

Trotter–Kato product formula and fractional powers of self-adjoint generators

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Abstract

Let A and B be non-negative self-adjoint operators in a Hilbert space such that their densely defined form sum $H = A + B$ obeys $\text{dom}(H^\alpha) \subseteq \text{dom}(A^\alpha) \cap \text{dom}(B^\alpha)$ for some $\alpha \in (1/2, 1)$. It is proved that if, in addition, A and B satisfy $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$, then the symmetric and non-symmetric Trotter–Kato product formula converges in the operator norm:

$$\| (e^{-tB/2n} e^{-tA/n} e^{-tB/2n})^n - e^{-tH} \| = O(n^{-(2\alpha-1)})$$

$$\| (e^{-tA/n} e^{-tB/n})^n - e^{-tH} \| = O(n^{-(2\alpha-1)})$$

uniformly in $t \in [0, T]$, $0 < T < \infty$, as $n \rightarrow \infty$, both with the same optimal error bound. The same is valid if one replaces the exponential function in the product by functions of the Kato class, that is, by real-valued Borel measurable functions $f(\cdot)$ defined on the non-negative real axis obeying $0 \leq f(x) \leq 1$, $f(0) = 1$ and $f'(+0) = -1$, with some additional smoothness property at zero. The present result improves previous ones relaxing the smallness of B^α with

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respect to A^x to the milder assumption $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ and extending essentially the admissible class of Kato functions.

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1. Introduction

In the present paper we deal with the operator-norm convergence of the Trotter–Kato product formula, which may have applications in quantum and in statistical mechanics.

Let A and B be two non-negative self-adjoint operators in a Hilbert space \mathfrak{H} such that $\text{dom}(A^{1/2}) \cap \text{dom}(B^{1/2})$ is dense in \mathfrak{H} . By H we denote the form-sum of A and B , i.e.

$$H = A \dot{+} B$$

which is a non-negative self-adjoint operator in the Hilbert space \mathfrak{H} . Obviously, one has $\text{dom}(H^{1/2}) = \text{dom}(A^{1/2}) \cap \text{dom}(B^{1/2})$. Further, we consider the Kato functions: they are real-valued Borel measurable functions f on $[0, \infty)$ satisfying

$$\begin{aligned} 0 \leq f(x) \leq 1, \quad x \in [0, \infty), \quad f(0) = 1, \quad f'(+0) = -1, \\ 0 \leq g(x) \leq 1, \quad x \in [0, \infty), \quad g(0) = 1, \quad g'(+0) = -1. \end{aligned}$$

Typical examples of the Kato functions are

$$f(x) = e^{-x} \quad \text{and} \quad f(x) = (1 + k^{-1}x)^{-k}, \quad k > 0. \quad (1.1)$$

In two remarkable papers [6,7], Kato has shown that these assumptions are enough to prove

$$s\text{-}\lim_{n \rightarrow \infty} (f(tA/n)g(tB/n))^n = e^{-tH}$$

uniformly in $t \in [0, T]$, $0 < T < \infty$. In the following we call a relation of this type a Trotter–Kato product formula. Naturally the question arises whether under suitable conditions the strong convergence of the Trotter–Kato product formula can be improved to the operator-norm convergence with a convergence rate estimate. Indeed this is possible. Beginning with Rogava [16] the operator-norm convergence with different convergence rates was verified in [8–14]. However, all these convergence rates are not optimal except the cases studied in [12,14]. In [11,12] some of ideas of Chernoff [2,3], have been used to prove that in the case $\text{dom}(H) \subseteq \text{dom}(A) \cap \text{dom}(B)$, which means that the algebraic sum $H = A + B$ is self-adjoint, the optimal error bound is $O(n^{-1})$. In [14] the conditions are formulated on

the fractional powers A^α, B^α , for $\alpha \in (1/2, 1]$. They lead to the optimal convergence rate $O(n^{-2(\alpha-1)})$.

The aim of the present paper is to study, in a sense, an intermediate case. We assume

$$\text{dom}(H^\alpha) \subseteq \text{dom}(A^\alpha) \cap \text{dom}(B^\alpha) \tag{1.2}$$

for some $\alpha \in (1/2, 1)$. This assumption is stronger than the natural condition $\text{dom}(H^{1/2}) \subseteq \text{dom}(A^{1/2}) \cap \text{dom}(B^{1/2})$, which is always satisfied, but weaker than the assumption $\text{dom}(H) \subseteq \text{dom}(A) \cap \text{dom}(B)$ used in [11] and [12]. Notice that comparing to [14] we do not demand the smallness of B^α with respect to A^α . We assume that the Kato functions further fulfill the conditions

$$|f|_{2\alpha} := \sup_{x>0} \frac{|f(x) - 1 + x|}{x^{2\alpha}} < +\infty, \quad \alpha \in (0, 1], \tag{1.3}$$

$$|g|_{2\alpha} := \sup_{x>0} \frac{|g(x) - 1 + x|}{x^{2\alpha}} < +\infty, \quad \alpha \in (0, 1]. \tag{1.4}$$

It is easily seen that (1.3) and (1.4) are in fact conditions at the neighborhood of zero. We set

$$m_f(x) := \sup_{y \in [x, \infty)} f(y), \quad x > 0.$$

Notice that examples (1.1) satisfy condition (1.3) and $m_f(x) < 1$ for $x > 0$. The aim of this note is to prove the following statement:

Theorem 1.1. *Let A and B be non-negative self-adjoint operators and let $H := A \dot{+} B$ be their form sum. Assume that for some $\alpha \in (1/2, 1)$ condition (1.2) is satisfied. Further, let f and g be Kato functions which obey conditions (1.3) and (1.4). If in addition one has $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ and $m_f(x) < 1$ for $x > 0$, then for any finite interval $[0, T]$ there is a constant $C_{T,2\alpha-1} > 0$ such that*

$$\| (f(tA/2n)g(tB/n)f(tA/2n))^n - e^{-tH} \| \leq C_{T,2\alpha-1} \frac{1}{n^{2\alpha-1}} \tag{1.5}$$

for $t \in [0, T]$ and $n = 1, 2, \dots$.

Estimate (1.5) gives the ultimate error bound for the convergence rate, which can be seen from an example given in [17]. Formula (1.5) remains true with the same error bound, when $f(\tau A/2)g(\tau B)f(\tau A/2)$, where $\tau = t/n$, is replaced by related families of the form $f(\tau A)^{1/2}g(\tau B)f(\tau A)^{1/2}$, $g(\tau B)^{1/2}f(\tau A)g(\tau B)^{1/2}$, $g(\tau B)f(\tau A)$, $f(\tau A)g(\tau B)$, $g(\tau B/2)f(\tau A)g(\tau B/2)$ where $\tau \geq 0$. We note further that Theorem 1.1 improves the previous result in [14] where the same optimal convergence rate $O(n^{-(2\alpha-1)})$ was

obtained but under stronger conditions on the operators A and B and on the Kato functions f and g , for details see Section 7. Notice also that Theorem 1.1 treats a case which is not covered by [11,12]. Nevertheless, setting formally $\alpha = 1$ in Theorem 1.1 the assumptions and results of that theorem turn into those ones of [11,12], except for the assumption $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ which is superfluous there. In [8] it is conjectured that this condition can be dropped for $\alpha \in (1/2, 1]$. However, the proof of this conjecture remains still open. By the way condition $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ implies $\text{dom}(H^{1/2}) = \text{dom}(A^{1/2})$ which guarantees the density of $\text{dom}(H^{1/2})$ in \mathfrak{H} .

The proof of Theorem 1.1 relies again on ideas of Chernoff [2,3] and follows in many aspects [11,12]. Theorem 2.1 in Section 2 can be regarded as an operator-norm variant of Chernoff's approach, which, in contrast to [15], gives error-bound estimates. In particular, it generalizes a result of [11]. In Section 3 we prove some estimates which are necessary in the following using results from [11]. The main theorem announced in the introduction is proved in Section 4. In Section 6, an example is given which illustrates the situation of our main theorem. Finally, in Section 7 we make some remarks on the obtained result.

