. For an orthonormal basis $\{e_i\}_{n=1}^{\infty}$ of K, we have

$$\mathrm{Tr}(A) = \sum_{i}^{\infty} \langle Ae_i, e_i \rangle_K \leqslant \liminf_{n \to \infty} \sum_{i}^{\infty} \langle A_n e_i, e_i \rangle_K \leqslant M,$$

where we used Fatou's lemma in the first inequality. □

Lemma 2.2. *We have the following assertions* (i) *and* (ii):

(i) *Let A be a non-negative, self-adjoint operator of Hilbert–Schmidt class on $L^2(\mathbb{H})$ and let $a \in L^2(\mathbb{H} \times \mathbb{H})$ be the integral kernel of A, i.e., $Af(z) = \int f(z')a(z,z')m(dz')$. Moreover, assume that the kernel a is continuous on $\mathbb{H} \times \mathbb{H}$ and the integral $\int_H a(z,z)m(dz)$ is finite. Then A is of trace class and the inequality*

$$\mathrm{Tr}(A) \leqslant \int_{\mathbb{H}} a(z,z)m(dz).$$

holds.

(ii) *Let A be an operator of trace class on $L^2(\mathbb{H})$. Assume that there exists a continuous function a on $\mathbb{H} \times \mathbb{H}$ such that $Af(z) = \int_H a(z,z')f(z')m(dz)$ holds for all $f \in C_0^{\infty}(\mathbb{H})$. Then the trace of A is given by*

$$\mathrm{Tr}(A) = \int_{\mathbb{H}} a(z,z)m(dz).$$

Proof. This is essentially due to Brislawn [1, Theorems 3.1 and 4.3], which is formulated in the Euclidean space. We deduce the problem on $\mathbb{H}$ to the one on $\mathbb{R}^2$ via the unitary operator U from $L^2(\mathbb{H})$ to $L^2(\mathbb{R}^2)$ defined by $(Uf)(x,t) = f(x,e^t)e^{-t/2}$.

We show (i). It is clear that UAU^{-1} defines a non-negative, self-adjoint operator on $L^2(\mathbb{R}^2)$ and is of Hilbert–Schmidt class, in fact, a direct computation shows that

UAU^{-1} has the integral kernel $a(x, e^t; x', e^{t'})e^{-t/2}e^{-t'/2}$, which is continuous. Then Theorem 4.3 in [1] is applicable. The assertion (ii) follows similarly from Brislawn's result. $\square$

Next, we are concerned with some function spaces on $\mathbb{R}^n$. Let $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and the set of tempered distributions on $\mathbb{R}^n$, respectively. For each $k \in \mathbb{Z}$, we introduce the norms on $\mathscr{S}(\mathbb{R}^n)$ by $||f||_{2k} = ||(1 + |\cdot|^2 - \triangle/2)^k f||_\infty$. Here $\triangle$ stands for Laplacian on $\mathbb{R}^n$ and $||\cdot||_\infty$ stands for the supremum norm. We denote by $\mathscr{S}_{2k}$ the completion of $\mathscr{S}(\mathbb{R}^n)$ with respect to the norm $||\cdot||_{2k}$. Then it follows that

$$\bigcap_{k\in\mathbb{Z}} \mathscr{S}_{2k} = \mathscr{S}(\mathbb{R}^n) \quad \text{and} \quad \bigcup_{k\in\mathbb{Z}} \mathscr{S}_{2k} = \mathscr{S}'(\mathbb{R}^n).$$

For $r \in \mathbb{R}$ and $f \in \mathscr{S}(\mathbb{R}^n)$, we set

$$||f||_{2,r} = ||(1 - \triangle/2)^{r/2} f||_{L^2(\mathbb{R}^n)}.$$

We denote by $\mathscr{H}_{2,r}$ the completion of $\mathscr{S}(\mathbb{R}^n)$ with respect to the norm $||\cdot||_{2,r}$. Clearly, $\mathscr{H}_{2,r}$ is a subspace of $\mathscr{S}'(\mathbb{R}^n)$ for any $r \in \mathbb{R}$.

Lemma 2.3. *Let $f \in \mathscr{S}(\mathbb{R}^n)$, $r > 0$. For $K \in \mathrm{GL}(n;\mathbb{R})$, we set $f_K(x) = f(Kx)$. Then*

$$||f_K||^2_{2,-r} \leqslant |\det K|^{-1}(1 + ||K^{-1}||^2_{\mathrm{HS}})^r ||f||^2_{2,-r},$$

where $||\cdot||_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm on $\mathrm{GL}(n;\mathbb{R})$.

Proof. We use the Fourier transform

$$\begin{aligned}
\widehat{f_K}(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(Kx) \exp(\sqrt{-1}\langle x, \xi\rangle)\, dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \exp(\sqrt{-1}\langle K^{-1}x, \xi\rangle)|\det K|^{-1}\, dx \\
&= |\det K|^{-1} \hat{f}((K^t)^{-1}\xi),
\end{aligned}$$

where K^t denotes the transposed matrix of K.

Then we have

$$\begin{aligned}||f_K||_{2,-r}^2 &= |\det K|^{-2} \int_{\mathbb{R}^n} (1+|\xi|^2/2)^{-r} |\widehat{f}((K^t)^{-1}\xi)|^2\, d\xi \\ &= |\det K|^{-2} \int_{\mathbb{R}^n} (1+|K^t\xi|^2/2)^{-r} |\widehat{f}(\xi)|^2 |\det K^t|\, d\xi \\ &\leqslant |\det K|^{-1} \min\{1, ||(K^t)^{-1}||_{\mathrm{op}}^{-2}\}^{-r} \int_{\mathbb{R}^n} (1+|\xi|^2/2)^{-r} |\widehat{f}(\xi)|^2\, d\xi \\ &\leqslant |\det K|^{-1} (1+||K^{-1}||_{\mathrm{HS}}^2)^r ||f||_{2,-r}^2,\end{aligned}$$

where $||\cdot||_{\mathrm{op}}$ denotes the operator norm and we have used the fact that

$$||f||_{2,r} = ||(1+|\cdot|^2/2)^{r/2} \cdot \widehat{f}||_{L^2(\mathbb{R}^n)}$$

for any $r \in \mathbb{R}$. □

Lemma 2.4. *Let $\mathscr{H}_{2,r}$ and $\mathscr{S}_{2k}$ be as above. For any positive integer r, there exists a positive integer k such that $\mathscr{H}_{2,-r}$ is continuously embedded in $\mathscr{S}_{-2k}$.*

Proof. This follows from the general theory of globally hypoelliptic pseudo-differential operators (see [3, Chapter 1; 14, Chapter IV, Section 25]).

Let $h = -\triangle/2 + |x|^2$ be the n-dimensional harmonic oscillator with domain $\mathscr{S}(\mathbb{R}^n)$, which is essentially self-adjoint. For any real number k, we introduce an auxiliary weighted Sobelev space $\widetilde{B}_{2k} = \{u \in L^2(\mathbb{R}^n) \mid (1+\bar{h})^k u \in L^2(\mathbb{R}^n)\}$ equipped with the norm $||(1+\bar{h})^k u||_{L^2(\mathbb{R}^n)}$. Here we denote by $\bar{A}$ the operator closure of an operator A.

Then $h+1$ is a strictly positive, globally hypoelliptic pseudo-differential operator. Hence, for any non-negative integer k, the operator $\overline{(h+1)^k}$ coincides with the self-adjoint operator $(\bar{h}+1)^k$ with domain $\widetilde{B}_{2k}$ [3, Theorem 1.8.11] and the resolvent $(\bar{h}+1)^{-k}$ coincides with the operator $\overline{((h+1)^k)^{-1}}$, which can be constructed as a pseudo-differential operator of order $-2k$ [3, Theorem 1.11.1]. Moreover, it also holds that $(h+1)^k$ maps $\widetilde{B}_{2m}$ to $\widetilde{B}_{2(m-k)}$ continuously for any real numbers k, m [3, Proposition 1.6.3]. This is why we may identify h with $\bar{h}$ in the rest of the proof.

Given $r > 0$, we take and fix N such that $r \leqslant 2N$. Then, by the standard Calderón–Vaillancourt theorem, which says that the boundedness of a symbol implies the boundedness of the associated pseudo-differential operator, roughly speaking (see, e.g., [4, Theorem 5.3]), we can find that the operators $(1+h)^{-N}(1-\triangle/2)^{r/2}$ and $(1-\triangle/2)^{r/2}(1+h)^{-N}$ extend to bounded operators on $L^2(\mathbb{R}^n)$. This implies that the following estimates on $\mathscr{S}(\mathbb{R}^n)$:

$$||(1+h)^{-N}u||_{L^2(\mathbb{R}^n)} \leqslant c||(1-\triangle/2)^{-r/2}u||_{L^2(\mathbb{R}^n)},$$
$$||(1-\triangle/2)^{r/2}u||_{L^2(\mathbb{R}^n)} \leqslant ||(1+h)^N u||_{L^2(\mathbb{R}^n)},$$

which is equivalent to the facts that the embeddings $\mathscr{H}_{2,-r} \hookrightarrow \widetilde{B}_{-2N}$ and $\widetilde{B}_{2N} \hookrightarrow \mathscr{H}_{2,r}$ are continuous, respectively.

On the other hand, the usual Sobolev embedding theorem says that the embedding $\mathscr{H}_{2,m} \hookrightarrow L^\infty(\mathbb{R}^n)$ is continuous if we take and fix m such that $m > n/2$. Then the composition

$$\mathscr{H}_{2,-r} \hookrightarrow \widetilde{B}_{-2N} \xrightarrow{(1+h)^{-k}} \widetilde{B}_{2(k-N)} \hookrightarrow \mathscr{H}_{2,m} \hookrightarrow L^\infty(\mathbb{R}^n)$$

defines a continuous map if $m \leqslant 2(k-N)$. This show that, if we choose k as $2k > r + n/2$, then the estimate

$$||(h+1)^{-k}u||_\infty (= ||u||_{2k}) \leqslant c||u||_{2,-r}$$

holds for all $u \in \mathscr{S}(\mathbb{R}^n)$. This completes the proof. □

3. Brownian motion on the hyperbolic plane

3.1. Brownian motion on $\mathbb{H}$

In this subsection we introduce the Brownian motion on $\mathbb{H}$ and give the short time asymptotic behaviour of the heat kernel of $-\triangle_{\mathbb{H}}/2$. In what follows, we denote $z = x + y\sqrt{-1}$ by (x, y) via the inclusion $\mathbb{H} \hookrightarrow \mathbb{C} \cong \mathbb{R}^2$.

Let $W = \{w \in C([0,\infty);\mathbb{R}^2) \mid w(0) = 0\}$ be the two-dimensional Wiener space, where we denote by $C(\Omega;\Omega')$ the space of Ω'-valued continuous functions on Ω. Let

$$H = \left\{ h \in W \mid h \text{ is absolutely continuous and } ||h||_H^2 = \int_0^\infty |\dot{h}(s)|^2\, ds < \infty \right\}$$

be the Cameron–Martin subspace, where $\dot{}$ denotes the derivative, and let P be the Wiener measure on W. As usual we denote by $E[\cdot]$ the integration with respect to P. We denote by $\{w_t(= (w_t^1, w_t^2))\}_{t \geqslant 0}$ the canonical realization of the Wiener process.

