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A Bairstow's type method for trigonometric polynomials

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Summary. We construct in this paper an analogous method to Bairstow's one, for trigonometric polynomials with real coefficients. We also present some numerical examples which illustrate this method.

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The method of Bairstow for algebraic polynomials with real coefficients is well known and it is used in practice successfully. It is described in many books of numerical analysis and articles, e.g. [5, 6, 1]. In this paper we present an analogous method to that of Bairstow, for trigonometric polynomials with real coefficients. To this purpose, we prove the results which allow both the construction of the method and the elaboration of an algorithm and we also present a few numerical examples which prove the efficiency of the method.

Let us consider the sets $\widetilde{\mathbb{R}}[x]$ and $\widetilde{\widetilde{\mathbb{R}}}[x]$ defined by

$$\widetilde{\mathbb{R}}[x] := \left\{ u(x) \mid u(x) = \sum_{i=1}^{n} \left(a_i \cos \frac{2i-1}{2} x + b_i \sin \frac{2i-1}{2} x \right), \right. \\ \left. a_i, b_i \in \mathbb{R},\ \forall i = 1, 2, \ldots, n,\ \forall n \in \mathbb{N}^* \right\}$$

and

$$\widetilde{\widetilde{\mathbb{R}}}[x] := \left\{ u(x) \mid u(x) = a_0 + \sum_{i=1}^{n} (a_i \cos ix + b_i \sin ix), \right. \\ \left. a_0, a_i, b_i \in \mathbb{R},\ \forall i = 1, 2, \ldots, n,\ \forall n \in \mathbb{N}^* \right\}.$$

We call the elements of these sets trigonometric polynomials in the variable x of odd degree and even degree respectively, with real coefficients. By our method, we intend to determine the second degree divisors for the polynomials from the set $\widetilde{\mathbb{R}}[x] \cup \widetilde{\widetilde{\mathbb{R}}}[x]$.

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Note that for this it is sufficient to consider only the polynomials from the set $\widetilde{\mathbb{R}}[x]$ and to search second degree divisors of the type

$$-\alpha\cos x - \beta\sin x + \gamma \in \widetilde{\widetilde{\mathbb{R}}}[x], \quad \text{with} \quad \alpha^2 + \beta^2 = 1.$$

Theorem 1. *Let u be a trigonometric polynomial given by*

$$u(x) = \sum_{i=1}^{n}\left(a_i\cos\frac{2i-1}{2}x + b_i\sin\frac{2i-1}{2}x\right) \in \widetilde{\mathbb{R}}[x], \quad \textit{with} \quad n \geq 2,\ a_n^2 + b_n^2 \neq 0,$$

and let

$$p(x) = -\alpha\cos x - \beta\sin x + \gamma \in \widetilde{\widetilde{\mathbb{R}}}[x], \quad \textit{with} \quad \alpha^2 + \beta^2 \neq 0.$$

Then

$$u(x) = p(x)\,v(x) + A\cos\frac{x}{2} + B\sin\frac{x}{2}, \tag{1}$$

where

$$v(x) = \sum_{j=1}^{n-1}\left(c_j\cos\frac{2j-1}{2}x + d_j\sin\frac{2j-1}{2}x\right) \in \widetilde{\mathbb{R}}[x],$$

and the coefficients of the quotient and of the remainder after the division of u by p are determined recursively by the relations

$$\begin{cases} c_j = \dfrac{2}{\alpha^2+\beta^2}[\gamma(\alpha c_{j+1} + \beta d_{j+1}) - \alpha a_{j+1} - \beta b_{j+1}] \\ \qquad\qquad - \dfrac{1}{\alpha^2+\beta^2}[(\alpha^2-\beta^2)c_{j+2} + 2\alpha\beta d_{j+2}] \\ d_j = \dfrac{2}{\alpha^2+\beta^2}[\gamma(\alpha d_{j+1} - \beta c_{j+1}) + \beta a_{j+1} - \alpha b_{j+1}] \\ \qquad\qquad - \dfrac{1}{\alpha^2+\beta^2}[(\alpha^2-\beta^2)d_{j+2} - 2\alpha\beta c_{j+2}], \end{cases} \tag{2}$$

