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Subordination and superordination for univalent solutions for fractional differential equations

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ABSTRACT

In this article, we establish the existence and uniqueness of univalent solution for fractional differential equation. Moreover, we illustrate some properties of this solution containing differential and integral subordination properties.

1. Introduction and preliminaries

The class of fractional operator equations of various types plays very important role not only in mathematics but also in physics, control systems, dynamical systems and engineering. Naturally, such equations required to be solved. There are numerous books focused in this direction, that is concerning the linear and nonlinear problems involving different types of fractional derivatives as well as integral. There are several kinds of fractional derivatives, which are generalizations for derivative of integral order (see [3–5,8,13]). In [2], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 1.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta, \quad 0 \leqslant \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta, \quad 0 \leqslant \alpha < 1,$$

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where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Note that, $I_z^\alpha f(z) = f(z) \times \frac{z^{\alpha-1}}{\Gamma(\alpha)}$ for $z>0$, and 0 for $z \leqslant 0$ (see [4]).

Remark 1.1. From Definition 1.1, we have $D_z^0 f(z) = f(0)$, $\lim_{\alpha\to 0} I_z^\alpha f(z) = f(z)$ and $\lim_{\alpha\to 0} D_z^{1-\alpha} f(z) = f'(z)$. Moreover,

$$D_z^\alpha\{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\{z^{\mu-\alpha}\}, \quad \mu > -1,\ 0 \leqslant \alpha < 1,$$

and

$$I_z^\alpha\{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\{z^{\mu+\alpha}\}, \quad \mu > -1,\ 0 \leqslant \alpha < 1,\ z > 0.$$

In this paper, we study the existence and uniqueness of univalent solution for the fractional differential equation

$$D_z^\alpha u(z) = f\big(z, u(z)\big), \tag{1}$$

subject to the initial condition $u(0)=0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U := \{z\colon\ z \in \mathbb{C};\ |z|<1\}$ and $f : U \times \mathbb{C} \to \mathbb{C}$ is an analytic function in $z \in U$. The existence is obtained by applying Schauder fixed point theorem while the uniqueness is obtained by using Banach fixed point theorem. Moreover, we discuss some properties of this solution involving fractional differential subordination. The following definitions and results are used in the sequel.

Theorem 1.1 *(Arzela–Ascoli). (See [7].) Let E be a compact metric space and $\mathcal{C}(E)$ be the Banach space of real or complex valued continuous functions normed by*

$$\|f\| := \sup_{t\in E}\big|f(t)\big|.$$

If $A = \{f_n\}$ is a sequence in $\mathcal{C}(E)$ such that f_n is uniformly bounded and equi-continuous, then $\overline{A}$ is compact.

Let M be a subset of Banach space X and $A : M \to M$ an operator. The operator A is called *compact* on the set M if it carries every bounded subset of M into a compact set. If A is continuous on M (that is, it maps bounded sets into bounded sets) then it is said to be *completely continuous* on M. A mapping $A : X \to X$ is said to a contraction if there exists a real number ρ, $0 \leqslant \rho < 1$ such that $\|Ax - Ay\| \leqslant \rho\|x-y\|$ for all $x, y \in X$.

Theorem 1.2 *(Schauder). (See [6].) Let X be a Banach space, $M \subset x$ a nonempty closed bounded convex subset and $P : M \to M$ is compact. Then P has a fixed point.*

Theorem 1.3 *(Banach). (See [6].) If X is a Banach space and $P : X \to X$ is a contraction mapping then P has a unique fixed point.*

The paper is organized as follows: in Section 2 we illustrated some properties of the fractional differential and integral operators D_z^α, I_z^α. In Section 3, we study the fractional differential subordination

$$\big(I_z^\alpha g_1\big)^\beta(z) \prec \big(I_z^\alpha f\big)^\beta(z) \prec \big(I_z^\alpha g_2\big)^\beta(z)$$

and

$$\left(\frac{I_z^\alpha g_1}{z}\right)^\mu(z) \prec \left(\frac{I_z^\alpha f}{z}\right)^\mu(z) \prec \left(\frac{I_z^\alpha g_2}{z}\right)^\mu(z), \quad \mu \geqslant 1.$$

In Section 4, we discuss the existence and uniqueness of univalent solution for the problem (1) by applying Schauder and Banach fixed point theorems respectively.

2. General properties

In this section, we study some important properties of the fractional differential and integral operators D_z^α, I_z^α, which are useful in the next section.

Theorem 2.1. *For $\alpha, \in [0,1)$ and f is a continuous function, then*

(1) $$DI_z^\alpha f(z) = \frac{(z)^{\alpha-1}}{\Gamma(\alpha)} f(0) + I_z^\alpha Df(z), \quad D = \frac{d}{dz},$$

(2) $$I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z).$$

Proof. (1) By using integration by part, we get

$$I_z^\alpha Df(z) = \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} Df(\zeta)\,d\zeta = -\frac{(z)^{\alpha-1}}{\Gamma(\alpha)} f(0) + \int_0^z \frac{(z-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)} f(\zeta)\,d\zeta = -\frac{(z)^{\alpha-1}}{\Gamma(\alpha)} f(0) + DI_z^\alpha f(z).$$

Note that when $f(0)=0$ we have $DI_z^\alpha f(z) = I_z^\alpha Df(z)$.