We consider the following stochastic differential equation on $\mathbb{H}$:

$$dX(t) = Y(t)\, dw^1(t), \quad dY(t) = Y(t)\, dw^2(t),$$

with the initial condition $(X(0), Y(0)) = (x, y)$. The solution is explicitly written as follows (see [5, p. 69]):

$$X(t,z,w) = x + y \int_0^t \exp(w_s^2 - s/2)\, dw_s^1,$$
$$Y(t,z,w) = y \exp(w_t^2 - t/2). \tag{3.1}$$

We write $Z(t,z,w) = (X(t,z,w), Y(t,z,w))$ and when $z = \sqrt{-1}$ we simply write $Z(t,w) = (X(t,w), Y(t,w))$. Using Itô's formula (see [6]), one can find that $\{Z(t,z,w)\}_{t\geqslant 0}$ is the Brownian motion on $\mathbb{H}$, i.e., a diffusion process whose generator is $\triangle_{\mathbb{H}}/2$. It is well-known (see [6, Chapter 5, Section 3]) that the law of $Z(t,z,w)$ on $\mathbb{H}$ is given by $p(t,z,z')\, m(dz')$, where the heat kernel $p(t,z,z')$ of $e^{t\triangle_{\mathbb{H}}/2}$ is given by

$$p(t,z,z') = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_d^\infty \frac{be^{-b^2/2t}}{\sqrt{\cosh b - \cosh d}}\, db \tag{3.2}$$

with $d = d_{\mathbb{H}}(z,z')$ (see, e.g., [18]).

Lemma 3.1. *Let $p(t,z,z')$ be as above. Then*

$$\lim_{t\searrow 0} tp(t,z,z) = \frac{1}{2\pi} \tag{3.3}$$

holds for any $z \in \mathbb{H}$.

Proof. Although the short time asymptotics (3.3) for the heat kernel of the Laplace–Beltrami operator is well known (see, e.g., [6, p. 421]), we give an elementary proof in our case for the sake of completeness.

First we see that the quantity $C_0 = \sup_{x>0} x/\sqrt{\cosh x - 1}$ is finite since

$$\lim_{x\to 0} \frac{x^2}{\cosh x - 1} = 2, \quad \lim_{x\to\infty} \frac{x^2}{\cosh x - 1} = 0. \tag{3.4}$$

Set $f_t(x) = \sqrt{t}xe^{-x^2/2}/\sqrt{\cosh(\sqrt{t}x) - 1}$ for $t>0$ and $x>0$. Then we have the domination $|f_t(x)| \leqslant C_0 e^{-x^2/2}$ and the limit $\lim_{t\searrow 0} f_t(x) = \sqrt{2}e^{-x^2/2}$, where we used the first equality as in (3.4). By the dominated convergence theorem, we have

$$\lim_{t\searrow 0} \int_0^1 f_t(x)\, dx = \sqrt{2} \int_0^\infty e^{-x^2/2}\, dx = \sqrt{\pi}. \tag{3.5}$$

Then the result follows from

$$2\pi tp(t,z,z) = \frac{e^{-t}}{\sqrt{\pi}} \int_0^\infty f_t(x)\,dx$$
$$\to 1$$

as $t \searrow 0$, where we changed the variable $b \to \sqrt{t}x$ in the equality and used (3.5) in the last line. $\square$

3.2. Generalized expectations

Now we recall some basic definitions and results from the Malliavin calculus along the line of Ikeda and Watanabe [6, Chapter V, Sections 8 and 9]. Let (W,H,P) be the Wiener space as above. A function of the form $[h](w) = \sum_{i=1,2} \int_0^\infty \dot{h}^i(s) \cdot dw^i(s)$ is called *a measurable linear functional associated with* $h \in H$. The law of $[h]$ is the Gaussian measure with mean 0 and variance $||h||_H^2$. For any orthonormal elements $h_1, \dots, h_n \in H$ and $f \in \mathcal{S}(\mathbb{R}^n)$, a function of the form $F(w) = f([h_1](w), \dots, [h_n](w))$ is called *a cylindrical function*. The Ornstein–Uhlenbeck operator L is defined by

$$LF(w) = \sum_{i=1}^n (\partial_i^2 f([h_1](w), \dots, [h_n](w)) - [h_i](w) \cdot \partial_i f([h_1](w), \dots, [h_n](w)))$$

on the space of all cylindrical functions, which is a core for L. For $p \in (1, \infty)$ and $s \in \mathbb{R}$, the Sobolev space $\mathbf{D}_{p,s}$ is the completion of the space of cylindrical functions on W with respect to the norm $||F||_{p,s} = ||(I-L)^{s/2}F||_{L^p(W,P)}$. The spaces of *test functionals* and of *generalized Wiener functionals* are defined by $\mathbf{D}_\infty = \bigcap_{p>1, s \in R} \mathbf{D}_{p,s}$ and $\mathbf{D}_{-\infty} = \bigcup_{p>1,s\in R} \mathbf{D}_{p,s}$, respectively. (For a separable Hilbert space K, the Sobolev spaces of K-valued function(al)s are defined in a similar way. In that case we write $\mathbf{D}_{p,s}(K)$, $\mathbf{D}_\infty(K)$, etc.) Then the paring of $F \in \mathbf{D}_\infty$ and $\Psi \in \mathbf{D}_{-\infty}$ is defined in a canonical way and is denoted by $E[\Psi \cdot F]$. (We often write as $E[\Psi(w)F(w)]$.) The pairing $E[\Psi \cdot 1]$ is often denoted simply by $E[\Psi]$ or formally by $\int_W \Psi(w)P(dw)$. We call $E[\,\cdot\,]$ *the generalized expectation*.

Let $F = (F^1, \dots, F^n) \in \mathbf{D}_\infty(\mathbb{R}^n)$. We say that F *is non-degenerate in the sense of Malliavin* if the Malliavin covariance

$$\det(\langle DF^i, DF^j \rangle_H)^{-1}_{i,j=1,\dots,n}$$

belongs to $\bigcap_{p>1} L^p(W,P)$, where D denotes the H-derivative, i.e., the Gâtaux derivative in H-direction. If F is non-degenerate in the sense of Malliavin, then, for any Schwartz distribution $\psi \in \mathcal{S}'(\mathbb{R}^n)$, the composition $\psi \circ F$ is well defined and belongs to $\mathbf{D}_{-\infty}$. In fact, the mapping $\psi \mapsto \psi \circ F$ is bounded from $\mathcal{S}_{-2k}$ to $\mathbf{D}_{p,-2k}$ for every $p \in (1, \infty)$ and $k = 1, 2, \dots$ (see [6, Chapter V, Section 9] for detailed information on the pullback of the Schwartz distributions).

In what follows we denote by I_Ω the characteristic function on any set Ω.

Lemma 3.2. *Let $Z(t,z,\ \cdot\ )$ be as in* (3.1). *Then $Z(t,z,\ \cdot\ )$ belongs to $\mathbf{D}_\infty(\mathbb{R}^2)$ and is non-degenerate in the sense of Malliavin for all $t>0$ and $z\in\mathbb{H}$.*

Proof. Without loss of generality, we may assume that $z=\sqrt{-1}$ in (3.1). It follows from Proposition 10.1 in [6, Chapter V], that both $X_t=\int_0^t \exp(w_s^1-s/2)\,dw_s^1$ and $Y_t=\exp(w_t^2-t/2)$ belong to $\mathbf{D}_\infty$. A direct computation shows that

$$DX_t = I_1+I_2,\quad DY_t=\exp(w_t^2-t/2)k_t^2, \tag{3.6}$$

where we set

$$I_1(w)=\left(\int_0^{\cdot}\exp(w_s^2-s/2)I_{[0,t]}(s)\,ds,0\right),$$
$$I_2(w)=\int_0^t \exp(w_s^2-s/2)\,(0,k_s)\,dw_s^1 \tag{3.7}$$

and k_t is given by $\dot{k}_t(s)=I_{[0,t]}(s)$ and the last integral in (3.7) reads the H-valued stochastic integral. Since $\langle I_1(w),I_2(w)\rangle_H=\langle DY_t(w),I_1(w)\rangle_H=0$, the Malliavin covariance is given by

$$\begin{aligned}&\|DX_t(w)\|_H^2\|DY_t(w)\|_H^2-\langle DX_t(w),DY_t(w)\rangle_H^2\\ &\quad=(\|I_1(w)\|_H^2+\|I_2(w)\|_H^2)\|DY_t(w)\|_H^2-\langle I_2(w),DY_t(w)\rangle_H^2\\ &\quad\geqslant\|I_1(w)\|_H^2\|DY_t(w)\|_H^2,\end{aligned} \tag{3.8}$$

where we used the Schwarz inequality in the last inequality. It follows from (3.6) that

$$\|I_1(w)\|_H^2=\int_0^t\exp(2w_s^2-s)\,ds\geqslant t\exp\left(-2\sup_{0\leqslant u\leqslant t}|w_t|-t\right),$$
$$\|DY_t(w)\|_H^2=t\exp(2w_t^2-t)\geqslant t\exp\left(-2\sup_{1\leqslant u\leqslant t}|w_u|-t\right),$$

from which, using Fernique's theorem [6, p. 402], we see that $\|I_1\|_H^{-2}\|DY_t\|_H^{-2}$ belongs to $\bigcap_p L^p(W)$. Then we conclude from (3.8). □

In particular, the composition $\tilde{\delta}_{z'}\circ Z(t,z,\ \cdot\ )$ is well defined and belongs to $\mathbf{D}_{-\infty}$ for all $z'\in\mathbb{H}$, where $\tilde{\delta}_{z'}$ is the Dirac delta function at $z'\in\mathbb{H}$ with respect to the Riemannian measure m (see Remark 3.3). Note that $p(t,z,z')=E[\tilde{\delta}_{z'}(Z(t,z,\ \cdot\ ))]$ (see [5, Section 2]).

Remark 3.3. By the natural inclusion $\mathbb{H} \hookrightarrow \mathbb{C} \cong \mathbb{R}^2$, we regard $Z(t,z,w)$ as an $\mathbb{R}^2$-valued random variable. In what follows, we shall regard distributions on $\mathbb{H}$ with compact support as elements of $\mathscr{S}'(\mathbb{R}^2)$ in this way. We can easily see that $\widetilde{\delta}_z = y^2\delta_{(x,y)}$, where $\delta_{(x,y)}$ is the Dirac delta function at $(x,y) \in \mathbb{R}^2$ with respect to the Lebesgue measure. We can also see that

$$\int_{\mathbb{H}} f(z)\widetilde{\delta}_z m(dz) = \int_{\mathbb{R}^2} f(x,y)\delta_{(x,y)}\,dx\,dy = f$$

for $f \in C_0^\infty(\mathbb{H})$, where the integrals converge with respect to the topology on $\mathscr{S}'(\mathbb{R}^2)$.

It is known (see [17]) that, for every positive generalized Wiener functional Ψ, there exists a unique positive finite measure μ^Ψ on W such that

$$E[\Psi \cdot F] = \int_W \widetilde{F}(w)\mu^\Psi(dw) \quad (F \in \mathbf{D}_\infty),$$

where $\widetilde{F}$ stands for the $\mathbf{D}_\infty$-quasi-continuous modification of F (see [10, Chapter IV, Section 2, p. 94]). Note the $\widetilde{F}(w)$ is uniquely defined up to the measure μ^Ψ. The probability measure which corresponds to $\widetilde{\delta}_{z'}(Z(t,z,w))/E[\widetilde{\delta}_{z'}(Z(t,z,\ \cdot\ ))]$ is denoted by $\mu_t^{z,z'}$. (For detailed information on the quasi-sure analysis on the Wiener space, see [10, Chapters IV and V].)