$(j = n-1, n-2, \ldots, 1)$ *with* $c_{n+1} := d_{n+1} := c_n := d_n := 0$ *and*

$$\begin{cases} A = a_1 + \dfrac{\alpha}{2}(c_1 + c_2) + \dfrac{\beta}{2}(d_1 + d_2) - \gamma c_1 \\ B = b_1 - \dfrac{\alpha}{2}(d_1 - d_2) + \dfrac{\beta}{2}(c_1 - c_2) - \gamma d_1. \end{cases} \tag{3}$$

Proof. The equality (1) follows from Corollary 23 in [2]. By straightforward calculation of the right hand side of this equality, we obtain

$$u(x) = -\frac{\alpha}{2}\sum_{j=1}^{n-1} c_j\left(\cos\frac{2(j+1)-1}{2}x - \cos\frac{2(j-1)-1}{2}x\right)$$
$$-\frac{\alpha}{2}\sum_{j=1}^{n-1} d_j\left(\sin\frac{2(j+1)-1}{2}x + \sin\frac{2(j-1)-1}{2}x\right)$$

$$
\begin{aligned}
&- \frac{\beta}{2} \sum_{j=1}^{n-1} c_j \left(\sin \frac{2(j+1)-1}{2} x - \sin \frac{2(j-1)-1}{2} x \right) \\
&- \frac{\beta}{2} \sum_{j=1}^{n-1} d_j \left(\cos \frac{2(j-1)-1}{2} x - \cos \frac{2(j+1)-1}{2} x \right) \\
&+ \gamma \sum_{j=1}^{n-1} \left(c_j \cos \frac{2j-1}{2} x + d_j \sin \frac{2j-1}{2} x \right) \\
&+ A \cos \frac{x}{2} + B \sin \frac{x}{2}.
\end{aligned}
$$

By reordering the terms in the above relation, we obtain

$$
\begin{aligned}
u(x) = &\sum_{j=2}^{n} \left(-\frac{\alpha c_{j-1}}{2} - \frac{\alpha c_{j+1}}{2} + \frac{\beta d_{j-1}}{2} - \frac{\beta d_{j+1}}{2} + \gamma c_j \right) \cos \frac{2j-1}{2} x \\
&+ \sum_{j=2}^{n} \left(-\frac{\alpha d_{j-1}}{2} - \frac{\alpha d_{j+1}}{2} - \frac{\beta c_{j-1}}{2} + \frac{\beta c_{j+1}}{2} + \gamma d_j \right) \sin \frac{2j-1}{2} x \\
&+ \left(-\frac{\alpha c_1}{2} - \frac{\alpha c_2}{2} - \frac{\beta d_1}{2} - \frac{\beta d_2}{2} + \gamma c_1 + A \right) \cos \frac{x}{2} \\
&+ \left(\frac{\alpha d_1}{2} - \frac{\alpha d_2}{2} - \frac{\beta c_1}{2} + \frac{\beta c_2}{2} + \gamma d_1 + B \right) \sin \frac{x}{2}.
\end{aligned}
$$

By identification of the coefficients of the two polynomials in the last equality, we get for (c_j, d_j), $j = n-1, n-2, \ldots, 1$, the systems of linear equations

$$
\begin{cases}
\alpha c_j - \beta d_j = 2\gamma c_{j+1} - \alpha c_{j+2} - \beta d_{j+2} - 2a_{j+1} \\
\beta c_j + \alpha d_j = 2\gamma d_{j+1} - \alpha d_{j+2} + \beta c_{j+2} - 2b_{j+1}.
\end{cases}
$$