2. Convergence theorem

We start with a lemma, which is proven using the standard Dunford–Taylor operator calculus.

Lemma 1. *Let K and L be non-negative self-adjoint operators on the Hilbert space \mathfrak{H} . Then*

$$\|e^{-K} - e^{-L}\| \leq N \|(I + K)^{-1} - (I + L)^{-1}\|$$

with a constant $N > 0$ independent of operators K and L .

Proof. By the Dunford–Taylor representation for exponentials we find

$$e^{-K} - e^{-L} = \frac{1}{2\pi i} \int_{\Gamma} dz e^{-z} ((z - K)^{-1} - (z - L)^{-1}), \quad (2.1)$$

where the contour Γ is given by $\Gamma = \Gamma_0 \cup \Gamma_{\infty}$ with

$$\Gamma_0 = \{z \in \mathbb{C} : z = e^{i\varphi}, \pi/4 \leq \varphi \leq 2\pi - \pi/4\},$$

$$\Gamma_{\infty} = \{z \in \mathbb{C} : z = re^{\pm i\pi/4}, r \geq 1\}.$$

From (2.1) we find the representation

$$e^{-K} - e^{-L} = \frac{1}{2\pi i} \int_{\Gamma} dz e^{-z} (I + K)(z - K)^{-1} \times ((I + L)^{-1} - (I + K)^{-1})(I + L)(z - L)^{-1}. \tag{2.2}$$

Since

$$(I + K)(z - K)^{-1} = -I + (1 + z)(z - K)^{-1}$$

one gets the estimate

$$\|(I + K)(z - K)^{-1}\| \leq 1 + \frac{1 + |z|}{\text{dist}(z, \mathbb{R}_+)}.$$

Setting

$$N_{\Gamma} := \sup_{z \in \Gamma} \frac{1 + |z|}{\text{dist}(z, \mathbb{R}_+)} < \infty$$

we find

$$\sup_{z \in \Gamma} \|(I + K)(z - K)^{-1}\| \leq (1 + N_{\Gamma}), \tag{2.3}$$

where the constant N_{Γ} depends only on Γ but not on the operator K . Similarly, from (2.3) one also gets

$$\sup_{z \in \Gamma} \|(I + L)(z - L)^{-1}\| \leq (1 + N_{\Gamma}).$$

Using these estimates, we find from (2.2) that

$$\|e^{-K} - e^{-L}\| \leq N \|(I + K)^{-1} - (I + L)^{-1}\|$$

with a constant

$$N := \frac{1}{2\pi} (1 + N_{\Gamma})^2 \int_{\Gamma} |dz| |e^{-z}|$$

depending only on the contour Γ . \square

To prove the main theorem we shall use, as in [11,12], an operator norm version of Chernoff's theorem with error bound, which partly improves Lemma 2.1 of [11] in the case $\varrho < 1$.

Theorem 2.1. *Let $\{F(\tau)\}_{\tau \geq 0}$ be a family of non-negative self-adjoint contractions on \mathfrak{H} such that $F(0) = I$ and let*

$$S(\tau) := \frac{I - F(\tau)}{\tau}, \quad \tau > 0.$$

Assume $\varrho \in (0, 1)$. Then there is a constant $M_\varrho > 0$ such that the estimate

$$\|(I + tS(\tau))^{-1} - (I + tH)^{-1}\| \leq M_\varrho \left(\frac{\tau}{t}\right)^\varrho \quad (2.4)$$

holds for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$, if and only if there is a constant $C_\varrho > 0$ such that the estimate

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq C_\varrho \left(\frac{\tau}{t}\right)^\varrho \quad (2.5)$$

is valid for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$.

Proof. Assume (2.4). By Lemma 1 there is a constant $N > 0$ such that

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq N \|(I + tS(\tau))^{-1} - (I + tH)^{-1}\|$$

for $\tau, t > 0$. Using (2.4) we obtain

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq NM_\varrho \left(\frac{\tau}{t}\right)^\varrho$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq 1$. Since

$$\sup_{x \in [0, 1]} |x^r - e^{-r(1-x)}| \leq \frac{1}{r}, \quad r \geq 1,$$

we find

$$\|F(\tau)^{t/\tau} - e^{-tS(\tau)}\| \leq \frac{\tau}{t}$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. By the inequality

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq \|F(\tau)^{t/\tau} - e^{-tS(\tau)}\| + \|e^{-tS(\tau)} - e^{-tH}\|$$

we finally get

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq \frac{\tau}{t} + NM_\varrho \left(\frac{\tau}{t}\right)^\varrho$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. Since $\frac{\tau}{t} \leq \left(\frac{\tau}{t}\right)^\varrho$ for $0 < \tau \leq t$, $\varrho \in [0, 1]$, we obtain

$$\|F(\tau)^{t/\tau} - e^{-tH}\| \leq (1 + NM_\varrho) \left(\frac{\tau}{t}\right)^\varrho$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. Setting $C_\varrho := 1 + NM_\varrho$ we have verified (2.5).

To prove the converse we use the representation

$$(I + tS(\tau))^{-1} - (I + tH)^{-1} = \int_0^\infty dx e^{-x} (e^{-xtS(\tau)} - e^{-xtH})$$

for $\tau, t > 0$. We have

$$(I + tS(\tau))^{-1} - (I + tH)^{-1} = \sum_{n=0}^{\infty} \int_n^{n+1} dx e^{-x} (e^{-xtS(\tau)} - e^{-xtH})$$

for $\tau, t > 0$. By the substitution $x = y + n$ we obtain

$$\begin{aligned} & (I + tS(\tau))^{-1} - (I + tH)^{-1} \\ &= \sum_{n=0}^{\infty} e^{-n} \int_0^1 dy e^{-y} (e^{-(y+n)tS(\tau)} - e^{-(y+n)tH}) \end{aligned}$$

for $\tau, t > 0$. Since

$$\begin{aligned} & e^{-(y+n)tS(\tau)} - e^{-(y+n)tH} \\ &= (e^{-ntS(\tau)} - e^{-ntH})e^{-ytS(\tau)} + e^{-ntH}(e^{-ytS(\tau)} - e^{-ytH}) \end{aligned}$$

and

$$e^{-ntS(\tau)} - e^{-ntH} = \sum_{k=0}^{n-1} e^{-ktS(\tau)} (e^{-tS(\tau)} - e^{-tH}) e^{-(n-k-1)tH}$$

we get

$$\begin{aligned} & (I + tS(\tau))^{-1} - (I + tH)^{-1} \\ &= \sum_{n=0}^{\infty} e^{-n} \left\{ \sum_{k=0}^{n-1} e^{-ktS(\tau)} (e^{-tS(\tau)} - e^{-tH}) e^{-(n-k-1)tH} \int_0^1 dy e^{-y} e^{-ytS(\tau)} \right. \\ & \quad \left. + \int_0^1 dy e^{-y} e^{-ntH} (e^{-ytS(\tau)} - e^{-ytH}) \right\}. \end{aligned}$$

Hence we obtain the estimate

$$\begin{aligned} & \|(I + tS(\tau))^{-1} - (I + tH)^{-1}\| \\ & \leq \sum_{n=0}^{\infty} e^{-n} \left\{ n \|e^{-tS(\tau)} - e^{-tH}\| + \int_0^1 dy e^{-y} \|e^{-ytS(\tau)} - e^{-ytH}\| \right\} \end{aligned} \tag{2.6}$$

for $\tau, t > 0$. By assumption (2.5) we have the estimate

$$\|e^{-tS(\tau)} - e^{-tH}\| \leq C_\varrho \left(\frac{\tau}{t}\right)^\varrho \tag{2.7}$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. Further, we use the decomposition

$$\begin{aligned} & \int_0^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \\ &= \int_{\tau/t}^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| + \int_0^{\tau/t} dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\|. \end{aligned} \tag{2.8}$$