Lemma 3.4. *For* $z_i = x_i + y_i\sqrt{-1} \in \mathbb{H}$ $(i = 0,1)$, *we set* $\tau(z_0,z_1) = (x_1 - x_0)/y_0 + (y_1/y_0)\sqrt{-1}$. *Then, for any* $t > 0$, *we have*

$$\widetilde{\delta}_{z_1}(Z(t,z_0,w)) = \widetilde{\delta}_{\tau(z_0,z_1)}(Z(t,w)) \quad \text{and} \quad \mu_t^{z_0,z_1} = \mu_t^{\sqrt{-1},\tau(z_0,z_1)}.$$

In particular, neither $\widetilde{\delta}_z(Z(t,z,w))$ *nor* $\mu_t^{z,z}$ *depends on* $z \in \mathbb{H}$.

Proof. Take a sequence $\{f_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{H})$ such that $f_n \to \widetilde{\delta}_{\sqrt{-1}}$ in $\mathscr{S}'$ as $n \to \infty$. Then the sequence $f_n((x-x_0)/y_0, y/y_0)$ converges to $\widetilde{\delta}_{x_0+y_0\sqrt{-1}}$. Indeed, for any $g \in C_0^\infty(\mathbb{H})$,

$$\int_{\mathbb{H}} f_n((x-x_0)/y_0, y/y_0)g(x,y)y^{-2}\,dx\,dy = \int_{\mathbb{H}} f_n(x,y)g(x_0+y_0x, y_0y)(y_0y)^{-2}y_0^2\,dx\,dy$$
$$\to g(x_0,y_0) \quad \text{as } n \to \infty.$$

Noting that $X(t, z_0, w) = x_0 + y_0 X(t, w)$ and $Y(t, z_0, w) = y_0 Y(t, w)$, we have

$$\begin{aligned}\widetilde{\delta}_{z_1}(Z(t, z_0, w)) &= \lim_{n\to\infty} f_n((X(t, z_0, w) - x_1)/y_1, Y(t, z_0, w)/y_1)\\ &= \lim_{n\to\infty} f_n(\{X(t, w) - (x_1 - x_0)/y_0\}/(y_1/y_0), Y(t, w)/(y_1/y_0))\\ &= \widetilde{\delta}_{\tau(z_0,z_1)}(Z(t, w)).\end{aligned}$$

The rest of the statement is trivial. □

3.3. Pinned Wiener measures

Now we introduce the pinned Wiener measure on $\mathbb{H}$. Let $T > 0$ and $z, z' \in \mathbb{H}$. Set $W_T(\mathbb{H}) = C([0, T]; \mathbb{H})$ and $\mathscr{L}_T^{z,z'}(\mathbb{H}) = \{l \in W_T(\mathbb{H}) \mid l(0) = z,\ l(T) = z'\}$. We equip the space $W_T(\mathbb{H})$ with the distance $d(l, l') = \sup\{d_{\mathbb{H}}(l_s, l'_s) \mid 0 \leqslant s \leqslant T\}$. Then $W_T(\mathbb{H})$ is a complete separable metric space and $\mathscr{L}_T^{z,z'}(\mathbb{H})$ is a closed subspace. The pinned Wiener measure $P_T^{z,z'}$ on $\mathbb{H}$ is defined by the probability measure on $\mathscr{L}_T^{z,z'}(\mathbb{H})$ which satisfies

$$\int_{\mathscr{L}_T^{z,z'}(\mathbb{H})} \prod_{i=1}^n f_i(l_{t_i}) P_T^{z,z'}(dl) = p(T, z, z')^{-1} \int_{\mathbb{H}^n} \prod_{i=1}^n (m(dz_i) f_i(z_i)) \times \prod_{i=1}^{n+1} p(t_i - t_{i-1}, z_{i-1}, z_i) \tag{3.9}$$

for any partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ of $[0, T]$ and any $f_1, \ldots, f_n \in C_0^\infty(\mathbb{H})$. Here, $z_0 = z$ and $z_{n+1} = z'$ by convention. The right-hand side of (3.9) is equal to

$$p(T, z, z')^{-1} E[\widetilde{\delta}_{z'}(Z(T, z, \cdot)) \prod_{i=1}^n f_i(Z(t_i, z, \cdot))] = \int_W \prod_{i=1}^n f_i(\widetilde{Z}(t_i, z, w)) \mu_T^{z,z'}(dw).$$

Here $\{\widetilde{Z}(t, z, w)\}_{0 \leqslant t \leqslant T}$ is the modification of $\{Z(t, z, w)\}_{0 \leqslant t \leqslant T}$ as a $W_T(\mathbb{H})$-valued random variable in the following sense (the stating point z is arbitrarily fixed):

1. For each $t \in [0, T]$, we have $\widetilde{Z}(t, z, w) = Z(t, z, w)$ for P-almost surely.
2. For each $t \in [0, T]$, the Wiener functional $\widetilde{Z}(t, z, \cdot)$ is $\mathbf{D}_\infty$-quasi continuous.
3. For quasi-sure $w \in W$, the map $t \mapsto \widetilde{Z}(t, z, w)$ is continuous.

(For the existence of such a modification, see Theorem 4.2 in [11], for example.) Then we can deduce that $P_T^{z,z'}$ is the law of $\mu_T^{z,z'}$ by the $W_T(\mathbb{H})$-valued random variable $\widetilde{Z}(\cdot, z, w)$.

Now we recall the pinned Wiener measure on $\mathbb{R}$. Let $T > 0$ and $x, x' \in \mathbb{R}$. Define $W_T(\mathbb{R})$ and $\mathscr{L}_T^{x,x'}(\mathbb{R})$ in a similar way as in the case of $\mathbb{H}$. The pinned Wiener measure $\nu_T^{x,x'}$ on $\mathbb{R}$ is defined as the measure on $\mathscr{L}_T^{x,x'}(\mathbb{R})$ which satisfies

$$\int_{\mathscr{L}_T^{x,x'}(\mathbb{R})} \prod_{i=1}^n f_i(l_{t_i}) \nu_T^{x,x'}(dl) = q(T,x,x')^{-1} \int_{\mathbb{R}^n} \prod_{i=1}^n (dx_i f_i(x_i)) \times \prod_{i=1}^{n+1} q(t_i - t_{i-1}, x_{i-1}, x_i) \tag{3.10}$$

for any partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ of $[0,T]$ and any $f_1, \ldots, f_n \in C_0^\infty(\mathbb{R})$. Here, $x_0 = x$ and $x_{n+1} = x'$ by convention and

$$q(t,x,x') = (2\pi t)^{-1/2} \exp\{-(x-x')^2/(2t)\} \tag{3.11}$$

is the Gaussian kernel. For the standard one-dimensional Brownian motion $\{b_t\}_{t\geqslant 0}$, we see that

$$E\left[\delta_{x'}(x+b_T) \prod_{i=1}^n f_i(x+b_{t_i})\right] = q(T,x,x') \int_{\mathscr{L}_T^{x,x'}(\mathbb{R})} \prod_{i=1}^n f_i(l_{t_i}) \nu_T^{x,x'}(dl) \tag{3.12}$$

and that, for any bounded Borel function F of n-variables,

$$\int_{\mathscr{L}_T^{0,0}(\mathbb{R})} F(l_{t_1}, \ldots, l_{t_n}) \nu_T^{0,0}(dl) = \int_{\mathscr{L}_1^{0,0}(\mathbb{R})} F(\sqrt{T} l_{t_1/T}, \ldots, \sqrt{T} l_{t_n/T}) \nu_1^{0,0}(dl) \tag{3.13}$$

by the Brownian scaling property.

4. Preliminary stochastic analysis

The purpose of this section is to prove a Fubini-like lemma for the generalized expectations (Lemma 4.4 below) and show the continuity of the pinned Wiener measure $P_T^{z,z'}$ with respect to $(z,z') \in \mathbb{H} \times \mathbb{H}$ (Lemma 4.6 below).

In Lemmas 4.1, 4.2 and 4.5, we consider one dimensional Wiener space (W_1, H_1, P_1). The canonical realization of the Brownian motion is denoted by $\{w_t\}_{t\geqslant 0}$.

In what follows we denote the expectation and the space of test functionals with respect to (W_1, H_1, P_1) also by $E[\,\cdot\,]$ and $\mathbf{D}_\infty$, respectively.

Lemma 4.1. *Let $T>0$, $0 \leqslant s < t \leqslant T$, $\varepsilon > 0$ and $\gamma \in \mathbb{R}$. Set $F(w) = \int_s^t \exp(\varepsilon w_u)\, du$. Then the power $(F)^\gamma$ belongs to $\mathbf{D}_\infty$. Moreover, if γ is non-negative integer, the estimate*

$$\|(F)^\gamma\|_{p,2} \leqslant C_{\varepsilon,\gamma,T} |t-s|^\gamma$$

holds for some $C_{\varepsilon,\gamma,T} > 0$.

Proof. We show the lemma for the case $\varepsilon = 1$, $s = 0$ and $t = 1$. The general case can be done in the same way. First we show that $F(w) = \int_0^1 \exp(w_u)\,du$ belongs to $\mathbf{D}_\infty$. Fernique's theorem (see [6, p. 402]) implies that both F and F^{-1} belong to $\bigcap_{p>1} L^p$. Using the closability of D, we obtain

$$DF(w) = \int_0^1 \exp(w_u) k_u\,du, \tag{4.1}$$

where k_u is given by $\dot{k}_u(t) = I_{[0,u]}(t)$ and belongs to H_1. Since $||k_u||_{H_1} = \sqrt{u}$, we have $||DF||_{H_1} \in \bigcap_{p>1} L^p$. Inductively, we have

$$D^n F(w) = \int_0^1 \exp(w_u)(k_u \otimes \cdots \otimes k_u)\,du \tag{4.2}$$

for $n = 1, 2, \ldots$. So we have $||D^n F||_{H_1^{\otimes n}} \in \bigcap_{p>1} L^p$ and $F \in \mathbf{D}_\infty$. Then, using the derivation property of D and Fernique's theorem, one can deduce from (4.2) the conclusion. $\square$

Lemma 4.2. *Let* $0 = t_0 < t_1 < \cdots < t_n = T$ *be a partition and set*

$$A_t(w) = \int_0^t \exp(2w_u)\,du. \tag{4.3}$$

Let $g \in \mathscr{S}(\mathbb{R}^{n-1})$, $r > 1/2$ *and* $\zeta \in \mathbb{R}$. *Take* $f_j \in \mathscr{S}(\mathbb{R})$ $(j = 1, 2, \ldots)$ *so that* $||f_j - \delta_\zeta||_{2,-r}^2 \to 0$ *as* $j \to \infty$. *For each* $j = 1, 2, \ldots$, *we set*

$$G_j(w) = \int_{\mathbb{R}^n} g(x_1, \ldots, x_{n-1}) f_j(x_n) \prod_{i=1}^n q(A_{t_i}(w) - A_{t_{i-1}}(w), x_{i-1}, x_i)\,dx_1 \cdots dx_n,$$

where $x_0 = 0$ *by convention and* q *is as in* (3.11).