Each of them has the determinant $\alpha^2 + \beta^2 \neq 0$. Solving recursively these systems we obtain the relations (2). Then we also obtain, for the coefficients A and B of the remainder, the relations (3). □

We want to determine those values of α, β and γ for which p is a divisor of u, or equivalently the remainder after the division of u by p is the zero polynomial. Therefore, if in the relation (1) we consider A and B as functions of α, β and γ, the problem is reduced to the one of solving the system of nonlinear equations

$$
\begin{cases}
A(\alpha, \beta, \gamma) = 0 \\
B(\alpha, \beta, \gamma) = 0 \\
\alpha^2 + \beta^2 - 1 = 0.
\end{cases} \tag{4}
$$

Let us now notice two characteristic aspects of Bairstow's method. The first one is the solution of the system (4) by Newton's method. Therefore, it is necessary to introduce the Jacobi matrix $J_{A,B}(\alpha, \beta, \gamma)$ of this system, i.e.

$$J_{A,B}(\alpha,\beta,\gamma) = \begin{pmatrix} \frac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma) & \frac{\partial A}{\partial \beta}(\alpha,\beta,\gamma) & \frac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma) \\ \frac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma) & \frac{\partial B}{\partial \beta}(\alpha,\beta,\gamma) & \frac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma) \\ 2\alpha & 2\beta & 0 \end{pmatrix}, \tag{5}$$

and to establish the conditions under which it is nonsingular. The second aspect concerns the calculation of these matrix elements; they are determined from the remainders after the division of u and v by p.

Lemma. *Let α, β and γ real numbers, with $\alpha^2 + \beta^2 = 1$. If z_1 and z_2 are the roots of the polynomial $-\alpha\cos x - \beta \sin x + \gamma$, then*

$$\alpha = \pm\cos\frac{z_1 + z_2}{2}, \quad \beta = \pm\sin\frac{z_1 + z_2}{2}, \quad \gamma = \pm\cos\frac{z_1 - z_2}{2}. \tag{6}$$

Theorem 2. *Let us suppose that*

$$v(x) = p(x)\,w(x) + C(\alpha,\beta,\gamma)\cos\frac{x}{2} + D(\alpha,\beta,\gamma)\sin\frac{x}{2}. \tag{7}$$

Then

$$\begin{cases} \frac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma) = \frac{1}{\alpha^2+\beta^2}[(\alpha\gamma + \beta^2)C(\alpha,\beta,\gamma) + \beta(\gamma - \alpha)D(\alpha,\beta,\gamma)] \\ \frac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma) = \frac{1}{\alpha^2+\beta^2}[(\alpha\gamma - \beta^2)D(\alpha,\beta,\gamma) - \beta(\gamma + \alpha)C(\alpha,\beta,\gamma)], \end{cases} \tag{8}$$

$$\begin{cases} \frac{\partial A}{\partial \beta}(\alpha,\beta,\gamma) = \frac{\gamma - \alpha}{\alpha^2+\beta^2}[\beta C(\alpha,\beta,\gamma) - \alpha D(\alpha,\beta,\gamma)] \\ \frac{\partial B}{\partial \beta}(\alpha,\beta,\gamma) = \frac{\gamma + \alpha}{\alpha^2+\beta^2}[\beta D(\alpha,\beta,\gamma) + \alpha C(\alpha,\beta,\gamma)] \end{cases} \tag{9}$$

and

$$\frac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma) = -C(\alpha,\beta,\gamma), \quad \frac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma) = -D(\alpha,\beta,\gamma). \tag{10}$$