Setting $t' = yt$ one has $\tau \leq t'$ if $\tau/t \leq y$. Hence by assumption (2.5) we obtain

$$\|e^{-yS(\tau)} - e^{-yH}\| \leq C_\varrho \left(\frac{\tau}{ty}\right)^\varrho$$

for $\tau, t, y \in (0, 1]$ and $\tau/t \leq y$. This yields the estimate

$$\int_{\tau/t}^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \leq C_\varrho \int_0^1 dy e^{-y} y^{-\varrho} \left(\frac{\tau}{t}\right)^\varrho \tag{2.9}$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. For $\varrho < 1$ one obviously has

$$\int_0^{\tau/t} dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \leq 2 \left(\frac{\tau}{t}\right)^\varrho \tag{2.10}$$

for $\tau, t \in [0, 1]$ with $0 < \tau \leq t$. Taking into account (2.9) and (2.10) we obtain from (2.8) the estimate

$$\int_0^1 dy e^{-y} \|e^{-yS(\tau)} - e^{-yH}\| \leq \left(C_\varrho \int_0^1 dy e^{-y} y^{-\varrho} + 2 \right) \left(\frac{\tau}{t}\right)^\varrho \tag{2.11}$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. Finally, using (2.7) and (2.11) we get from (2.6) the estimate

$$\|(I + tS(\tau))^{-1} - (I + tH)^{-1}\| \leq \sum_{n=0}^\infty e^{-n} \left\{ nC_\varrho + C_\varrho \int_0^1 dy e^{-y} y^{-\varrho} + 2 \right\} \left(\frac{\tau}{t}\right)^\varrho$$

for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. Setting

$$M_\varrho := \sum_{n=0}^\infty e^{-n} \left\{ nC_\varrho + C_\varrho \int_0^1 dy e^{-y} y^{-\varrho} + 2 \right\}$$

we have verified (2.4). \square

We note that in [11] it was shown that for $\varrho = 1$ condition (2.4) implies (2.5).

It is unclear whether in this case the converse is also true. Note also that setting $\tau = t/n, n = 1, 2, \dots$, estimate (2.5) transforms into

$$\|F(t/n)^n - e^{-tH}\| \leq C_\varrho \frac{1}{n^\varrho}$$

for $t \in (0, 1]$ and $n = 1, 2, \dots$. However, this is nothing else but an operator norm estimate for a chosen family of contractions $F(\tau)$.

3. Auxiliary estimates

We are going to apply Theorem 2.1 to the family

$$F(\tau) := f(\tau A/2)g(\tau B)f(\tau A/2), \quad \tau > 0.$$

In the following we use notations which essentially go back to [11,12] but which are slightly modified. We set

$$\begin{aligned} A_\tau &:= \frac{I - f(\tau A)}{\tau}, \quad \tau > 0, \\ B_\tau &:= \frac{I - g(\tau B)}{\tau}, \quad \tau > 0, \\ K_\tau &:= B_\tau + A_{\tau/2} - \frac{\tau}{4}A_{\tau/2}^2, \quad \tau > 0. \end{aligned}$$

One has $K_\tau \geq 0$ and

$$S(\tau) = K_\tau + \frac{\tau^2}{4}A_{\tau/2}B_\tau A_{\tau/2} - \frac{\tau}{2}(B_\tau A_{\tau/2} + A_{\tau/2}B_\tau)$$

for $\tau > 0$. We set

$$\begin{aligned} Q_\tau &:= \frac{\tau^2}{4}(I + K_\tau)^{-1/2}A_{\tau/2}B_\tau A_{\tau/2}(I + K_\tau)^{-1/2} \\ &\quad - \frac{\tau}{2}(I + K_\tau)^{-1/2}(B_\tau A_{\tau/2} + A_{\tau/2}B_\tau)(I + K_\tau)^{-1/2}, \quad \tau > 0, \end{aligned}$$

so that

$$I + S(\tau) = (I + K_\tau)^{1/2}(I + Q_\tau)(I + K_\tau)^{1/2}, \quad \tau > 0.$$

Our next aim is to prove several estimates which we need for the proof of the main theorem.

Lemma 2. *Let A and B be non-negative self-adjoint operators. If f and g are Kato functions, then one has*

$$\|B_\tau^{1/2}(I + K_\tau)^{-1/2}\| \leq \|(I + B_\tau)^{1/2}(I + K_\tau)^{-1/2}\| \leq 1$$

and

$$\|A_{\tau/2}^{1/2}(I + K_{\tau})^{-1/2}\| \leq \sqrt{2} \left\| \left(I + \frac{1}{2}A_{\tau/2} \right)^{1/2} (I + K_{\tau})^{-1/2} \right\| \leq \sqrt{2}$$

for $\tau > 0$. Moreover, the operator $I + Q_{\tau}$ has a bounded inverse for each $\tau > 0$ and the norm of its inverse operator is uniformly estimated by

$$\|(I + Q_{\tau})^{-1}\| \leq \frac{1}{2}(3 + \sqrt{5})$$

for $\tau > 0$.

Proof. The proof can be obtained from [11] by making the replacements $A \leftrightarrow B$ and $f \leftrightarrow g$. \square

Lemma 3. Let A and B be non-negative self-adjoint operators. If f and g are Kato functions and $m_f(x) < 1$ for $x > 0$, then there are constants $C_A > 0$ and $C_0 > 0$ such that

$$\|(I + S(\tau))^{-1/2}u\| \leq C_A \|(I + A)^{-1/2}u\| + C_0 \|u\| \tau^{1/2} \quad (3.1)$$

for $u \in \mathfrak{H}$ and $\tau > 0$.

Proof. Using Lemma 2 we get

$$\|(I + S(\tau))^{-1/2}u\| \leq \sqrt{\frac{3 + \sqrt{5}}{2}} \|(I + K_{\tau})^{-1/2}u\|$$

for $u \in \mathfrak{H}$ and $\tau > 0$. Since $K_{\tau} \geq \frac{1}{2}A_{\tau/2}$ we find

$$\|(I + S(\tau))^{-1/2}u\| \leq \sqrt{3 + \sqrt{5}} \|(I + A_{\tau/2})^{-1/2}u\|$$

for $u \in \mathfrak{H}$ and $\tau > 0$. Obviously, there is a constant $\delta > 0$ such that $|(1 - f(x))x^{-1} - 1| > 1/2$ for $x \in (0, \delta)$. Hence

$$1 - f(x) \geq \frac{1}{2}x$$

for $x \in (0, \delta)$, which yields

$$1 - f(x) \geq \frac{1}{2}x \chi_{(0, \delta)}(x) + (1 - m_f(\delta)) \chi_{[\delta, \infty)}(x)$$

for $x > 0$, where $\chi_{(0,\delta)}(\cdot)$ and $\chi_{[\delta,\infty)}(\cdot)$ are the characteristic functions of the intervals $(0, \delta)$ and $[\delta, \infty)$, respectively. Hence we find

$$A_{\tau/2} \geq \frac{1}{2} A E_A([0, 2\delta/\tau]) + 2 \frac{1 - m_f(\delta)}{\tau} E_A([2\delta/\tau, \infty))$$

for $\tau > 0$, where $E_A(\cdot)$ is the spectral measure of A . Therefore we obtain

$$\begin{aligned} & \| (I + S(\tau))^{-1/2} u \| \\ & \leq \sqrt{2(3 + \sqrt{5})} \| (I + A)^{-1/2} E_A([0, 2\delta/\tau]) u \| \\ & \quad + \sqrt{\frac{3 + \sqrt{5}}{2(1 - m_f(\delta))}} \| E_A([2\delta/\tau, \infty)) u \| \tau^{1/2} \end{aligned}$$

for $u \in \mathfrak{H}$ and $\tau > 0$. Setting $C_A := \sqrt{2(3 + \sqrt{5})}$ and $C_0 := \sqrt{\frac{3 + \sqrt{5}}{2(1 - m_f(\delta))}}$ we prove (3.1). \square

Lemma 4. *Let A and B be non-negative self-adjoint operators such that their form sum obeys (1.2) for some $\alpha \in (1/2, 1)$. Further, let f and g be Kato functions which satisfy (1.3) and (1.4).*

(i) *If $p, q \in [0, \alpha]$ and $p + q \geq 1$, then there is a constant $D_{p,q} > 0$ such that*

$$\| (I + H)^{-p} (H - S(\tau)) (I + H)^{-q} \| \leq D_{p,q} \tau^{p+q-1} \tag{3.2}$$

for $\tau > 0$.