Then G_j *converges to*

$$q(A_T(\cdot), 0, \zeta) \int_{\mathscr{L}_T^{0,\zeta}(\mathbb{R})} g(b_{A_{t_1}(\cdot)}, \ldots, b_{A_{t_{n-1}}(\cdot)}) \nu_{A_T(\cdot)}^{0,\zeta}(db) \tag{4.4}$$

as $j \to \infty$ *in* $\mathbf{D}_\infty$, *as well as in the sense of pointwise convergence. In particular, the right-hand side of* (4.4) *defines an element of* $\mathbf{D}_\infty$.

Proof. First, we define two matrices $K_1, K_2(w) \in \mathrm{GL}(n, \mathbb{R})$ by

$$K_1 = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix}, \tag{4.5}$$

$$K_2(w) = \mathbf{diag}[A_{t_1}(w)^{1/2}, \{A_{t_2}(w) - A_{t_1}(w)\}^{1/2}, \dots, \{A_{t_n}(w) - A_{t_{n-1}}(w)\}^{1/2}], \quad (4.6)$$

where **diag** denotes the diagonal matrix. We set $K(w) = K_1^{-1}K_2(w)$. Then it follows from Lemma 4.1 that $\det K(\cdot)$, $\det K(\cdot)^{-1}$ and all the entries of $K(\cdot)$ are elements of $\mathbf{D}_\infty$, so are all the entries of $K(\cdot)^{-1}$. Applying the change of variable formula, we have

$$G_j(w) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (g \otimes f_j)\,(K(w)y)e^{-|y|^2/2}\,dy. \quad (4.7)$$

By differentiating the both sides of (4.7), we can find that, for any positive integer k,

$$D^k G_j(w) = \sum_{|\alpha| \leqslant k} \sum_{i=1}^{i_\alpha} X_{\alpha,i}(w) \int_{\mathbb{R}^n} (\partial^\alpha(g \otimes f_j))\,(K(w)y)p_{\alpha,i}(y)e^{-|y|^2/2}\,dy \quad (4.8)$$

holds for some non-negative integers i_α, polynomials $p_{\alpha,i}(y)$ and $H^{\otimes k}$-valued $\mathbf{D}_\infty$-functionals $X_{\alpha,i}$. Here, for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, we used the standard notations $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Take $N = k + r$. Then we have

$$\begin{aligned}
&||D^k G_j(w) - D^k G_{j'}(w)||_{H^{\otimes k}} \\
&\leqslant c \sum_{\alpha,i} ||X_{\alpha,i}(w)||_{H^{\otimes k}} \cdot ||\partial^\alpha(g \otimes f_j - g \otimes f_{j'})(K(w)\ \cdot\ )||_{2,-N} \\
&\leqslant c|\det K(w)|^{-1/2}(1 + ||K^{-1}(w)||_{HS}^2)^{N/2} \\
&\quad \times \sum_{\alpha,i} ||X_{\alpha,i}(w)||_{H^{\otimes k}} \cdot ||\partial^\alpha(g \otimes f_j - g \otimes f_{j'})||_{2,-N} \\
&\leqslant c|\det K(w)|^{-1/2}(1 + ||K^{-1}(w)||_{HS}^2)^{N/2} \\
&\quad \times \sum_{\alpha,i} ||X_{\alpha,i}(w)||_{H^{\otimes k}} \cdot ||g \otimes f_j - g \otimes f_{j'}||_{2,-r}
\end{aligned}$$

for some $c > 0$, where we used Lemma 2.3. Since $||g \otimes f_j - g \otimes f_{j'}||_{2,-r} \to 0$ as $j, j' \to \infty$ by the assumption, we have

$$E[||D^k G_j - D^k G_{j'}||_{H^{\otimes k}}^p] \to 0 \quad \text{as } j, j' \to \infty.$$

This implies $\{G_j\}_{j=1,2,\dots}$ is a $\mathbf{D}_\infty$-Cauchy sequence.

On the other hand, it is easy to see that the sequence G_j converges to (4.4) for each fixed w. The two limits (in $\mathbf{D}_\infty$ and in the pointwise sense) must coincide. Hence we obtain the lemma. □

We prepare a Fubini-like lemma for the generalized expectation. We regard $W = W_1 \times W_1$ and $P(dw) = P_1(dw^1) \otimes P_1(dw^2)$.

Lemma 4.3. *Let* $(\zeta,\theta)\in\mathbb{R}^2$, $g\in\mathscr{S}(\mathbb{R}^{n-1})$. *Let* $F\in\mathbf{D}_\infty$ *and assume that* F *is independent of the coordinate* w^1. *We set* $\beta_t(w)=\int_0^t \exp(w_s^2)\,dw_s^1$. *Then, for any partition* $0<t_1<\cdots<t_n=T$, *we have*

$$\begin{aligned}&\int_W g(\beta_{t_1}(w),\dots,\beta_{t_{n-1}}(w))F(w^2)\delta_{(\zeta,\theta)}(\beta_T(w),w_T^2)P(dw)\\&\quad=\int_{W_1}P_1(dw^2)F(w^2)\delta_\theta(w_T^2)\\&\qquad\times\left[q(A_T(w^2),0,\zeta)\int_{\mathscr{L}_T^{0,\zeta}(\mathbb{R})}g(b_{A_{t_1}(w^2)},\dots,b_{A_{t_{n-1}}(w^2)})\nu^{0,\zeta}_{A_T(w^2)}(db)\right].\end{aligned}$$

Proof. Since $\beta_t(w)$ is a martingale whose quadratic variational process (see [6, Chapter II, Section 2, p. 53]) is $A_t(w^2)$, we have, for each fixed w^2, $\{\beta_t(w)\}_{t\geqslant 0}$ under the probability measure $P_1(dw^1)$ has the same law as $\{b_{A_t(w^2)}\}_{t\geqslant 0}$, where $\{b_t\}_{t\geqslant 0}$ is a one-dimensional standard Brownian motion starting at 0. As in the proof of Lemma 3.2, we can show that (β_t,w_t^2) belongs to $\mathbf{D}_\infty$ and is non-degenerate. Let $f_j,\widetilde{f}_j\in\mathscr{S}(\mathbb{R})$ such that $f_j\to\delta_\zeta$ and $\widetilde{f}_j\to\delta_\theta$ in $\mathscr{H}_{2,-1}$ as $j\to\infty$. Since $f_j(x)\widetilde{f}_j(y)\to\delta_{(\zeta,\theta)}(x,y)$ in $\mathscr{S}'(\mathbb{R}^2)$ by Lemma 2.4, we have

$$\begin{aligned}&\int_W g(\beta_{t_1}(w),\dots,\beta_{t_{n-1}}(w))F(w^2)\delta_{(\zeta,\theta)}(\beta_T(w),w_T^2)P(dw)\\&\quad=\lim_{j\to\infty}\int_W g(\beta_{t_1}(w),\dots,\beta_{t_{n-1}}(w))F(w^2)f_j(\beta_T(w))\widetilde{f}_j(w_T^2)P(dw)\\&\quad=\lim_{j\to\infty}\int_{W_1}P_1(dw^2)F(w^2)\widetilde{f}_j(w_T^2)\left(\int_{W_1}P_1(db)g\left(b_{A_{t_1}(w^2)},\dots,b_{A_{t_{n-1}}(w^2)}\right)f_j\left(b_{A_T(w^2)}\right)\right)\\&\quad=\lim_{j\to\infty}\int_{W_1}P_1(dw^2)F(w^2)\widetilde{f}_j(w_T^2)G_j(w^2).\end{aligned}$$

By Lemma 2.4 and the continuity of the pullback of Schwartz distributions (see [6, Chapter V, Section 9, Theorem 9.1 and its Corollary]), we see that $\widetilde{f}_j(w_T^2)\to\delta_\theta(w_T^2)$ as $j\to\infty$ in $\mathbf{D}_{-\infty}$. By Lemma 4.2, we have

$$G_j(w^2)\to q(A_T(w^2),0,\zeta)\int_{\mathscr{L}_T^{0,\zeta}(\mathbb{R})}g(b_{A_{t_1}(w^2)},\dots,b_{A_{t_{n-1}}(w^2)})\nu^{0,\zeta}_{A_T(w^2)}(db)$$

in $\mathbf{D}_\infty$ as $j\to\infty$. This completes the proof. □

Now we show a key lemma, which shall be used later.

Lemma 4.4. *For* $g\in C_0^\infty(\mathbb{R}^{n-1})$ *and* $h\in C_0^\infty((0,\infty)^{n-1})$, *we set*

$$f(z_1,\dots,z_{n-1})=g(x_1,\dots,x_{n-1})h(y_1,\dots,y_{n-1}).$$

Let $0<t_1<\cdots<t_n=T$ *be a partition of* $[0,T]$. *Then, for any* $c=a+b\sqrt{-1}\in\mathbb{H}$, *we have*

$$
\begin{aligned}
&p(T,\sqrt{-1},c)\int_{\mathscr{L}_T^{\sqrt{-1},c}(\mathbb{H})} f(l_{t_1},\dots,l_{t_{n-1}})P_T^{\sqrt{-1},c}(dl)\\
&\quad= e^{-T/8}\sqrt{b}\,q(T,0,\log b)\int_{\mathscr{L}_T^{0,\log b}(\mathbb{R})}\nu_T^{0,\log b}(dw)h(\exp(w_{t_1}),\dots,\exp(w_{t_{n-1}}))\\
&\qquad\times\left[q(A_T(w),0,a)\int_{\mathscr{L}^{0,a}_{A_T(w)}(\mathbb{R})} g(b_{A_{t_1}(w)},\dots,b_{A_{t_{n-1}}(w)})\nu^{0,a}_{A_T(w)}(db)\right]. \qquad (4.9)
\end{aligned}
$$

Proof. First note that $\widetilde{\delta}_c(x,e^y)=b\delta_{(a,\log b)}(x,y)$ in $\mathscr{S}'(\mathbb{R}^2)$. Then the left-hand side of (4.9) is equal to

$$
\begin{aligned}
&E[\widetilde{\delta}_c(Z(T,w))f(Z(t_1,w),\dots,Z(t_{n-1},w))]\\
&\quad= bE[\delta_{(a,\log b)}(X(T,w),w_T^2-T/2)\cdot g(X(t_1,w),\dots,X(t_{n-1},w))\\
&\qquad\times h(\exp(w_{t_1}^2-t_1/2),\dots,\exp(w_{t_{n-1}}^2-t_{n-1}/2))]. \qquad (4.10)
\end{aligned}
$$