Proof. Derivation of (1) with respect to α, β and γ results in

$$\begin{cases} 0 = p(x)\frac{\partial v}{\partial \alpha}(x) - v(x)\cos x + \frac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma)\cos\frac{x}{2} + \frac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma)\sin\frac{x}{2} \\ 0 = p(x)\frac{\partial v}{\partial \beta}(x) - v(x)\sin x + \frac{\partial A}{\partial \beta}(\alpha,\beta,\gamma)\cos\frac{x}{2} + \frac{\partial B}{\partial \beta}(\alpha,\beta,\gamma)\sin\frac{x}{2} \\ 0 = p(x)\frac{\partial v}{\partial \gamma}(x) + v(x) + \frac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma)\cos\frac{x}{2} + \frac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma)\sin\frac{x}{2}. \end{cases} \tag{11}$$

Let us denote the roots of the polynomial p by z_1 and z_2. Firstly, we consider the case when $z_1 \neq z_2$. Substituting these roots into (7) and (11), then combining the obtained relations, we find for the derivatives $\frac{\partial A}{\partial \alpha}, \frac{\partial B}{\partial \alpha}, \ldots, \frac{\partial B}{\partial \gamma}$ the following systems of linear equations:

$$\begin{cases} \dfrac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} + \dfrac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} = \left(C\cos\dfrac{z_1}{2} + D\sin\dfrac{z_1}{2}\right)\cos z_1 \\ \dfrac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma)\cos\dfrac{z_2}{2} + \dfrac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma)\sin\dfrac{z_2}{2} = \left(C\cos\dfrac{z_2}{2} + D\sin\dfrac{z_2}{2}\right)\cos z_2, \end{cases}$$

$$\begin{cases} \dfrac{\partial A}{\partial \beta}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} + \dfrac{\partial B}{\partial \beta}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} = \left(C\cos\dfrac{z_1}{2} + D\sin\dfrac{z_1}{2}\right)\sin z_1 \\ \dfrac{\partial A}{\partial \beta}(\alpha,\beta,\gamma)\cos\dfrac{z_2}{2} + \dfrac{\partial B}{\partial \beta}(\alpha,\beta,\gamma)\sin\dfrac{z_2}{2} = \left(C\cos\dfrac{z_2}{2} + D\sin\dfrac{z_2}{2}\right)\sin z_2 \end{cases}$$

and

$$\begin{cases} \dfrac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} + \dfrac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} = -C\cos\dfrac{z_1}{2} - D\sin\dfrac{z_1}{2} \\ \dfrac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma)\cos\dfrac{z_2}{2} + \dfrac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma)\sin\dfrac{z_2}{2} = -C\cos\dfrac{z_2}{2} - D\sin\dfrac{z_2}{2}\,. \end{cases}$$

Each of these systems has the determinant $\Delta := \sin\dfrac{z_2 - z_1}{2} \neq 0$. For exemple, let us determine $\dfrac{\partial A}{\partial \alpha}$. Solving the first of the above systems and using the relations (6), we obtain

$$\begin{aligned} &\frac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma) = \frac{1}{\Delta}\left[\left(C\cos\frac{z_1}{2} + D\sin\frac{z_1}{2}\right)\cos z_1 \sin\frac{z_2}{2} - \left(C\cos\frac{z_2}{2} + D\sin\frac{z_2}{2}\right)\right. \\ &\left.\times \cos z_2 \sin\frac{z_1}{2}\right] = \frac{1}{\Delta}\left[(\cos z_1 - \cos z_2)\sin\frac{z_1}{2}\sin\frac{z_2}{2}D + C\cos z_1\cos\frac{z_1}{2}\sin\frac{z_2}{2}\right. \\ &\left. - C\cos z_2\cos\frac{z_2}{2}\sin\frac{z_1}{2}\right] = \frac{1}{2\Delta}\left[\left(\cos\frac{z_1 - z_2}{2} - \cos\frac{z_1 + z_2}{2}\right)(\cos z_1 - \cos z_2)D\right. \\ &\left. + C\cos z_1\left(\sin\frac{z_1 + z_2}{2} - \sin\frac{z_1 - z_2}{2}\right) - C\cos z_2\left(\sin\frac{z_1 + z_2}{2} + \sin\frac{z_1 - z_2}{2}\right)\right] \\ &= -\frac{1}{\Delta}\left[D\sin\frac{z_1 + z_2}{2}\sin\frac{z_1 - z_2}{2}\left(\cos\frac{z_1 - z_2}{2} - \cos\frac{z_1 + z_2}{2}\right) + C\sin\frac{z_1 + z_2}{2}\right. \\ &\left.\times \sin\frac{z_1 + z_2}{2}\sin\frac{z_1 - z_2}{2} + C\sin\frac{z_1 - z_2}{2}\cos\frac{z_1 + z_2}{2}\cos\frac{z_1 - z_2}{2}\right] \\ &= -\frac{1}{(\alpha^2 + \beta^2)\Delta}\sin\frac{z_1 - z_2}{2}\left[(\alpha\gamma + \beta^2)C + \beta(\gamma - \alpha)D\right]. \end{aligned}$$