(ii) *There is a constant $D_\alpha > 0$ such that*

$$\| S(\tau) (I + H)^{-\alpha} \| \leq D_\alpha \tau^{\alpha-1} \tag{3.3}$$

for $\tau > 0$.

Proof. (i) First we note that if $p \in [0, \alpha]$, then condition (1.2) implies $\text{dom}(H^p) \subseteq \text{dom}(A^p) \cap \text{dom}(B^p)$. Hence $(I + A)^p (I + H)^{-p}$ and $(I + B)^p (I + H)^{-p}$ are bounded operators. Since

$$\begin{aligned} I - F(\tau) &= I - f(\tau A/2)^2 + (I - g(\tau B)) \\ & \quad + (I - f(\tau A/2))(I - g(\tau B))(I - f(\tau A/2)) \\ & \quad - (I - f(\tau A/2))(I - g(\tau B)) - (I - g(\tau B))(I - f(\tau A/2)), \end{aligned} \tag{3.4}$$

we find

$$\begin{aligned}
 & (I + H)^{-p}(H - S(\tau))(I + H)^{-q} \\
 &= (I + H)^{-p} \left(A - \frac{I - f(\tau A/2)^2}{\tau} \right) (I + H)^{-q} \\
 &+ (I + H)^{-p} \left(B - \frac{I - g(\tau B)}{\tau} \right) (I + H)^{-q} \\
 &- (I + H)^{-p}(I - f(\tau A/2))(I - g(\tau B))(I - f(\tau A/2))(I + H)^{-q}\tau^{-1} \\
 &+ (I + H)^{-p}(I - f(\tau A/2))(I - g(\tau B))(I + H)^{-q}\tau^{-1} \\
 &+ (I + H)^{-p}(I - g(\tau B))(I - f(\tau A/2))(I + H)^{-q}\tau^{-1},
 \end{aligned}$$

which gives the estimate

$$\begin{aligned}
 & \| (I + H)^{-p}(H - S(\tau))(I + H)^{-q} \| \\
 & \leq \left\| \left((I + H)^{-p} \left(A - \frac{I - f(\tau A/2)^2}{\tau} \right) (I + H)^{-q} \right) \right\| \tag{3.5}
 \end{aligned}$$

$$\begin{aligned}
 & + \left\| \left((I + H)^{-p} \left(B - \frac{I - g(\tau B)}{\tau} \right) (I + H)^{-q} \right) \right\| \\
 & + \| (I - f(\tau A/2))(I + H)^{-p} \| \| (I - f(\tau A/2))(I + H)^{-q} \| \tau^{-1} \\
 & + \| (I - f(\tau A/2))(I + H)^{-p} \| \| (I - g(\tau B))(I + H)^{-q} \| \tau^{-1} \\
 & + \| (I - g(\tau B))(I + H)^{-p} \| \| (I - f(\tau A/2))(I + H)^{-q} \| \tau^{-1} \tag{3.6}
 \end{aligned}$$

for $\tau > 0$. To estimate (3.5) we use

$$\begin{aligned}
 & \left\| \left((I + H)^{-p} \left(A - \frac{I - f(\tau A/2)^2}{\tau} \right) (I + H)^{-q} \right) \right\| \\
 & \leq \| (I + A)^p (I + H)^{-p} \| \| (I + A)^q (I + H)^{-q} \| \\
 & \quad \times \left\| \left((I + A)^{-p} \left(A - \frac{I - f(\tau A/2)^2}{\tau} \right) (I + A)^{-q} \right) \right\|
 \end{aligned}$$

and the functional calculus which yield

$$\begin{aligned} & \left\| (I + H)^{-p} \left(A - \frac{I - f(\tau A/2)^2}{\tau} \right) (I + H)^{-q} \right\| \\ & \leq \| (I + A)^p (I + H)^{-p} \| \| (I + A)^q (I + H)^{-q} \| \\ & \quad \times \left\{ |f|_{p+q} + \frac{1}{2} \gamma_f((p + q)/2)^2 \right\} \tau^{p+q-1} \end{aligned}$$

for $\tau > 0$, where

$$\gamma_h(r) := \sup_{x>0} \frac{1 - h(x)}{x^r}, \quad r \in (0, 1],$$

and h is a Kato function. Similarly, one estimates (3.6) to get the inequality

$$\begin{aligned} & \left\| (I + H)^{-p} \left(B - \frac{I - g(\tau B)}{\tau} \right) (I + H)^{-q} \right\| \\ & \leq \| (I + B)^p (I + H)^{-p} \| \| (I + B)^q (I + H)^{-q} \| \|g\|_{p+q} \tau^{p+q-1} \end{aligned}$$

for $\tau > 0$. Since

$$\| (1 - f(\tau A/2))(I + H)^{-r} \| \leq 2^{-r} \| (I + A)^r (I + H)^{-r} \| \gamma_f(r) \tau^r \tag{3.7}$$

and

$$\| (I - g(\tau B))(I + H)^{-r} \| \leq \| (I + B)^r (I + H)^{-r} \| \gamma_g(r) \tau^r \tag{3.8}$$

for $\tau > 0$ and $r \in (0, \alpha]$, we finally obtain the estimate

$$\begin{aligned} & \| (I + H)^{-p} (H - S(\tau))(I + H)^{-q} \| \leq \| (I + A)^p (I + H)^{-p} \| \\ & \quad \times \| (I + A)^q (I + H)^{-q} \| \left\{ |f|_{p+q} + \frac{1}{2} \gamma_f((p + q)/2)^2 \right\} \tau^{p+q-1} \\ & \quad + \| (I + B)^p (I + H)^{-p} \| \| (I + B)^q (I + H)^{-q} \| \|g\|_{p+q} \tau^{p+q-1} \\ & \quad + 2^{-(p+q)} \gamma_f(p) \gamma_f(q) \| (I + A)^p (I + H)^{-p} \| \| (I + A)^q (I + H)^{-q} \| \tau^{p+q-1} \\ & \quad + 2^{-p} \gamma_f(p) \gamma_g(q) \| (I + A)^p (I + H)^{-p} \| \| (I + B)^q (I + H)^{-q} \| \tau^{p+q-1} \\ & \quad + 2^{-q} \gamma_f(q) \gamma_g(p) \| (I + A)^q (I + H)^{-q} \| \| (I + B)^p (I + H)^{-p} \| \tau^{p+q-1} \end{aligned}$$

for $\tau > 0$. Setting

$$\begin{aligned}
 D_{p,q} := & \| (I + A)^p (I + H)^{-p} \| \\
 & \times \| (I + A)^q (I + H)^{-q} \| \left\{ |f|_{p+q} + \frac{1}{2} \gamma_f ((p+q)/2)^2 \right\} \\
 & + \| (I + B)^p (I + H)^{-p} \| \| (I + B)^q (I + H)^{-q} \| |g|_{p+q} \\
 & + 2^{-(p+q)} \gamma_f(p) \gamma_f(q) \| (I + A)^p (I + H)^{-p} \| \| (I + A)^q (I + H)^{-q} \| \\
 & + 2^{-p} \gamma_f(p) \gamma_g(q) \| (I + A)^p (I + H)^{-p} \| \| (I + B)^q (I + H)^{-q} \| \\
 & + 2^{-q} \gamma_f(q) \gamma_g(p) \| (I + A)^q (I + H)^{-q} \| \| (I + B)^p (I + H)^{-p} \|
 \end{aligned}$$

for $\tau > 0$ we prove the estimate (3.2).