Now we consider the Girsanov transform (see [6, Chapter IV, Section 4, p. 190]). The process $(\widetilde{w}_t^1,\widetilde{w}_t^2)=(w_t^1,w_t^2-t/2)$ is a two-dimensional standard Brownian motion with respect to the measure

$$\widetilde{P}|_{\mathscr{B}_T}=\exp[\frac{1}{2}w_T^2-T/8]\cdot P|_{\mathscr{B}_T},$$

where $\mathscr{B}_T$ is the σ-field generated by $\{w(s)\,|\,0\leqslant s\leqslant T\}$. We denote by $\widetilde{E}$ the expectation with respect to $\widetilde{P}|_{\mathscr{B}_T}$. Let $\beta_t(w)$ be as in the previous lemma. Then the right-hand side of (4.10) is equal to

$$
\begin{aligned}
&b\widetilde{E}\Big[\delta_{(a,\log b)}(\beta_T(\widetilde{w}),\widetilde{w}_T^2)\cdot g(\beta_{t_1}(\widetilde{w}),\dots,\beta_{t_{n-1}}(\widetilde{w}))\\
&\qquad\times h(\exp(\widetilde{w}_{t_1}^2),\dots,\exp(\widetilde{w}_{t_{n-1}}^2))\exp\left[-\frac{1}{2}\widetilde{w}_T^2-T/8\right]\Big]\\
&\quad= e^{-T/8}\sqrt{b}\widetilde{E}[\delta_{(a,\log b)}(\beta_T(\widetilde{w}),\widetilde{w}_T^2)\cdot g(\beta_{t_1}(\widetilde{w}),\dots,\beta_{t_{n-1}}(\widetilde{w}))\\
&\qquad\times h(\exp(\widetilde{w}_{t_1}^2),\dots,\exp(\widetilde{w}_{t_{n-1}}^2))]\\
&\quad= e^{-T/8}\sqrt{b}\int_{W_1}P_1(dw^2)h(\exp(w_{t_1}^2),\dots,\exp(w_{t_{n-1}}^2))\delta_{\log b}(w_T^2)\\
&\qquad\times\left[q(A_T(w^2),0,a)\int_{\mathscr{L}_T^{0,a}(\mathbb{R})} g(b_{A_{t_1}(w^2)},\dots,b_{A_{t_{n-1}}(w^2)})\nu^{0,a}_{A_T(w^2)}(db)\right],
\end{aligned}
$$

where we used Lemma 4.3 for the second equality. Then the integrand with respect to the measure $P_1(dw^2)\delta_{\log b}(w_T^2)$ is a continuous $\mathbf{D}_\infty$-functionals which depends only on the coordinate w^2 and is measurable with respect to the σ-field generated by $\{w^2(s) \mid 0 \leqslant s \leqslant T\}$. This proves the lemma. □

Lemma 4.5. *For any positive integer n, there exists a positive constant C_n such that*

$$q(T,0,x)\int_{\mathscr{L}_T^{0,x}(\mathbb{R})} |w_t - w_s|^{2n} \nu_T^{0,x}(dw) \leqslant C_n T^{-1/2} |t-s|^n \tag{4.11}$$

holds for all s, t, T with $0 \leqslant s < t \leqslant T$ and for all $x \in \mathbb{R}$.

Proof. Let (W_1, P_1) be the one-dimensional Wiener space as above and $\{w_t\}_{t \geqslant 0}$ be the canonical realization of the one-dimensional Brownian motion. The left-hand side of (4.11) is equal to

$$E[\delta_x(w_T)|w_t - w_s|^{2n}] = T^{n-1/2} E[\delta_{x/T}(w_1)|w_{t/T} - w_{s/T}|^{2n}], \tag{4.12}$$

where we used the scaling property of Brownian motion.

Note that $\{\delta_x \mid x \in \mathbb{R}\}$ is bounded in $\mathscr{S}_{-2}(\mathbb{R})$ (see [6, Lemma 9.1] and its proof). Hence, by the continuity of the pullback of Schwartz distributions, we see that $\{\delta_x(w_1) \mid x \in \mathbb{R}\}$ is bounded in $\mathbf{D}_{2,-2}$.

So we consider $\mathbf{D}_{2,2}$-norm of $|w_{t/T} - w_{s/T}|^{2n}$. Set $F(w) = w_t - w_s$ and let k_t be as in (4.1). Then $DF(w) = k_t - k_s$, $D^2F(w) = 0$ and $||DF(w)||_H = \sqrt{t-s}$. Hence, we have $E[F^{4n}] \leqslant C_n|t-s|^{2n}$,

$$E[||D(F^{2n})||_H^2] = (2n)^2(t-s)E[F^{4n-2}] \leqslant C_n|t-s|^{2n}$$

and

$$E[||D^2(F^{2n})||_{H\otimes H}^2] = (2n)^2(2n-1)^2(t-s)^2 E[F^{4n-4}] \leqslant C_n|t-s|^{2n}.$$

Hence, $\mathbf{D}_{2,2}$-norm of $|w_{t/T} - w_{s/T}|^{2n}$ is less than $C_n|(t-s)/T|^n$. Combining this with (4.12), we obtain (4.11). □

Now we show the tightness of the pinned Wiener measures. We recall the notion of tightness. A family Λ of probability measures on a separable metric space S is called *tight on S* if for any $\varepsilon > 0$ there exists a compact subset K of S such that $\inf_{P \in \Lambda} P(K) \geqslant 1 - \varepsilon$ holds.

Lemma 4.6. *Let $P_T^{z,z'}$ be the pinned Wiener measure defined as in* (3.9). *Then, for each* $T>0$, *we have the following assertions* 1 *and* 2:

1. *For any compact subset K in $\mathbb{H}\times\mathbb{H}$, the family of probability measures $\{P_T^{z,z'} \mid (z,z')\in K\}$ is tight on $W_T(\mathbb{H})$.*
2. *The mapping $(z,z')\mapsto P_T^{z,z'}$ is continuous with respect to the topology of the weak convergence of probability measures on $W_T(\mathbb{H})$.*

Proof. First we consider another distance on $W_T(\mathbb{H})$. Set $d'_{\mathbb{H}}(z,z')^2=(x-x')^2+(\log y-\log y')^2$ for $z=x+\sqrt{-1}y$ and $z'=x'+\sqrt{-1}y'$ and set $d'(l,l')=\sup\{d'_{\mathbb{H}}(l_s,l'_s)\mid 0\leqslant s\leqslant T\}$ for $l,l'\in W_T(\mathbb{H})$. Then the distance d' on $W_T(\mathbb{H})$ induces the topology which coincides with the one by $d_{\mathbb{H}}$ instead of $d'_{\mathbb{H}}$. So it suffices to show the tightness with respect to d'.

By Ikeda and Watanabe [6, Theorems I-4.3 and I-4.2] and its proof, it is enough to show that, for any $T>0$ and for any compact set K in $\mathbb{H}\times\mathbb{H}$, there exists a positive constant C such that

$$\int_{\mathscr{L}_T^{z,z'}(\mathbb{H})} d'_{\mathbb{H}}(l_s,l_t)^4 P_T^{z,z'}(dl)=\int_W d'_{\mathbb{H}}(\widetilde{Z}(s,z,w),\widetilde{Z}(t,z,w))^4\mu_T^{z,z'}(dw)\leqslant C(t-s)^2 \tag{4.13}$$

for all s,t with $0\leqslant s<t\leqslant T$ and for all $(z,z')\in K$. Note that

$$d'_{\mathbb{H}}(\widetilde{Z}(s,z,w),\widetilde{Z}(t,z,w))^2=y^2|\widetilde{X}(s,w)-\widetilde{X}(t,w)|^2+|\log\widetilde{Y}(s,w)-\log\widetilde{Y}(t,w)|^2.$$

Let $\tau(z,z')$ be as in Lemma 3.4. If we write $\tau(z,z')$ as $c=a+b\sqrt{-1}$, it follows from Lemma 3.4 that the left-hand side of (4.13) is dominated by

$$C_1\int_W\{|\widetilde{X}(s,w)-\widetilde{X}(t,w)|^4+|\log\widetilde{Y}(s,w)-\log\widetilde{Y}(t,w)|^4\}\mu_T^{\sqrt{-1},c}(dw), \tag{4.14}$$

where $C_1=2\max\{y^4\mid (z,z')\in K\}$.

First we estimate the second term in (4.14). It follows from Lemma 4.4 and the monotone convergence theorem that

$$\begin{aligned}
&\int_W|\log\widetilde{Y}(s,w)-\log\widetilde{Y}(t,w)|^4\mu_T^{\sqrt{-1},c}(dw)\\
&\leqslant C_2q(T,0,\log b)\int_{\mathscr{L}_T^{0,\log b}(\mathbb{R})}|w_s-w_t|^4(2\pi A_T(w))^{-1/2}\nu_T^{0,\log b}(dw)\\
&\leqslant C_3\left[q(T,0,\log b)\int_{\mathscr{L}_T^{0,\log b}(\mathbb{R})}|w_s-w_t|^8\nu_T^{0,\log b}(dw)\right]^{1/2}\cdot E[\delta_{\log b}(w_T)A_T(w)^{-1}]^{1/2}\\
&\leqslant C_4|t-s|^2, \qquad (4.15)
\end{aligned}$$

where the constant C_2 takes the form $ce^{-T/8}\max\{\sqrt{b}p(T,\sqrt{-1},c)^{-1} \mid c\in\tau(K)\}$ and we used Lemmas 4.1 and 4.5 and the boundedness of $\{\delta_{\log b}(w_T) \mid c\in\tau(K)\}$ in the last inequality.

Next we estimate the first term in (4.14). In the same way as above we have

$$\begin{aligned}
&\int_W |\widetilde{X}(s,w)-\widetilde{X}(t,w)|^4 \mu_T^{\sqrt{-1},c}(dw)\\
&\leqslant C_2 q(T,0,\log b)\int_{\mathscr{L}_T^{0,\log b}(\mathbb{R})} v_T^{0,\log b}(dw)\\
&\quad\times q(A_T(w),0,a)\int_{\mathscr{L}^{0,a}_{A_T(w)}(\mathbb{R})} |b_{A_s(w)}-b_{A_t(w)}|^4 v^{0,a}_{A_T(w)}(db)\\
&\leqslant C_5 q(T,0,\log b)\int_{\mathscr{L}_T^{0,\log b}(\mathbb{R})} v_T^{0,\log b}(dw)|A_t(w)-A_s(w)|^2 A_T^{-1/2}(w)\\
&= C_5 E[\delta_{\log b}(w_T)|A_t(w)-A_s(w)|^2 A_T^{-1/2}(w)]\\
&\leqslant C_6|t-s|^2,
\end{aligned}\tag{4.16}$$

where we used Lemma 4.5 in the second inequality and we used Lemma 4.1, Hölder's inequality and the boundedness of $\delta_{\log b}(w_T)$ and $A_T(w)^{-1/2}$ in the last inequality. Here the constant C_6 may depends on T. Then (4.14)–(4.16) imply (4.13), from which the first assertion follows.

Finally we show the second assertion. Assume that (z_n,z_n') converges to (z,z') as $n\to\infty$. Then $\{P_T^{z_n,z_n'}\}_{n=1}^{\infty}$ is tight by the first assertion, hence, is relatively compact in the topology of weak convergence of probability measures [6, Theorem 2.6]. Let P' be any cluster point. Then, we see from (3.9) that the finite-dimensional distribution of P' and $P_T^{z,z'}$ coincide. This shows that $\{P_T^{z_n,z_n'}\}_{n=1}^{\infty}$ has the unique cluster point $P_T^{z,z'}$, hence $P_T^{z,z'}$ is continuous in (z,z'). This completes the proof. □

5. Estimate of the trace of the heat semigroup e^{-tH_V}

5.1. The heat kernel for H_V

In this subsection we establish the Feynman–Kac representation of the integral kernel of the heat semigroup e^{-tH_V}. Without loss of generality, we may assume that V is non-negative, considering $V+|\inf V|$ instead of V if necessary.