Similarly, we can obtain the other partial derivatives of A and B.

It remains to prove the statement when $z_1 = z_2$. In this case p has a double root, therefore z_1 is also the root of its derivative. Let us derive (7) and (11) with respect to x. Substituting z_1 and combining the obtained relations, we find for $\dfrac{\partial A}{\partial \alpha}, \dfrac{\partial B}{\partial \alpha}, \ldots, \dfrac{\partial B}{\partial \gamma}$ the following systems of linear equations:

$$\begin{cases} \dfrac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} - \dfrac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} = 2\left(C\cos\dfrac{z_1}{2} + D\sin\dfrac{z_1}{2}\right)\sin z_1 \\ \qquad\qquad + \left(C\sin\dfrac{z_1}{2} - D\cos\dfrac{z_1}{2}\right)\cos z_1 \\ \dfrac{\partial A}{\partial \alpha}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} + \dfrac{\partial B}{\partial \alpha}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} = \left(C\cos\dfrac{z_1}{2} + D\sin\dfrac{z_1}{2}\right)\cos z_1, \end{cases}$$

$$\begin{cases} \dfrac{\partial A}{\partial \beta}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} - \dfrac{\partial B}{\partial \beta}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} = -2\left(C\cos\dfrac{z_1}{2} + D\sin\dfrac{z_1}{2}\right)\cos z_1 \\ \qquad\qquad + \left(C\sin\dfrac{z_1}{2} - D\cos\dfrac{z_1}{2}\right)\sin z_1 \\ \dfrac{\partial A}{\partial \beta}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} + \dfrac{\partial B}{\partial \beta}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} = \left(C\cos\dfrac{z_1}{2} + D\sin\dfrac{z_1}{2}\right)\sin z_1 \end{cases}$$

and

$$\begin{cases} \dfrac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} - \dfrac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} = -C\sin\dfrac{z_1}{2} + D\cos\dfrac{z_1}{2} \\ \dfrac{\partial A}{\partial \gamma}(\alpha,\beta,\gamma)\cos\dfrac{z_1}{2} + \dfrac{\partial B}{\partial \gamma}(\alpha,\beta,\gamma)\sin\dfrac{z_1}{2} = -C\cos\dfrac{z_1}{2} - D\sin\dfrac{z_1}{2}. \end{cases}$$

Each of the above systems has the determinant $\cos^2\dfrac{z_1}{2} + \sin^2\dfrac{z_1}{2} \neq 0$. Solving these systems we also obtain the solutions (8)–(10). □

Theorem 3. *Let α, β and γ be real numbers, with $\alpha^2 + \beta^2 \neq 0$. The Jacobi matrix $J_{A,B}(\alpha,\beta,\gamma)$ introduced in* (5) *is nonsingular if and only if the polynomial $-\alpha\cos x - \beta\sin x + \gamma$ and the polynomial v defined by* (1) *have no common root, i.e. are relative prime.*