(ii) Using decomposition (3.4) we find the estimate

$$\begin{aligned}
 & \| S(\tau)(I + H)^{-\alpha} \| \\
 & \leq 4 \| (I - f(\tau A/2))(I + H)^{-\alpha} \| \tau^{-1} + \| (I - g(\tau B))(I + H)^{-\alpha} \| \tau^{-1}
 \end{aligned}$$

for $\tau > 0$. Taking into account (3.7) and (3.8) we obtain

$$\begin{aligned}
 & \| S(\tau)(I + H)^{-\alpha} \| \leq 2^{2-\alpha} \| (I + A)^\alpha (I + H)^{-\alpha} \| \gamma_f(\alpha) \tau^{\alpha-1} \\
 & + \| (I + B)^\alpha (I + H)^{-\alpha} \| \gamma_g(\alpha) \tau^{\alpha-1}
 \end{aligned}$$

for $\tau > 0$. Setting

$$D_\alpha := 2^{2-\alpha} \| (I + A)^\alpha (I + H)^{-\alpha} \| \gamma_f(\alpha) + \| (I + B)^\alpha (I + H)^{-\alpha} \| \gamma_g(\alpha)$$

we prove (3.3) for $\tau > 0$. \square

4. Error estimate

In order to prove the main theorem we need the following two lemmas.

Lemma 5. *Let A and B be non-negative self-adjoint operators such that for some $\alpha \in (1/2, 1)$ condition (1.2) is satisfied. Further, let f and g be Kato functions which obey conditions (1.3) and (1.4). If in addition one has $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ and $m_f(x) < 1$*

for $x > 0$, then there is a constant $S_{x-1/2} > 0$ such that

$$t \|(I + tS(\tau))^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \leq S_{x-1/2} \left(\frac{\tau}{t}\right)^{\alpha-1/2} \tag{4.1}$$

for $t, \tau \in (0, 1]$ with $0 < \tau \leq t$.

Proof. By the functional calculus one has

$$\begin{aligned} & t \|(I + tS(\tau))^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \\ & \leq t^{1/2} \|(I + S(\tau))^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \end{aligned}$$

for $t \in (0, 1]$ and $\tau > 0$. By Lemma 3 we find

$$\begin{aligned} & t \|(I + tS(\tau))^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \\ & \leq C_A \|(I + A)^{-1/2} (H - S(\tau))(I + tH)^{-1}\| t^{1/2} \\ & \quad + C_0 \|(H - S(\tau))(I + tH)^{-1}\| \tau^{1/2} t^{1/2} \end{aligned} \tag{4.2}$$

for $t \in (0, 1]$ and $\tau > 0$. Since $\text{dom}(H^{1/2}) = \text{dom}(A^{1/2})$, we get

$$\begin{aligned} & \|(I + A)^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \\ & \leq \|(I + H)^{1/2} (I + A)^{-1/2}\| \|(I + H)^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \end{aligned}$$

for $t \in (0, 1]$ and $\tau > 0$. Using again the functional calculus we obtain

$$\begin{aligned} & \|(I + A)^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \\ & \leq \|(I + H)^{1/2} (I + A)^{-1/2}\| \|(I + H)^{-1/2} (H - S(\tau))(I + H)^{-\alpha}\| t^{-\alpha} \end{aligned}$$

for $t \in (0, 1]$ and $\tau > 0$. By Lemma 4(i) there is a constant $D_{\frac{1}{2}, \alpha} > 0$ such that

$$\begin{aligned} & \|(I + A)^{-1/2} (H - S(\tau))(I + tH)^{-1}\| \\ & \leq D_{\frac{1}{2}, \alpha} \|(I + H)^{1/2} (I + A)^{-1/2}\| \tau^{\alpha-1/2} t^{-\alpha}, \end{aligned}$$

which yields

$$\begin{aligned} & \|(I + A)^{-1/2} (H - S(\tau))(I + tH)^{-1}\| t^{1/2} \\ & \leq D_{\frac{1}{2}, \alpha} \|(I + H)^{1/2} (I + A)^{-1/2}\| \left(\frac{\tau}{t}\right)^{\alpha-1/2} \end{aligned} \tag{4.3}$$

for $t \in (0, 1]$ and $\tau > 0$.

To estimate the second term of the right-hand side of (4.2), we use again the functional calculus to find that

$$\|(H - S(\tau))(I + tH)^{-1}\| \leq \frac{1}{t} + \|S(\tau)(I + H)^{-\alpha}\| t^{-\alpha}$$

for $t \in (0, 1]$ and $\tau > 0$. By virtue of Lemma 4(ii) there is a constant $D_\alpha > 0$ such that

$$\|S(\tau)(I + tH)^{-1}\| \leq D_\alpha \tau^{\alpha-1} t^{-\alpha}$$

for $t \in (0, 1]$ and $\tau > 0$, which yields

$$\|S(\tau)(I + tH)^{-1}\| \tau^{1/2} t^{1/2} \leq D_\alpha \left(\frac{\tau}{t}\right)^{\alpha-1/2} \tag{4.4}$$

for $t \in (0, 1]$ and $\tau > 0$. Hence we obtain

$$\|(H - S(\tau))(I + tH)^{-1}\| \tau^{1/2} t^{1/2} \leq (1 + D_\alpha) \left(\frac{\tau}{t}\right)^{\alpha-1/2}$$

for $t, \tau \in (0, 1]$ and $0 < \tau \leq t$. Applying estimates (4.3) and (4.4) we find from (4.2) that

$$\begin{aligned} t\|(I + tS(\tau))^{-1/2}(H - S(\tau))(I + tH)^{-1}\| \\ \leq \{C_A D_{\frac{1}{2},\alpha}\|(I + H)^{1/2}(I + A)^{-1/2}\| + C_0(1 + D_\alpha)\} \left(\frac{\tau}{t}\right)^{\alpha-1/2} \end{aligned}$$

for $t, \tau \in (0, 1]$ with $0 < \tau \leq t$. Setting

$$S_{\alpha-1/2} := C_A D_{\frac{1}{2},\alpha} \|(I + H)^{1/2}(I + A)^{-1/2}\| + C_0(1 + D_\alpha)$$

we get (4.1). \square

Lemma 6. *Let A and B be non-negative self-adjoint operators such that for some $\alpha \in (1/2, 1)$ condition (1.2) is satisfied. Further, let f and g be Kato functions which obey conditions (1.3) and (1.4). Then there is a constant $G_{2\alpha-1} > 0$ such that*

$$t\|(I + tH)^{-1}(H - S(\tau))(I + tH)^{-1}\| \leq G_{2\alpha-1} \left(\frac{\tau}{t}\right)^{2\alpha-1} \tag{4.5}$$

for $t \in (0, 1]$ and $\tau > 0$.

Proof. By the functional calculus we get

$$\begin{aligned} t\|(I + tH)^{-\alpha}(H - S(\tau))(I + tH)^{-\alpha}\| \\ \leq \|(I + H)^{-\alpha}(H - S(\tau))(I + H)^{-\alpha}\| t^{-(2\alpha-1)} \end{aligned}$$

for $t \in (0, 1]$ and $\tau > 0$. Applying Lemma 4 we find a constant $G_{2\alpha-1} > 0$ such that

$$t \|(I + tH)^{-\alpha}(H - S(\tau))(I + tH)^{-\alpha}\| \leq G_{2\alpha-1} \left(\frac{\tau}{t}\right)^{2\alpha-1}$$

for $t \in (0, 1]$ and $\tau > 0$, proving (4.5). \square

We are now going to prove the main theorem mentioned in the Introduction.