Let p be the kernel as in (3.2). For $t>0$ and $z,z'\in\mathbb{H}$, we define

$$\begin{aligned}
p_V(t,z,z') &= p(t,z,z')\int_W \exp\left(-\int_0^t V(\widetilde{Z}(s,z,w))\,ds\right)\mu_t^{z,z'}(dw)\\
&= p(t,z,z')\int_{\mathscr{L}_t^{z,z'}(\mathbb{H})} \exp\left(-\int_0^t V(l_s)\,ds\right)P_t^{z,z'}(dl).
\end{aligned}$$

Proposition 5.1. *Assume that V satisfies the condition* (A.0). *Then, for each fixed $t>0$, the function $p_V(t,z,z')$ is continuous in the variable $(z,z')\in\mathbb{H}\times\mathbb{H}$ and gives the integral kernel of the heat semigroup e^{-tH_V}, i.e., the identity*

$$(e^{-tH_V}f)(z)=\int_{\mathbb{H}}f(z')p_V(t,z,z')m(dz') \tag{5.1}$$

holds for any $f\in L^2(\mathbb{H})$.

Proof. The function $W_t(\mathbb{H})\ni l\mapsto\exp(-\int_0^t V(l_s)\,ds)\in\mathbb{R}$ is continuous and bounded. Hence, using Lemma 4.6, we can find that $p_V(t,z,z')$ is continuous in (z,z').

Note that the proposition is true when $V=0$. By the obvious domination $p_V(t,z,z')\leqslant p(t,z,z')$, we see that the right-hand side of (5.1) defines a bounded operator on $L^2(\mathbb{H})$.

First we consider the case $V\in C_0^\infty(\mathbb{H})$. The following Feynman–Kac formula:

$$(e^{-tH_V}f)(z)=E\left[f(Z(t,z,\ \cdot\ ))\exp\left(-\int_0^t V(Z(s,z,\ \cdot\ ))\,ds\right)\right] \tag{5.2}$$

holds for $f\in C_0^\infty(\mathbb{H})$, since both V and f are smooth functions with compact support. (Eq. (5.2) can be shown by the reversibility of the Brownian motion on $\mathbb{H}$ with respect to the Riemannian measure $m(dz)$ and by Theorem 3.2 in Ikeda and Watanabe [6, Chapter V, Section 3].)

On the other hand, in the same way as in [6, p. 414], we see that $\exp(-\int_0^t V(Z(s,z,\ \cdot\ ))\,ds)$ belongs to $\mathbf{D}_\infty$. Let $\mathscr{P}=\{0=t_0<t_1<\cdots<t_n=t\}$ be a partition of $[0,t]$ and $|\mathscr{P}|$ be its scale of mesh. Noting that $\exp[-\sum_i V(Z(t_i,z,\ \cdot\ ))(t_i-t_{i-1})]$ converges in $\mathbf{D}_\infty$-topology to $\exp(-\int_0^t V(Z(s,z,\ \cdot\ ))\,ds)$ as $|\mathscr{P}|$ tends to zero, we have the following:

$$\begin{aligned}
&E\left[\widetilde{\delta}_{z'}(Z(t,z,\ \cdot\ ))\exp\left(-\int_0^t V(Z(s,z,\ \cdot\ ))\,ds\right)\right]\\
&=\lim_{|\mathscr{P}|\to0}E\left[\widetilde{\delta}_{z'}(Z(t,z,\ \cdot\ ))\exp\left[-\sum_i V(Z(t_i,z,\ \cdot\ ))(t_i-t_{i-1})\right]\right]\\
&=p(t,z,z')\lim_{|\mathscr{P}|\to0}\int_W\exp\left[-\sum_i V(\widetilde{Z}(t_i,z,w))(t_i-t_{i-1})\right]\mu_t^{z,z'}(dw)\\
&=p(t,z,z')\lim_{|\mathscr{P}|\to0}\int_{\mathscr{L}_t^{z,z'}(\mathbb{H})}\exp\left[-\sum_i V(l_{t_i})(t_i-t_{i-1})\right]P_t^{z,z'}(dl)\\
&=p_V(t,z,z').
\end{aligned} \tag{5.3}$$

By Remark 3.3 and the continuity of the pullback of Schwartz distributions, we see that

$$\int_{\mathbb{H}} m(dz')f(z')\widetilde{\delta}_{z'}(Z(t,z,w)) = f(Z(t,z,w)) \tag{5.4}$$

holds, where the integral converges in $\mathbf{D}_{-\infty}$. By (5.2)–(5.4), the identity (5.1) holds for all $f \in C_0^\infty(\mathbb{H})$, which is dense.

Now we consider general V's. First we take non-negative $V_n \in C_0^\infty(\mathbb{H})$ such that $V_n \nearrow V$ as $n \to \infty$ uniformly on each compact subset. Then the dominated convergence theorem yields that $p_{V_n}(t,z,z') \to p_V(t,z,z')$ for each $z, z' \in \mathbb{H}$ and that

$$\int_{\mathbb{H}} f(z')p_{V_n}(t,z,z')m(dz') \to \int_{\mathbb{H}} f(z')p_V(t,z,z')m(dz')$$

as $n \to \infty$ for each $z \in \mathbb{H}$ and $f \in C_0^\infty(\mathbb{H})$. On the other hand, for each $t > 0$, $e^{-tH_{V_n}}$ converges to e^{-tH_V} as $n \to \infty$ in the strong operator topology, since $H_{V_n}f$ converges to $H_V f$ as $n \to 0$ for each $f \in C_0^\infty(\mathbb{H})$, which is a common core for H_V and H_{V_n}'s (see [13, Theorems VIII.21 and VIII.25]). Then we obtain (5.1) for $f \in C_0^\infty(\mathbb{H})$ since an L^2-convergent sequence $\{e^{-tH_{V_n}}\}_{n=1}^\infty$ has a subsequence which converges almost everywhere. By the boundedness of the operators, identity (5.1) still holds for any $f \in L^2(\mathbb{H})$. □

5.2. *Upper bound for* $\mathrm{Tr}(e^{-tH_V})$

In this and the next subsection we derive upper and lower bound estimates for $\mathrm{Tr}(e^{-tH_V})$, respectively.

Set

$$J(t) = E\left[\widetilde{\delta}_{\sqrt{-1}}(Z(t,w)) \cdot \int_0^t \exp(w_s^2 - s/2)\, ds\right]. \tag{5.5}$$

Lemma 5.2. *Let $J(t)$ be as above. Then we have* $\lim_{t \searrow 0} J(t) = (2\pi)^{-1}$.

Proof. Applying the Girsanov transform in the same way as in the proof of Lemma 4.4, we have

$$\begin{aligned} J(t) &= \widetilde{E}\left[\widetilde{\delta}_{\sqrt{-1}}(\beta_t(\widetilde{w}), \exp(\widetilde{w}_t^2)) \int_0^t \exp(\widetilde{w}_s^2)\, ds \exp\left[-\frac{1}{2}\widetilde{w}_t^2 - t/8\right]\right] \\ &= e^{-t/8}\widetilde{E}\left[\delta_{(0,0)}(\beta_t(\widetilde{w}), \widetilde{w}_t^2) \int_0^t \exp(\widetilde{w}_s^2)\, ds\right] \\ &= e^{-t/8} \int_{W_1} P_1(dw^2)\delta_0(w_t^2)\left(\int_0^t \exp(w_s^2)\, ds\right)\{2\pi A_t(w^2)\}^{-1/2}, \end{aligned} \tag{5.6}$$

where $\widetilde{E}$ is as in the proof of Lemma 4.4 and we used Lemma 4.3 and the monotone convergence theorem in the last equality. Note that $\{w_{tu}^2\}_{u\geqslant 0}$ and $\{\sqrt{t}b_u\}_{u\geqslant 0}$ have the same law, where $\{b_u\}_{u\geqslant 0}$ is a standard realization of the one-dimensional Brownian motion. Then we have from (5.6) that

$$\begin{aligned} J(t) &= \frac{e^{-t/8}}{\sqrt{2\pi}} E\left[t^{-1/2}\delta_0(b_1)\left(t\int_0^1 \exp(\sqrt{t}b_u)\,du\right)\left(t\int_0^1 \exp(2\sqrt{t}b_u)\,du\right)^{-1/2}\right] \\ &= \frac{e^{-t/8}}{2\pi}\int_{\mathscr{L}_1^{0,0}(\mathbb{R})}\left(\int_0^1 \exp(\sqrt{t}b_u)\,du\right)\left(\int_0^1 \exp(2\sqrt{t}b_u)\,du\right)^{-1/2} v_1^{0,0}(db). \end{aligned} \tag{5.7}$$

By the Schwarz inequality

$$\int_0^1 \exp(\sqrt{t}b_u)\,du \leqslant \left(\int_0^1 \exp(2\sqrt{t}b_u)\,du\right)^{1/2},$$

we can find that the integrand (with respect to $v_1^{0,0}$) on the right-hand side of (5.7) is less than 1 and converges to 1 as $t\searrow 0$ for each fixed b. Then the dominated convergence theorem completes the proof. $\square$

Lemma 5.3. *Assume that V satisfies assumption* (A.1). *Let $J(t)$ be as in* (5.5). *Then e^{-tH_V} is of trace class for all $t>0$. Moreover, we have the upper bound estimate*

$$\mathrm{Tr}(e^{-tH_V}) \leqslant \frac{J(t)}{t}\int_{\mathbb{H}} \exp(-tV(z))m(dz) \tag{5.8}$$

$$= \frac{J(t)}{2\pi}\int_{\mathbb{H}\times\mathbb{R}^2} \exp\left[-t\left(\frac{y^2}{2}(\xi^2+\eta^2) + V(x,y)\right)\right] dx\,dy\,d\xi\,d\eta, \tag{5.9}$$

where $(x,y;\xi,\eta)\in\mathbb{H}\times\mathbb{R}^2$ and $dx\,dy\,d\xi\,d\eta$ is the four-dimensional Lebesgue measure.

Proof. Note that the last equality in (5.9) follows from the identity

$$\frac{ty^2}{2\pi}\int_{\mathbb{R}^2} \exp\left[-\frac{ty^2}{2}(\xi^2+\eta^2)\right] d\xi\,d\eta = 1. \tag{5.10}$$

We show (5.8). Take any $f\in C_0^\infty(\mathbb{H})$ such that $0\leqslant f\leqslant 1$. Then the operator $fe^{-tH_V}f$ is non-negative, of Hilbert–Schmidt class, where f denotes the multiplication operator by the function f, and has the integral kernel $f(z)p_V(t,z,z')f(z')$. It follows from Proposition 5.1 and Lemma 2.2 that

$$
\begin{aligned}
\mathrm{Tr}(fe^{-tH_V}f) &\leqslant \int_{\mathbb{H}} f(z)^2 p_V(t,z,z)m(dz) \\
&\leqslant \int_{\mathbb{H}} m(dz)p(t,z,z) \int_W \mu_t^{z,z}(dw) \exp\left(-\int_0^t V(\widetilde{Z}(s,z,w))\,ds\right) \\
&= p(t,\sqrt{-1},\sqrt{-1}) \int_W \mu_t^{\sqrt{-1},\sqrt{-1}}(dw) \\
&\quad \times \int_{\mathbb{H}} m(dz) \exp\left(-\int_0^t V(x+y\widetilde{X}(s,w), y\widetilde{Y}(s,w))\,ds\right), \qquad (5.11)
\end{aligned}
$$

where p is as in (3.2) and we used Lemma 3.4 and Fubini's theorem in the last equality. Here we also used the fact that $p(t,z,z)$ is independent of $z \in \mathbb{H}$.