Proof. Expanding the determinant of the matrix $J_{A,B}(\alpha,\beta,\gamma)$ and using the relations (8)–(10), we have

$$\det(J_{A,B}(\alpha,\beta,\gamma)) = 2\alpha\left(\frac{\partial A}{\partial \beta}\frac{\partial B}{\partial \gamma} - \frac{\partial A}{\partial \gamma}\frac{\partial B}{\partial \beta}\right) + 2\beta\left(\frac{\partial A}{\partial \gamma}\frac{\partial B}{\partial \alpha} - \frac{\partial A}{\partial \alpha}\frac{\partial B}{\partial \gamma}\right)$$
$$= \frac{2}{\alpha^2+\beta^2}[(\gamma+\alpha)C^2 + 2\beta CD + (\gamma-\alpha)D^2].$$

Using now the relations (6) between the coefficients α, β, γ and the roots z_1, z_2 of the polynomial $-\alpha\cos x - \beta\sin x + \gamma$, we get

$$\det(J_{A,B}(\alpha,\beta,\gamma)) = \frac{2}{\sqrt{\alpha^2+\beta^2}}\left[\left(\cos\frac{z_1-z_2}{2} + \cos\frac{z_1+z_2}{2}\right)C^2\right.$$
$$\left. + 2CD\sin\frac{z_1+z_2}{2} + \left(\cos\frac{z_1-z_2}{2} - \cos\frac{z_1+z_2}{2}\right)D^2\right] = \frac{4}{\sqrt{\alpha^2+\beta^2}}$$
$$\times\left[C^2\cos\frac{z_1}{2}\cos\frac{z_2}{2} + \left(\sin\frac{z_1}{2}\cos\frac{z_2}{2} + \cos\frac{z_1}{2}\sin\frac{z_2}{2}\right)CD + D^2\sin\frac{z_1}{2}\sin\frac{z_2}{2}\right]$$
$$= \frac{4}{\sqrt{\alpha^2+\beta^2}}\left(C\cos\frac{z_1}{2} + D\sin\frac{z_1}{2}\right)\left(C\cos\frac{z_2}{2} + D\sin\frac{z_2}{2}\right) = \frac{4v(z_1)v(z_2)}{\sqrt{\alpha^2+\beta^2}}.$$

Consequently, the matrix $J_{A,B}(\alpha,\beta,\gamma)$ is nonsingular if and only if we have $v(z_1)\,v(z_2) \neq 0$, and thus results the theorem. □

Remark. From the above theorems it follows that by applying the Newton's method we can determine for the trigonometric polynomial $u(x)$ second degree divisors of the type $-\alpha\cos x - \beta\sin x + \gamma$ with $\alpha^2 + \beta^2 = 1$ that are relatively prime to $v(x)$. Using the local convergence theorem for Newton's method [6] we conclude that the sequence $(\alpha^{(k)},\beta^{(k)},\gamma^{(k)})$ produced by Bairstow's method will converge (quadratically) to (α,β,γ), the solution of the system (4), on condition that the initial approximation $(\alpha^{(0)},\beta^{(0)},\gamma^{(0)})$ to be sufficiently closed to (α,β,γ).

In the following, we present some numerical examples obtained by application of the method described above. Given a trigonometric polynomial $u(x)$, the algorithm starts with initial approximation $(\alpha^{(0)},\beta^{(0)},\gamma^{(0)})$, then the next approximations $(\alpha^{(k)},\beta^{(k)},\gamma^{(k)})$ and the errors $err_1^{(k)} := |A^{(k)}| + |B^{(k)}|$, $err_2^{(k)} := (\alpha^{(k)})^2 + (\beta^{(k)})^2 - 1$ are computed. The algorithm stops when k, the number of the accomplished iterations, exceeds *kmax* or when *err*$_1^{(k)} \le$ *err* and *err*$_2^{(k)} \le$ *err* ($kmax$ = 100 and err = 10^{-13}). Finally, the roots z_1, z_2 of the obtained second degree factor $-\alpha^{(k)}\cos x - \beta^{(k)}\sin x + \gamma^{(k)}$ are also computed.