Proof of Theorem 1.1. By the resolvent identities

$$\begin{aligned} (I + tS(\tau))^{-1} - (I + tH)^{-1} &= t(I + tS(\tau))^{-1}(H - S(\tau))(I + tH)^{-1} \\ &= t(I + tH)^{-1}(H - S(\tau))(I + tS(\tau))^{-1} \end{aligned}$$

one gets

$$\begin{aligned} (I + tS(\tau))^{-1} - (I + tH)^{-1} &= t(I + tH)^{-1}(H - S(\tau))(I + tH)^{-1} \\ &\quad + t^2(I + tH)^{-1}(H - S(\tau))(I + tS(\tau))^{-1} \\ &\quad \times (H - S(\tau))(I + tH)^{-1} \end{aligned}$$

for $\tau, t > 0$. Hence we find the estimate

$$\begin{aligned} &\|(I + tS(\tau))^{-1} - (I + tH)^{-1}\| \\ &\leq t \|(I + tH)^{-1}(H - S(\tau))(I + tH)^{-1}\| \\ &\quad + t^2 \|(I + tH)^{-1}(H - S(\tau))(I + tS(\tau))^{-1}(H - S(\tau))(I + tH)^{-1}\| \end{aligned}$$

for $\tau, t > 0$, which can be written as

$$\begin{aligned} &\|(I + tS(\tau))^{-1} - (I + tH)^{-1}\| \\ &\leq t \|(I + tH)^{-1}(H - S(\tau))(I + tH)^{-1}\| \\ &\quad + t^2 \|(I + tS(\tau))^{-1/2}(H - S(\tau))(I + tH)^{-1}\|^2. \end{aligned}$$

Taking into account Lemmas 5 and 6 we get

$$\|(I + tS(\tau))^{-1} - (I + tH)^{-1}\| \leq G_{2\alpha-1} \left(\frac{\tau}{t}\right)^{2\alpha-1} + S_{\alpha-1/2}^2 \left(\frac{\tau}{t}\right)^{2\alpha-1}$$

for $t, \tau \in (0, 1]$ with $0 < \tau \leq t$. Setting $M_{2\alpha-1} := G_{2\alpha-1} + S_{\alpha-1/2}^2$ one gets a constant such that condition (2.4) holds for $\tau, t \in (0, 1]$ with $0 < \tau \leq t$. By Theorem 2.1 there is a constant $C_{2\alpha-1} > 0$ such (2.5) is valid for $q = 2\alpha - 1$ and $\tau, t \in (0, 1]$ and $0 < \tau \leq t$. Setting $C_{1,2\alpha-1} := C_{2\alpha-1}$ and $\tau = t/n, n = 1, 2, \dots$, we immediately verify (1.5) for $t \in [0, 1]$.

To extend the result to $t \in [0, T]$, $0 < T < \infty$, we set $A_s := sA$ and $B_s := sB$ where $s > 0$. Obviously, one has $H_s := A_s + B_s = sH$. If A and B are non-negative self-adjoint operators such that for some $\alpha \in (1/2, 1)$ condition (1.2) is satisfied, then, of course, one has $\text{dom}(H_s^\alpha) \subseteq \text{dom}(A_s^\alpha) \cap \text{dom}(B_s^\alpha)$ for $s > 0$. Similarly, the condition $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ implies $\text{dom}(A_s^{1/2}) \subseteq \text{dom}(B_s^{1/2})$ for $s > 0$. Hence, there is a constant $C_{2\alpha-1}(s) > 0$ such that

$$\|(f(tA_s/2n)g(tB_s/n)f(tA_s/2n))^n - e^{-tH_s}\| \leq C_{2\alpha-1}(s) \frac{1}{n^{2\alpha-1}}$$

for $t \in [0, 1]$ and $n = 1, 2, \dots$, which yields

$$\|(f(tA/2n)g(tB/n)f(tA/2n))^n - e^{-tH}\| \leq C_{2\alpha-1}(s) \frac{1}{n^{2\alpha-1}}$$

for $t \in [0, s]$ and $n = 1, 2, \dots$. Choosing $s = T$ and setting $C_{T,2\alpha-1} := C_{2\alpha-1}(T)$ we get that for any finite interval $[0, T]$ there is a constant $C_{T,2\alpha-1} > 0$ such that (1.5) holds. \square

5. Related families

Let us show that estimate (6.1) holds not only for the family $F(\tau) = f(\tau A/2)g(\tau B)f(\tau A/2)$ but also for the families

$$F_1(\tau) := f(\tau A)^{1/2}g(\tau B)f(\tau A)^{1/2},$$

$$F_2(\tau) := g(\tau B)^{1/2}f(\tau A)g(\tau B)^{1/2},$$

$$F_3(\tau) := g(\tau B)f(\tau A),$$

$$F_4(\tau) := f(\tau A)g(\tau B),$$

$$F_5(\tau) := g(\tau B/2)f(\tau A)g(\tau B/2),$$

where $\tau \geq 0$. To this end we prove the following.

Lemma 7. *Let A and B be non-negative self-adjoint operators such that for some $\alpha \in (1/2, 1)$ condition (1.2) is satisfied. Further, let f and g be Kato functions which obey (1.3) and (1.4). Then there is a constant $C_F > 0$ such that*

$$\|(I - F(t/n))e^{-tH}\| \leq C_F \frac{e^t}{n^\alpha} \quad (5.1)$$

for $t \geq 0$ and $n = 1, 2, \dots$.

Proof. We use the representation

$$(I - F(t/n))e^{-tH} = f(tA/2n)g(\tau B)(I - f(tA/2n))e^{-tH} + f(tA/2n)(I - g(tB/n))e^{-tH} + (I - f(tA/2n))e^{-tH},$$

which yields the estimate

$$\|(I - F(t/n))e^{-tH}\| \leq 2\|(I - f(tA/2n))e^{-tH}\| + \|(I - g(tB/n))e^{-tH}\|.$$

Since

$$(I - f(tA/2n))e^{-tH} = (I - f(tA/2n))(I + A)^{-\alpha}(I + A)^\alpha(I + H)^{-\alpha}(I + H)^\alpha e^{-(I+H)t} e^t,$$

we find the estimate

$$\begin{aligned} & \|(I - f(tA/2n))e^{-tH}\| \\ & \leq \|(I - f(tA/2n))(I + A)^{-\alpha}\| \|(I + A)^\alpha(I + H)^{-\alpha}\| \|(I + H)^\alpha e^{-(I+H)t}\| e^t. \end{aligned}$$

Since

$$\|(I - f(tA/2n))(I + A)^{-\alpha}\| \leq \sup_{\lambda \geq 0} \frac{1 - f(t\lambda/2n)}{(1 + \lambda)^\alpha} \leq \sup_{\lambda > 0} \frac{1 - f(\lambda)}{\lambda^\alpha} \left(\frac{t}{2n}\right)^\alpha \leq \gamma_f(\alpha) \left(\frac{t}{2n}\right)^\alpha,$$

we obtain the estimate

$$\|(I - f(tA/2n))e^{-tH}\| \leq 2^{-\alpha} \gamma_f(\alpha) b(\alpha) \|(I + A)^\alpha(I + H)^{-\alpha}\| \frac{e^t}{n^\alpha}$$

for $n = 1, 2, \dots$, where $b(\alpha) := \sup_{\lambda > 0} \lambda^\alpha e^{-\lambda}$. Similarly, we prove that

$$\|(I - g(tB/n))e^{-tH}\| \leq \gamma_g(\alpha) b(\alpha) \|(I + B)^\alpha(I + H)^{-\alpha}\| \frac{e^t}{n^\alpha}$$

for $n = 1, 2, \dots$. Setting

$$C_F := 2^{1-\alpha} \gamma_f(\alpha) b(\alpha) \|(I + A)^\alpha(I + H)^{-\alpha}\| + \gamma_g(\alpha) b(\alpha) \|(I + B)^\alpha(I + H)^{-\alpha}\|$$

one gets estimate (5.1). \square

Theorem 5.1. *Let A and B be non-negative self-adjoint operators such that for some $\alpha \in (1/2, 1)$ condition (1.2) is satisfied. Further, let f and g be Kato functions which satisfy conditions (1.3) and (1.4). If in addition one has $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$ and $m_f(x) < 1$ for $x > 0$, then for any finite interval $[0, T]$ there are constants $C_{T, 2\alpha-1}^{(j)} > 0$,*

$j = 1, 2, 3, 4, 5$, such that

$$\|F_j(t/n)^n - e^{-tH}\| \leq C_{T,2\alpha-1}^{(j)} \frac{1}{n^{2\alpha-1}}, \quad j = 1, 2, 3, 4, 5 \quad (5.2)$$

for $t \in [0, T]$ and $n = 1, 2, \dots$.