By Jensen's inequality, we have

$$
\begin{aligned}
&\exp\left(-\int_0^t V(x+y\widetilde{X}(s,w), y\widetilde{Y}(s,w))\,ds\right) \\
&\quad \leqslant \int_0^t \frac{ds}{t} \exp(-tV(x+y\widetilde{X}(s,w), y\widetilde{Y}(s,w))). \qquad (5.12)
\end{aligned}
$$

For any function F and for any $a>0$ and $b \in \mathbb{R}$, it follows that

$$
\int_{\mathbb{H}} F(x+by, ay)m(dz) = a \int_{\mathbb{H}} F(x,y)m(dz). \qquad (5.13)
$$

From (5.12) and (5.13), we see that

the right-hand side of (5.11)

$$
\begin{aligned}
&\leqslant p(t,\sqrt{-1},\sqrt{-1}) \int_W \mu_t^{\sqrt{-1},\sqrt{-1}}(dw) \int_0^t \frac{ds}{t} \widetilde{Y}(s,w) \int_{\mathbb{H}} \exp(-tV(z))m(dz) \\
&= \frac{J(t)}{t} \int_{\mathbb{H}} \exp(-tV(z))m(dz),
\end{aligned}
$$

where we used expressions (3.1) and (5.5).

Since $fe^{-tH_V}f$ converges strongly to e^{-tH_V} as $f \nearrow 1$, we can apply Lemma 2.1 to obtain (5.8). This completes the proof. □

5.3. *Lower bound for* $\mathrm{Tr}(e^{-tH_V})$

For $\varepsilon, t > 0$ and $z \in \mathbb{H}$, we set

$$
\Delta(z;\varepsilon) = \{z' \in \mathbb{H} \mid d_{\mathbb{H}}(z,z') \leqslant \varepsilon\}
$$

and

$$Q(\varepsilon;t) = P_t^{z,z}(\{l \in \mathscr{L}_t^{z,z}(\mathbb{H}) \mid l_s \in \Delta(z;\varepsilon) \text{ for all } s \in [0,t]\}). \tag{5.14}$$

Lemma 5.4. *For every* ε, $t>0$, *the probability* $Q(\varepsilon;t)$ *is independent of* z.

Proof. Using (1.2), we have $d_{\mathbb{H}}(\widetilde{Z}(t,z,w),z) = d_{\mathbb{H}}(\widetilde{Z}(t,w),\sqrt{-1})$, since $\widetilde{X}(t,z,w) = x + y\widetilde{X}(t,w)$ and $\widetilde{Y}(t,z,w) = y\widetilde{Y}(t,w)$. By Lemma 3.4, we have

$$\mu_t^{z,z}(\{w \in W \mid \widetilde{Z}(t,z,w) \in \Delta(z;\varepsilon) \text{ for all } s \in [0,t]\})$$
$$= \mu_t^{\sqrt{-1},\sqrt{-1}}(\{w \in W \mid \widetilde{Z}(t,w) \in \Delta(\sqrt{-1};\varepsilon) \text{ for all } s \in [0,t]\}).$$

This shows the lemma. □

Lemma 5.5. *Let* $p(t,\sqrt{-1},\sqrt{-1})$ *and* $Q(\varepsilon;t)$ *be as above. Then we have*

$$\lim_{t \searrow 0} tp(t,\sqrt{-1},\sqrt{-1})Q(\varepsilon;t) = \frac{1}{2\pi} \tag{5.15}$$

for each fixed $\varepsilon>0$.

Proof. Set $\widetilde{\Delta}_\varepsilon = \{x + y\sqrt{-1} \in \mathbb{H} \mid |x| \leqslant \varepsilon, e^{-\varepsilon} \leqslant y \leqslant e^{\varepsilon}\}$ and

$$\widetilde{Q}(\varepsilon;t) = P_t^{\sqrt{-1},\sqrt{-1}}(\{l \in \mathscr{L}_t^{\sqrt{-1},\sqrt{-1}}(\mathbb{H}) \mid l_s \in \widetilde{\Delta}_\varepsilon \text{ for all } s \in [0,t]\}).$$

It is sufficient to show that $\lim_{t \searrow 0} \widetilde{Q}(\varepsilon;t) = 1$ holds for each $\varepsilon>0$, since for any $\varepsilon>0$ there exist $\varepsilon'>0$, $\varepsilon''>0$ such that $\widetilde{Q}(\varepsilon',t) \leqslant Q(\varepsilon,t) \leqslant \widetilde{Q}(\varepsilon'',t)$.

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$. We apply Lemma 4.4 with $c = \sqrt{-1}$, $g(x_1,\dots,x_{n-1}) = \prod_{i=1}^{n-1} \chi(x_i)$ and $h(y_1,\dots,y_{n-1}) = \prod_{i=1}^{n-1} \chi(\log y_i)$. Letting $\chi(x) \to I_{[-\varepsilon,\varepsilon]}(x)$ for all $x \in \mathbb{R}$, we obtain

$$p(t,\sqrt{-1},\sqrt{-1}) \int_{\mathscr{L}_t^{\sqrt{-1},\sqrt{-1}}(\mathbb{H})} \prod_{i=1}^{n-1} I_{\{l_{t_i} \in \widetilde{\Delta}_\varepsilon\}}(l) P_t^{\sqrt{-1},\sqrt{-1}}(dl)$$
$$= e^{-t/8}(2\pi t)^{-1/2} \int_{\mathscr{L}_t^{0,0}(\mathbb{R})} v_t^{0,0}(dw) \prod_{i=1}^{n-1} I_{\{|w_{t_i}| \leqslant \varepsilon\}}(w)$$
$$\times \left[(2\pi A_t(w))^{-1/2} \int_{\mathscr{L}_{A_t(w)}^{0,0}(\mathbb{R})} \prod_{i=1}^{n-1} I_{\{|b_{t_i}| \leqslant \varepsilon\}}(b) v_{A_t(w)}^{0,0}(db)\right]. \tag{5.16}$$

By letting the mesh of the partition $0<t_1<\cdots<t_{n-1}<t$ go down to zero, we have

$$p(t,\sqrt{-1},\sqrt{-1})\widetilde{Q}(\varepsilon;t) = e^{-t/8}(2\pi t)^{-1/2}\int_{\mathscr{L}_t^{0,0}(\mathbb{R})} v_t^{0,0}(dw) I_{\{\|w\|_\infty \leqslant \varepsilon\}}(w) \times \left[(2\pi A_t(w))^{-1/2}\int_{\mathscr{L}^{0,0}_{A_t(w)}(\mathbb{R})} I_{\{\|b\|_\infty \leqslant \varepsilon\}}(b) v^{0,0}_{A_t(w)}(db)\right], \tag{5.17}$$

where $\|w\|_\infty = \sup_{0<s<1}|w_s|$ etc. Now we use the scaling property for both w and b. Then, the right-hand side of (5.17) is equal to

$$e^{-t/8}(2\pi t)^{-1/2}\int_{\mathscr{L}_1^{0,0}(\mathbb{R})} v_1^{0,0}(dw) I_{\{\|w\|_\infty \leqslant \varepsilon/\sqrt{t}\}}(w) \times \left[(2\pi t C_t(w))^{-1/2}\int_{\mathscr{L}_1^{0,0}(\mathbb{R})} I_{\{\|b\|_\infty \leqslant \varepsilon/t_{1/2}C_t(w)\}}(b) v_1^{0,0}(db)\right], \tag{5.18}$$

where $C_t(w) = \int_0^1 \exp(2\sqrt{t}w_u)\,du$.

Note that, if $t\leqslant 1$, $C_t^{-1/2}$ is dominated as

$$\left(\int_0^1 \exp(2\sqrt{t}w_u)\,du\right)^{-1/2} \leqslant \left(\int_0^1 \exp\left(-2\sqrt{t}\sup_{0\leqslant u\leqslant 1}|w_u|\right)du\right)^{-1/2} \leqslant \exp\left(\sup_{0\leqslant u\leqslant 1}|w_u|\right).$$

Similarly we can estimate C_t^{-1}. Then the lemma follows from the dominated convergence theorem, (3.3), (5.17) and (5.18). □

Now we give a lower bound for the trace of e^{-tH_V}.

Lemma 5.6. *Assume that V satisfies* (A.1). *Then, for each ε, $t>0$, we have the lower bound estimate*

$$\mathrm{Tr}(e^{-tH_V}) \geqslant \frac{t}{2\pi} p(t,\sqrt{-1},\sqrt{-1})Q(\varepsilon;t) \times \int_{\mathbb{H}\times\mathbb{R}^2} \exp\left[-t\left(\frac{y^2}{2}(\xi^2+\eta^2) + V_\varepsilon(x,y)\right)\right] dx\,dy\,d\xi\,d\eta,$$

where $(x,y;\xi,\eta)\in\mathbb{H}\times\mathbb{R}^2$ and $dx\,dy\,d\xi\,d\eta$ is the four dimensional Lebesgue measure.

Proof. It follows from Proposition 5.1 and Lemma 5.3 that e^{-tH_V} is of trace class and has the continuous integral kernel $p_V(t,z,z')$. Then Lemma 2.2(ii) implies that the trace of e^{-tH_V} is given by the integration of the kernel over the diagonal. Hence, we have

$$\begin{aligned}
\mathrm{Tr}(e^{-tH_V}) &= \int_{\mathbb{H}} m(dz) p_V(t,z,z) \\
&= \int_{\mathbb{H}} m(dz) p(t,z,z) \int_{\mathscr{L}_t^{z,z}(\mathbb{H})} \exp\left(-\int_0^t V(l_s)\,ds\right) P_t^{z,z}(dl) \\
&\geqslant \int_{\mathbb{H}} m(dz) p(t,z,z) \int_{\Omega} \exp\left(-\int_0^t V(l_s)\,ds\right) P_t^{z,z} \\
&\geqslant p(t,\sqrt{-1},\sqrt{-1}) Q(\varepsilon;t) \int_{\mathbb{H}} \exp(-tV_\varepsilon(z)) m(dz) \\
&= \frac{t}{2\pi} p(t,\sqrt{-1},\sqrt{-1}) Q(\varepsilon;t) \\
&\quad \times \int_{\mathbb{H}\times\mathbb{R}^2} \exp\left[-t\left(\frac{y^2}{2}(\xi^2+\eta^2) + V_\varepsilon(x,y)\right)\right] dx\,dy\,d\xi\,d\eta,
\end{aligned}$$

where Ω in the third line is as in (5.14) and we used (3.11) in the last equality. □

6. Proof of Theorem 1.3

In this section we complete the proof of Theorem 1.3 as in the proof of Theorem 10.5 in [16]. To the aim, we recall the celebrated Abelian and Tauberian theorems.

Abelian Theorem (Simon [16, Theorem 10.2]). Let μ be a positive Borel measure on $[0,\infty)$. Assume that there exist $C>0$, $\gamma>0$ such that $\lim_{\lambda\to\infty} \lambda^{-\gamma}\mu([0,\lambda)) = C$ holds. Then we have $\lim_{t\searrow 0} t^\gamma \int_0^\infty e^{-tx}\mu(dx) = C\Gamma(\gamma+1)$. Here $\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}\,dx$ is the gamma function.