Example 1. Let us consider as a model problem the polynomial

$$u(x) = 3\cos\frac{7}{2}x - 3\sin\frac{7}{2}x + 13\cos\frac{5}{2}x + 13\sin\frac{5}{2}x - 13\cos\frac{3}{2}x + 13\sin\frac{3}{2}x - 3\cos\frac{1}{2}x - 3\sin\frac{1}{2}x.$$

For different initial approximations $(\alpha^{(0)},\beta^{(0)},\gamma^{(0)})$, the following three numerical experiments illustrate the convergence of the algorithm when the roots z_1, z_2 of the polynomial $u(x)$ are distinct, equal or complex.

k	$\alpha^{(k)}$	$\beta^{(k)}$	$\gamma^{(k)}$	$err_1^{(k)}$	$err_2^{(k)}$
0	1.0000000000	−2.0000000000	3.0000000000	$2.0 \cdot 10^{+02}$	$4.0 \cdot 10^{+00}$
1	0.8126026409	−1.0936986796	0.9024482161	$3.2 \cdot 10^{+01}$	$8.6 \cdot 10^{-01}$
2	0.8016487367	−0.7102761124	0.7967004114	$1.6 \cdot 10^{+01}$	$1.5 \cdot 10^{-01}$
3	0.7082985201	−0.7120608504	0.7095748320	$5.9 \cdot 10^{-01}$	$8.7 \cdot 10^{-03}$
4	0.7071216318	−0.7071102399	0.7071245807	$2.6 \cdot 10^{-03}$	$2.6 \cdot 10^{-05}$
5	0.7071067813	−0.7071067812	0.7071067812	$5.3 \cdot 10^{-09}$	$2.3 \cdot 10^{-10}$
6	0.7071067812	−0.7071067812	0.7071067812	$1.4 \cdot 10^{-14}$	$-2.0 \cdot 10^{-17}$

z_1 = 0.0000000000, z_2 = 4.7123889804

k	$\alpha^{(k)}$	$\beta^{(k)}$	$\gamma^{(k)}$	$err_1^{(k)}$	$err_2^{(k)}$
0	1.0000000000	2.0000000000	3.0000000000	$2.8 \cdot 10^{+01}$	$4.0 \cdot 10^{+00}$
1	0.0975077052	1.4512461474	1.4730971678	$4.3 \cdot 10^{+00}$	$1.1 \cdot 10^{+00}$
2	−0.0260094300	1.0751778319	1.0693683461	$1.6 \cdot 10^{+00}$	$1.6 \cdot 10^{-01}$
3	0.0016876821	1.0029836870	1.0033592890	$1.1 \cdot 10^{-01}$	$6.0 \cdot 10^{-03}$
4	−0.0000053291	1.0000058668	1.0000046812	$3.4 \cdot 10^{-04}$	$1.2 \cdot 10^{-05}$
5	0.0000000000	1.0000000000	1.0000000000	$1.8 \cdot 10^{-09}$	$6.3 \cdot 10^{-11}$
6	0.0000000000	1.0000000000	1.0000000000	$1.1 \cdot 10^{-15}$	$0.0 \cdot 10^{+00}$