Proof. We set

$$f_0(x) := f(2x)^{1/2}, \quad x \geq 0.$$

The function $f_0(x)$ is also a Kato function which satisfies $|f_0|_{2\alpha} < \infty$ and $m_{f_0}(x) < 1$ for $x > 0$. One has

$$f_0(\tau A/2) = f(\tau A)^{1/2}, \quad \tau \geq 0. \quad (5.3)$$

We set

$$F_0(\tau) := f_0(\tau A/2)g(\tau B)f_0(\tau A/2), \quad \tau \geq 0.$$

By Theorem 1.1 there is a constant $C_{T,2\alpha-1}^{(0)} > 0$ such that

$$\|F_0(t/n)^n - e^{-tH}\| \leq C_{T,2\alpha-1}^{(0)} \frac{1}{n^{2\alpha-1}}$$

for $t \in [0, T]$ and $n = 1, 2, \dots$. By (5.3) one has $F_0(\tau) = F_1(\tau)$ for $\tau \geq 0$, which proves the assertion for $j = 1$. We note that

$$F_4(\tau)^n = f(\tau A)^{1/2} F_1(\tau)^{n-1} f(\tau A)^{1/2} g(\tau B), \quad \tau \geq 0, \quad n = 1, 2, \dots$$

Using the representation

$$\begin{aligned} F_4(\tau)^n - e^{-tH} &= f(\tau A)^{1/2} (F_1(\tau)^{n-1} (I - F_1(\tau)) f(\tau A)^{1/2} g(\tau B) \\ &\quad + f(\tau A)^{1/2} (F_1(\tau)^n - e^{-tH}) f(\tau A)^{1/2} g(\tau B) \\ &\quad + (f(\tau A)^{1/2} - I) e^{-tH} f(\tau A)^{1/2} f(\tau A)^{1/2} g(\tau B) \\ &\quad + e^{-tH} (f(\tau A)^{1/2} - I) g(\tau B) + e^{-tH} (g(\tau B) - I), \end{aligned}$$

where $\tau = t/n$, $t \geq 0$ and $n = 1, 2, \dots$, we find the estimate

$$\begin{aligned} \|F_4(\tau)^n - e^{-tH}\| &\leq \|F_1(\tau)^{n-1} (I - F_1(\tau))\| \\ &\quad + \|F_1(\tau)^n - e^{-tH}\| + 2\|(I - f(\tau A)^{1/2})e^{-tH}\| + \|(I - g(\tau B))e^{-tH}\|. \end{aligned}$$

Since

$$\|(I - f(\tau A)^{1/2})e^{-tH}\| \leq \gamma_f(\alpha)b(\alpha)\|(I + A)^\alpha(I + H)^{-\alpha}\|\frac{e^t}{n^\alpha} \tag{5.4}$$

and

$$\|(I - g(\tau B)^{1/2})e^{-tH}\| \leq \gamma_g(\alpha)b(\alpha)\|(I + B)^\alpha(I + H)^{-\alpha}\|\frac{e^t}{n^\alpha} \tag{5.5}$$

as well as the estimate $\|F_1(\tau)^{n-1}(I - F_1(\tau))\| \leq \frac{1}{n}$, $n = 1, 2, \dots$, we obtain from the statement for $j = 1$ the existence of a constant $C_{T,2\alpha-1}^{(4)} > 0$ such that (5.2) holds for $j = 4$, $t \in [0, T]$ and $n = 1, 2, \dots$. Next, using

$$F_3(\tau)^n = g(\tau B)F_4(\tau)^{n-1}f(\tau A)$$

and

$$F_2(\tau)^n = g(\tau B)^{1/2}F_4(\tau)^{n-1}f(\tau A)g(\tau B)^{1/2}$$

$\tau \geq 0$, $n = 1, 2, \dots$, as well as estimates (5.4) and (5.5) one verifies (5.2) for $j = 2, 3$ in the same way as above. To prove the statement for $j = 5$ we introduce the function

$$g_0(x) := g(x/2)^2, \quad x \geq 0.$$

Then one gets $F_5(\tau) = g_0(\tau B)^{1/2}f(\tau A)g_0(\tau B)^{1/2}$ for $\tau \geq 0$. The function g_0 is again a Kato function which satisfies (1.4). Hence, applying here the result for $F_2(\tau)$ we prove the case $j = 5$. \square

6. Example

Let $\mathfrak{H} := L_2(\Omega)$ where Ω is a bounded domain in \mathbb{R}^l , $l = 1, 2, \dots$, with boundary $\partial\Omega$ of C^∞ -class. By A we denote the negative half Laplacian with Dirichlet boundary conditions on $L_2(\Omega)$, i.e $A = -\frac{1}{2}\Delta_D$. The domain is given by

$$\text{dom}(A) := H_2^2(\Omega) \cap \mathring{H}_2^1(\Omega),$$

where $\mathring{H}_2^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Lebesgue space $H_2^1(\Omega)$, cf. Definitions 4.2.1/1 and 4.2.1/2 of [18]. Using the space $H_{2,B_D}^2(\Omega)$,

$$H_{2,B_D}^2(\Omega) := \{u \in H_2^2(\Omega) : B_D u|_{\partial\Omega} = 0\},$$

cf. Definition 4.3.3/2 of [18], where B_D is given by

$$B_D u := u|_{\partial\Omega}$$

for $u \in H_2^2(\Omega)$, we obtain

$$\text{dom}(A) := H_{2,B_D}^2(\Omega).$$

Further, let B be the negative half Laplacian with Neumann boundary conditions, i.e $B := -\frac{1}{2}\Delta_N$. One has

$$\text{dom}(B) := H_{2,B_N}^2(\Omega),$$

where

$$H_{2,B_N}^2(\Omega) := \{u \in H_2^2(\Omega) : B_N u|_{\partial\Omega} = 0\},$$

cf. Definition 4.3.3/2 of [18]. The boundary operator B_N is given by

$$(B_N u)(x) = \frac{\partial}{\partial v(x)} u(x), \quad x \in \partial\Omega,$$

for $u \in H_2^2(\Omega)$ where $v(x)$ is the outer unit normal to the boundary $\partial\Omega$ at the point $x \in \partial\Omega$. Since $\text{dom}(A^{1/2}) = \mathring{H}_2^1(\Omega)$ and $\text{dom}(B^{1/2}) = H_2^1(\Omega)$ one gets

$$\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2}).$$

Hence

$$\text{dom}(H^{1/2}) = \text{dom}(A^{1/2}) \cap \text{dom}(B^{1/2}) = \mathring{H}_2^1(\Omega).$$

and $H := A + B = 2A = -\Delta_D$. Therefore $\text{dom}(H^\alpha) = \text{dom}(A^\alpha)$ for arbitrary $\alpha \in [0, 1]$.