Tauberian theorem (Simon [16, Theorem 10.3]). Let μ be a positive Borel measure on $[0,\infty)$. Assume that $\int_0^\infty e^{-tx}\mu(dx)$ is finite for each $t>0$ and assume that there exist $D>0$, $\gamma>0$ such that $\lim_{t\searrow 0} t^\gamma \int_0^\infty e^{-tx}\mu(dx) = D$ holds. Then we have $\lim_{\lambda\to\infty} \lambda^\gamma \mu([0,\lambda)) = D/\Gamma(\gamma+1)$.

Proposition 6.1. *Assume that V satisfies* (A.1) *and* (A.2). *Let C_V be the constant as in* (A.2). *Then we have*

$$\lim_{t\searrow 0} t^\gamma\,\mathrm{Tr}(e^{-tH_V}) = C_V\Gamma(\gamma+1). \tag{6.1}$$

Proof. We apply the Abelian theorem to the measure on $[0,\infty)$ defined by

$$\mu([0,\lambda)) = \left|\left\{(x,y;\xi,\eta)\in\mathbb{H}\times\mathbb{R}^2 \,\middle|\, \frac{y^2}{2}(\xi^2+\eta^2)+V(x,y)<\lambda\right\}\right|.$$

Then (1.4) means that the assumption as in the Abelian theorem is fulfilled with $C=(2\pi)^2C_V$, hence we have the conclusion

$$\lim_{t\searrow 0} t^\gamma \int_0^\infty e^{-tx}\mu(dx) = \lim_{t\searrow 0} t^\gamma \int_{\mathbb{H}\times\mathbb{R}^2} \exp\left[-t\left(\frac{y^2}{2}(\xi^2+\eta^2)+V(x,y)\right)\right] dx\,dy\,d\xi\,d\eta$$
$$=(2\pi)^2 C_V\Gamma(\gamma+1). \tag{6.2}$$

By Lemmas 5.3, 5.2 and (6.2), we have

$$\limsup_{t\searrow 0} t^\gamma \operatorname{Tr}(e^{-tH_V}) \leqslant C_V\Gamma(\gamma+1). \tag{6.3}$$

Similarly, by Lemma 5.6, Lemma 5.5, (1.5) and the Abelian theorem, we have

$$\liminf_{t\searrow 0} t^\gamma \operatorname{Tr}(e^{-tH_V}) \geqslant C_{V,\varepsilon}\Gamma(\gamma+1). \tag{6.4}$$

Then the lemma follows from (6.3) and (6.4) since $\lim_{\varepsilon\searrow 0} C_{V,\varepsilon}=C_V$ holds by (A.2). □

Now we apply the Tauberian theorem to the measure on $[0,\infty)$ defined by $\mu([0,\lambda))=N(H_V<\lambda)$. Proposition 6.1 says that the assumption as in the Tauberian theorem with $D=C_V/\Gamma(\gamma+1)$. Then we have the conclusion.

7. Examples

In this section we show the existence of a scalar potential V which satisfies conditions (A.1) and (A.2).

Let $\mathbb{D}=\{z=re^{\theta\sqrt{-1}}\in\mathbb{C}\mid 0\leqslant r<1,\ 0\leqslant\theta<2\pi\}$ be the Poincaré disk endowed with Riemannian measure $m(dz)=4r(1-r^2)^{-2}\,dr\,d\theta$. The distance between 0 and $re^{i\theta}\in\mathbb{D}$ is given by

$$d_{\mathbb{D}}(0,re^{\theta\sqrt{-1}})=\log(1+r)(1-r)^{-1}. \tag{7.1}$$

For any $z\in\mathbb{H}$, let $C(z)=(z-\sqrt{-1})(z+\sqrt{-1})^{-1}$ be the Cayley transform, which defines an isometric diffeomorphism from $\mathbb{H}$ to $\mathbb{D}$. In what follows we often abbreviate $d_{\mathbb{D}}(0,re^{\theta\sqrt{-1}})$ to d, and we identify any function V on $\mathbb{D}$ with the function $V(C(\cdot))$ on $\mathbb{H}$.

For any $\alpha>0$, $\delta>0$, we take $V(z)=\alpha(1-r^2)^{-\delta}$ for $z=re^{\theta\sqrt{-1}}\in\mathbb{D}$ and we show that such V has the desired property. Using the facts that (7.1) is equivalent to the relation $r = (e^d - 1)/(e^d + 1) = \tanh(d/2)$ and that $1 - \tanh^2 = 1/\cosh^2$, we can write

$$V(z) = \alpha(\cosh d/2)^{2\delta}. \tag{7.2}$$

Lemma 7.1. *Let V be as above. Then V satisfies* (A.1).

Proof. Note that $C_{R_0,N} = \sup_{R>R_0} R^N e^{-R}$ is finite for any $R_0 > 0$ and for any integer N. Fix $t > 0$. Then, setting $R_0 = t\alpha$, $R = t\alpha(1 - r^2)^{-\delta}$, we have

$$\begin{aligned} e^{-t\alpha(1-r^2)^{-\delta}} &= (1 - r^2)^N (1 - r^2)^{-N} e^{-t\alpha(1-r^2)^{-\delta}} \\ &= (1 - r^2)^N (R/t\alpha)^{N/\delta} e^{-R} \\ &\leqslant C_{R_0,N}(t\alpha)^{-N/\delta}(1 - r^2)^N. \end{aligned}$$

Hence we have

$$\begin{aligned} \int_{\mathbb{D}} e^{-tV(z)} m(dz) &= \int\int e^{-t\alpha(1-r^2)^{-\delta}} \frac{4r\,dr\,d\theta}{(1 - r^2)^2} \\ &\leqslant 2\pi C_{R_0,N}(t\alpha)^{-N/\delta} \int_0^1 (1 - r^2)^{N-2} 4r\,dr. \end{aligned} \tag{7.3}$$

The right-hand side of (7.3) is finite for large N. This completes the proof. □

Lemma 7.2. *Let V be as in* (7.2). *Then V satisfies* (1.4) *with* $C_V = \frac{2\delta}{\delta+1}\alpha^{-1/\delta}$ *and* $\gamma = 1 + 1/\delta$.

Proof. By considering the change of variables $(\xi, \eta) \to (\sqrt{\alpha}\xi, \sqrt{\alpha}\eta)$ and $\lambda \to \lambda/\alpha$, we may assume that $\alpha = 1$. Then the left-hand side of (1.4) is equal to

$$\begin{aligned} &\int\int_{\mathbb{H}\cap\{V<\lambda\}} dx\,dy \int\int_{\mathbb{R}^2} I_{\{\xi^2+\eta^2<2(\lambda-V)/y^2\}}(\xi, \eta)\,d\xi\,d\eta \\ &= \int\int_{\mathbb{H}\cap\{V<\lambda\}} dx\,dy \int_0^{2\pi} d\varphi \int_0^{\sqrt{2(\lambda-V)}/y} \rho\,d\rho \\ &= 2\pi \int\int_{\mathbb{H}\cap\{V<\lambda\}} (\lambda - V(x, y))y^{-2}\,dx\,dy \\ &= 2\pi \int\int_{\mathbb{D}\cap\{V<\lambda\}} (\lambda - V(z))m(dz) \\ &= (2\pi)^2\lambda \int_{\{r^2<1-\lambda^{-1/\delta}\}} \frac{4r\,dr}{(1 - r^2)^2} - (2\pi)^2 \int_{\{r^2<1-\lambda^{-1/\delta}\}} (1 - r^2)^{-\delta-2} 4r\,dr \end{aligned}$$

$$= (-2(2\pi)^2\lambda + 2(2\pi)^2\lambda^{1+1/\delta}) - \left(-\frac{2(2\pi)^2}{1+\delta} + \frac{2(2\pi)^2\lambda^{1+1/\delta}}{1+\delta}\right)$$
$$= (2\pi)^2\frac{2\delta}{1+\delta}\lambda^{1+1/\delta} + O(\lambda)$$

as $\lambda \to \infty$. This proves the lemma. □

Lemma 7.3. *Let V be as in* (7.2). *Then V satisfies* (A.2) *with $C_V = \frac{2\delta}{\delta+1}\alpha^{-1/\delta}$ and $\gamma = 1 + 1/\delta$.*

Proof. Let $\varepsilon > 0$ be small enough. If $z, z' \in \mathbb{D}$ satisfy the relation $d(z,z') \leqslant \varepsilon$, then

$$\cosh((d(0,z) - \varepsilon)/2) \leqslant \cosh(d(0,z')) \leqslant \cosh((d(0,z) + \varepsilon)/2), \tag{7.4}$$

where we used the triangle inequality $d(0,z) - d(z,z') \leqslant d(0,z') \leqslant d(0,z) + d(z,z')$. Using the formula $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y = \cosh x(\cosh y + \tanh x \sinh y)$, we have

$$\begin{aligned}&\text{the right-hand side of (7.4)}\\ &\quad = \cosh(d(0,z)/2)\,[\cosh(\varepsilon/2) + \tanh(d(0,z)/2)\sinh(\varepsilon/2)]\\ &\quad \leqslant [\cosh(\varepsilon/2) + \sinh(\varepsilon/2)]\cosh(d(0,z)/2)\end{aligned} \tag{7.5}$$

and

$$\begin{aligned}&\text{the left-hand side of (7.4)}\\ &\quad = \cosh(d(0,z)/2)\,[\cosh(\varepsilon/2) - \tanh(d(0,z)/2)\sinh(\varepsilon/2)]\\ &\quad \geqslant [\cosh(\varepsilon/2) - \sinh(\varepsilon/2)]\cosh(d(0,z)/2).\end{aligned} \tag{7.6}$$

Taking the form (7.2) into account, we deduce from (7.4) to (7.6) that

$$(\cosh(\varepsilon/2) - \sinh(\varepsilon/2))^{2\delta}, V(z) \leqslant V(z') \leqslant (\cosh(\varepsilon/2) + \sinh(\varepsilon/2))^{2\delta} V(z)$$

provided $d(z,z') \leqslant \varepsilon$. Then, by the definition (1.3) of V_ε, we have

$$(\cosh(\varepsilon/2) - \sinh(\varepsilon/2))^{2\delta} V(z) \leqslant V_\varepsilon(z) \leqslant (\cosh(\varepsilon/2) + \sinh(\varepsilon/2))^{2\delta} V(z)$$

for all $z \in \mathbb{D}$.

Then it follows from Lemma 7.2 (and its proof) replaced α by $(\cosh(\varepsilon/2) \pm \sinh(\varepsilon/2))^{2\delta}\alpha$ that the limit $C_{V,\varepsilon}$ exists and the estimate

$$C_V(\cosh(\varepsilon/2) + \sinh(\varepsilon/2))^{-2} \leqslant C_{V,\varepsilon} \leqslant C_V(\cosh(\varepsilon/2) - \sinh(\varepsilon/2))^{-2}$$

holds. Letting $\varepsilon \to 0$, we complete the proof. □

Remark 7.4. By examining the proofs in this section we can easily find that the same conclusions as in Lemmas 7.1–7.3 hold if V satisfies (A.0) and the condition

$$\lim_{r \nearrow 1} \frac{V(re^{\theta\sqrt{-1}})}{\alpha(1-r^2)^{-\delta}} = 1$$

uniformly in θ for some $\alpha > 0$ and $\delta > 0$.

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