$$z_1 = 1.5707963268, \quad z_2 = 1.5707963268$$

k	$\alpha^{(k)}$	$\beta^{(k)}$	$\gamma^{(k)}$	$err_1^{(k)}$	$err_2^{(k)}$
0	0.0000000000	2.0000000000	3.0000000000	$6.0 \cdot 10^{+00}$	$3.0 \cdot 10^{+00}$
1	0.0000000000	1.2500000000	2.4750000000	$2.9 \cdot 10^{+01}$	$5.6 \cdot 10^{-01}$
2	0.0000000000	1.0250000000	1.7644981403	$3.3 \cdot 10^{+00}$	$5.1 \cdot 10^{-02}$
3	0.0000000000	1.0003048780	1.6712861271	$2.2 \cdot 10^{-01}$	$6.1 \cdot 10^{-04}$
4	0.0000000000	1.0000000465	1.6667004959	$1.8 \cdot 10^{-03}$	$9.3 \cdot 10^{-08}$
5	0.0000000000	1.0000000000	1.6666666691	$1.3 \cdot 10^{-07}$	$2.2 \cdot 10^{-15}$
6	0.0000000000	1.0000000000	1.6666666667	$7.9 \cdot 10^{-15}$	$0.0 \cdot 10^{+00}$

$$z_1 = 1.5707963268 + 1.0986122887\,\mathrm{i}, \quad z_2 = 1.5707963268 - 1.0986122887\,\mathrm{i}$$

Therefore, we obtain the following decomposition of the given polynomial:

$$u(x) = \left(-\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\right)(-\sin x + 1)\left(-\sin x + \frac{5}{3}\right) q(x),$$

where $q(x) = 24\sqrt{2}\cos\frac{x}{2} \in \widetilde{\mathbb{R}}[x]$.

Example 2. Let us consider the polynomial

$$u(x) = 11\cos\frac{21}{2}x + \cos\frac{19}{2}x + 10\cos\frac{17}{2}x + 2\cos\frac{15}{2}x + 9\cos\frac{13}{2}x + 3\cos\frac{1}{2}x + 8\cos\frac{9}{2}x + 4\cos\frac{7}{2}x + 7\cos\frac{5}{2}x + 5\cos\frac{3}{2}x + 6\cos\frac{1}{2}x.$$

k	$\alpha^{(k)}$	$\beta^{(k)}$	$\gamma^{(k)}$	$err_1^{(k)}$	$err_2^{(k)}$
0	1.0000000000	2.0000000000	3.0000000000	$6.6 \cdot 10^{+04}$	$4.0 \cdot 10^{+00}$
1	0.5848351737	1.2075824132	1.5937250841	$9.8 \cdot 10^{+03}$	$8.0 \cdot 10^{-01}$
2	0.4415194069	0.9456312389	1.1505401090	$2.3 \cdot 10^{+03}$	$8.9 \cdot 10^{-02}$
3	0.4089533825	0.9136944962	1.0534988126	$7.1 \cdot 10^{+02}$	$2.1 \cdot 10^{-03}$
4	0.3897567492	0.9211480569	1.0190504759	$2.3 \cdot 10^{+02}$	$4.2 \cdot 10^{-04}$
5	0.3651379903	0.9313345782	0.9996121526	$6.7 \cdot 10^{+01}$	$7.1 \cdot 10^{-04}$
6	0.3414150913	0.9402542593	0.9923383599	$1.4 \cdot 10^{+01}$	$6.4 \cdot 10^{-04}$
7	0.3329329079	0.9429926432	0.9907559944	$8.4 \cdot 10^{-01}$	$7.9 \cdot 10^{-05}$
8	0.3323423154	0.9431590332	0.9906333591	$3.8 \cdot 10^{-03}$	$3.8 \cdot 10^{-07}$
9	0.3323395777	0.9431597983	0.9906327854	$8.3 \cdot 10^{-08}$	$8.1 \cdot 10^{-12}$
10	0.3323395777	0.9431597983	0.9906327853	$2.5 \cdot 10^{-14}$	$5.1 \cdot 10^{-17}$

$$z_1 = 1.3689940994, \quad z_2 = 1.0950324172$$

In fact, the polynomial $u(x)$ has exactly 21 distinct real roots in the interval $[0, 2\pi)$ and it was studied and used in [4] to construct certain Gaussian quadratures of the trigonometric type.

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