Now we are going to calculate the domains $\text{dom}(A^\alpha)$ and $\text{dom}(B^\alpha)$ for $\alpha \in (1/2, 1)$. By Theorem 1 and Theorem 2 of [4] and Theorem 8.1 of [5] (see also Theorem 4.3.3 of [18]) we find

$$\text{dom}(A^\alpha) = H_{2,B_D}^{2\alpha}(\Omega), \quad \alpha \in (1/2, 1),$$

and

$$\text{dom}(B^\alpha) = \begin{cases} H_2^{2\alpha}(\Omega), & \alpha \in (1/2, 3/4), \\ H_{2,B_N}^{2\alpha}(\Omega), & \alpha \in (3/4, 1). \end{cases}$$

Since $H_{2,B_D}^{2\alpha}(\Omega) \subseteq H_2^{2\alpha}(\Omega)$ one gets

$$\text{dom}(H^\alpha) = \text{dom}(A^\alpha) \subseteq \text{dom}(B^\alpha)$$

for $\alpha \in (1/2, 3/4)$. Applying now Theorem 5.1 we find that

$$\|(e^{-t(-\Delta_D)/2n} e^{-t(-\Delta_N)/2n})^n - e^{-t(-\Delta_D)}\| = O(n^{-\kappa})$$

for any $\kappa := 2\alpha - 1 < \kappa_0 := \frac{1}{2}$ ($\alpha < \frac{3}{4}$) uniformly in $t \in [0, T]$ as $n \rightarrow \infty$.

If $\alpha = 3/4$, then it does not hold that $\text{dom}(A^{3/4}) \subseteq \text{dom}(B^{3/4})$. Hence $\text{dom}(A^{3/4}) \cap \text{dom}(B^{3/4})$ is a proper subset of $\text{dom}(H^{3/4})$ which does not allow one to apply Theorem 5.1.

If $\alpha \in (3/4, 1)$, then

$$\begin{aligned} \text{dom}(A^\alpha) \cap \text{dom}(B^\alpha) &= H_{2, \{B_D, B_N\}}^{2\alpha}(\Omega) \\ &:= \{u \in H_2^{2\alpha}(\Omega) : B_D u|_{\partial\Omega} = 0, B_N u|_{\partial\Omega} = 0\} \subseteq H_{2, B_D}^{2\alpha}(\Omega). \end{aligned}$$

This yields that $\text{dom}(A^\alpha) \cap \text{dom}(B^\alpha)$ is a proper subset of $\text{dom}(H^\alpha)$ which does not allow one to apply Theorem 5.1, either.

If $\alpha = 1$, then one gets that $\text{dom}(A) \cap \text{dom}(B)$ is a proper subset of $\text{dom}(H)$ too which yields $H \neq A + B$. Therefore the results of [11,12] are not applicable.

Notice that in contrast to [14] and to examples given there, here we have given an example when the operator B^α is not small with respect to A^α for $\alpha \in (1/2, 3/4)$.

7. Remarks

Let us make the following remarks:

- (i) By an example given in [17] the error bound estimate $O(n^{-(2\alpha-1)})$ in Theorems 1.1 and 5.1 cannot be improved, i.e. it is the *ultimate optimal* one.
- (ii) This optimal error bound was already found in [14] see Theorems 5.3 and 5.5. In comparison with [14] the conditions on the operators A and B are relaxed here. There it was assumed that B^α is small with respect to A^α , i.e. $\text{dom}(A^\alpha) \subseteq \text{dom}(B^\alpha)$ and

$$\|B^\alpha u\| \leq a \|A^\alpha u\| + b \|u\|, \quad u \in \text{dom}(A^\alpha), \tag{7.1}$$

for some $\alpha \in (1/2, 1)$ and $a \in (0, 1)$, $b > 0$. This condition is reduced in the present paper to the mild subordination condition $\text{dom}(A^{1/2}) \subseteq \text{dom}(B^{1/2})$, which (7.1) obviously implies. The yet open problem is to eliminate completely this subordination condition.

- (iii) The conditions on the Kato functions f and g are essentially relaxed compared with [14]. Indeed, in [14] the Kato functions f and g have to satisfy at zero a smoothness condition of the type $|f|_2, |g|_2 < \infty$ which is replaced here by $|f|_{2\alpha}, |g|_{2\alpha} < \infty$, $\alpha \in (1/2, 1)$. A behavior at infinity like $f(x) \sim x^{-2\alpha}$ is also demanded there.

- (iv) Theorem 1.1 shows that only relations between certain domains related to A , B and H are decisive for the *convergence rate* of the Trotter–Kato product formula.
- (v) Theorem 1.1 holds for $\alpha \in (1/2, 1)$. The method of the proof does not allow one to include the case $\alpha = 1$. However, this case was considered in [11,12]. It is remarkable that one does not need any subordination condition there.
- (vi) For $\alpha = 1/2$ we cannot expect operator-norm convergence in general, see [17]. However, if there is a subordination such that the operator B is relatively compact with respect to A , then operator-norm convergence holds, see [15].

The first version of this result was announced in [8] with a sketch of a proof which relies on an operator-norm estimate proved by Birman and Solomyak in [1]. In the present paper we have improved this previous result, by removing some restrictive condition on the Kato functions f and g imposed there as well as by giving a different, simpler proof which does not make use of operator-norm inequalities à la Birman–Solomyak.

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References

- [1] M.S. Birman, M.Z. Solomyak, Estimates for the difference of fractional powers of self-adjoint operators in case of unbounded perturbations, Zap. Nauchn. Sem. Leningrad Otdel Mat. Inst. Steklova 178 (1989) 120–145 (Russian), translated in J. Soviet Math. 61(2) (1992) 2018–2034.
- [2] P.R. Chernoff, Note on product formulas for operator semigroups, J. Funct. Anal. 2 (1968) 238–242.
- [3] P.R. Chernoff, Semigroup product formulas and addition of unbounded operators, Bull. Amer. Math. Soc. 76 (1970) 395–398.
- [4] D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad. 43 (1967) 82–86.
- [5] P. Grisvard, Caractérisation de quelques espaces d'interpolation, Arch. Rational Mech. Anal. 25 (1967) 40–63.
- [6] T. Kato, On the Trotter–Lie product formula, Proc. Japan Acad. 50 (1974) 694–698.
- [7] T. Kato, Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups, in: I. Gohberg, Mc. Kac (Eds.), Topics in Functional Analysis Advances in Mathematics, Supplementary Studies, Vol. 3, Academic Press, New York, 1978, pp. 185–195.

- [8] T. Ichinose, H. Neidhardt, V.A. Zagrebnov, Operator norm convergence of the Trotter–Kato product formula, Proceedings of the International Conference on Functional Analysis, Kiev, August 22–26, 2001, Ukraine Academic Press, Kiev, 2003, pp. 100–106.
- [9] T. Ichinose, Hideo Tamura, Error estimates in the operator norm for Trotter–Kato product formula, *Integral Equations Operator Theory* 27 (1997) 195–207.
- [10] T. Ichinose, Hideo Tamura, Error estimate in operator norm of exponential product formulas for propagators of parabolic evolution equations, *Osaka J. Math.* 35 (1998) 751–770.
- [11] T. Ichinose, Hideo Tamura, The norm convergence of the Trotter–Kato product formula with error bound, *Comm. Math. Phys.* 217 (2001) 489–502.
- [12] T. Ichinose, Hideo Tamura, Hiroshi Tamura, V.A. Zagrebnov, Note on the paper “The norm convergence of the Trotter–Kato product formula with error bound” by Ichinose and Tamura, *Comm. Math. Phys.* 221 (2001) 499–510.
- [13] H. Neidhardt, V.A. Zagrebnov, On error estimates for the Trotter–Kato product formula, *Lett. Math. Phys.* 44 (1998) 169–186.
- [14] H. Neidhardt, V.A. Zagrebnov, Fractional powers of self-adjoint operators and Trotter–Kato product formula, *Integral Equations Operator Theory* 35 (1999) 209–231.
- [15] H. Neidhardt, V.A. Zagrebnov, Trotter–Kato product formula and operator-norm convergence, *Commun. Math. Phys.* 205 (1999) 129–159.
- [16] D.L. Rogava, Error bounds for the Trotter–type formulas for self-adjoint operators, *Functional Anal. Appl.* 27 (3) (1993) 217–219.
- [17] Hiroshi Tamura, A remark on operator-norm convergence of Trotter–Kato product formula, *Integral Equations Operator Theory* 37 (2000) 350–356.
- [18] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Deutscher Verlag der Wissenschaften, Berlin, 1978.