(2) Since $D_z^\alpha f(z) = DI^{1-\alpha} f(z)$, then using part (1), and Remark 1.1, we get

$$I_z^\alpha D_z^\alpha f(z) = I_z^\alpha DI^{1-\alpha} f(z) = I_z^\alpha\left[\frac{(z)^{-\alpha}}{\Gamma(1-\alpha)} f(0) + I_z^{1-\alpha} Df(z)\right] = f(z).$$

On the other hand

$$D_z^\alpha I_z^\alpha f(z) = DI^{1-\alpha} I_z^\alpha f(z) = DI^1 f(z) = f(z). \qquad \square$$

Lemma 2.1. *Let $f(z)$ be a non-decreasing function then for $z_1 \leqslant z_2$ we have $I_{z_1}^\alpha f(z_1) \leqslant I_{z_2}^\alpha f(z_2)$.*

Proof.

$$I_{z_1}^\alpha f(z_1) = f(z_1) \times \frac{z_1^{\alpha-1}}{\Gamma(\alpha)} \leqslant f(z_2) \times \frac{z_2^{\alpha-1}}{\Gamma(\alpha)} = I_{z_2}^\alpha f(z_2). \qquad \square$$

Lemma 2.2.

$$I_z^\alpha f(z) \to I_z^n f(z), \quad \alpha \to n,\ n = 1, 2, 3, \ldots.$$

Proof.

$$\left|I_z^\alpha f(z) - I_z^n f(z)\right| = \left|\frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1}\,d\zeta - \frac{1}{\Gamma(n)} \int_0^z f(\zeta)(z-\zeta)^{n-1}\,d\zeta\right|$$

$$\leqslant \int_0^z \left|f(\zeta)\right| \left|\frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(z-\zeta)^{n-1}}{\Gamma(n)}\right| d\zeta$$

and since $\frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(z-\zeta)^{n-1}}{\Gamma(n)}$ as $\alpha \to n$, then $\|I_z^\alpha f(z) - I_z^n f(z)\| \to 0$ as $\alpha \to n$. $\square$

3. Subordination and superordination

Let F and G be analytic functions in the unit disk U. The function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. Or Given two functions $F(z)$ and $G(z)$, which are analytic in U, the function $F(z)$ is said to be subordination to $G(z)$ in U if there exists a function $h(z)$, analytic in U with

$$h(0) = 0 \quad \text{and} \quad \left|h(z)\right| < 1 \quad \text{for all } z \in U,$$

such that

$$F(z) = G\big(h(z)\big) \quad \text{for all } z \in U.$$

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$. Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$.

Definition 3.1. (See [9].) Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where $E(f) := \{\zeta \in \partial U\colon \lim_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Lemma 3.1. *(See [10].) Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in U, and*
(2) *$\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.*

If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3.2. *(See [11].) Let $q(z)$ be convex univalent in the unit disk U and ψ and $\gamma \in \mathbb{C}$ with $\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\}>0$. If $p(z)$ is analytic in U and $\psi p(z)+\gamma zp'(z) \prec \psi q(z)+\gamma zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 3.3. *(See [12].) Let $q(z)$ be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(1) *$zq'(z)\varphi(q(z))$ is starlike univalent in U, and*
(2) *$\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\}>0$ for $z \in U$.*

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$ and $\vartheta(p(z))+zp'(z)\varphi(z)$ is univalent in U and $\vartheta(q(z))+zq'(z)\varphi(q(z)) \prec \vartheta(p(z))+zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Lemma 3.4. *(See [9].) Let $q(z)$ be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$. Further, assume that $\Re\{\overline{\gamma}\}>0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(z)+\gamma zp'(z)$ is univalent in U then $q(z)+\gamma zq'(z) \prec p(z)+\gamma zp'(z)$ implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.*

Theorem 3.1. *Let f, g be analytic functions in U such that $f(0)=g(0)=0$, $g'(0)=1$, $(I_z^\alpha g)^\beta(z)$ be convex univalent in U satisfies*

$$\Re\left\{1+\frac{zg''(z)}{g'(z)}+(\beta-1)\frac{zg'(z)}{g(z)}+\frac{z}{g'(z)}+\frac{1}{\gamma}\right\}>0, \quad \beta \geqslant 1,\ g(z)\neq 0,\ g'(z)\neq 0,\ z\in U.$$

If $(I_z^\alpha f)^\beta \in \mathcal{A}$ and the subordination

$$\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta f^\beta(z)\left[1+\beta\gamma\frac{zf'(z)}{f(z)}\right] \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta g^\beta(z)\left[1+\beta\gamma\frac{zg'(z)}{g(z)}\right], \quad \beta\geqslant 1,\ \gamma\in\mathbb{C},$$

holds then $(I_z^\alpha f)^\beta(z) \prec (I_z^\alpha g)^\beta(z)$ and $(I_z^\alpha g)^\beta(z)$ is the best dominant.

Proof. Our aim is to apply Lemma 3.2. Setting

$$p(z) := \left(I_z^\alpha f\right)^\beta \quad \text{and} \quad q(z) := \left(I_z^\alpha g\right)^\beta.$$

First we show that

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\right\}>0.$$

By using Theorem 2.1 we obtain

$$\begin{aligned}\Re\left\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\right\} &= \Re\left\{1+\frac{z\beta(I_z^\alpha g)^{\beta-1}(z)[\frac{z^{\alpha-1}}{\Gamma(\alpha)}+I_z^\alpha g''(z)]}{\beta(I_z^\alpha g)^{\beta-1}(z)I_z^\alpha g'(z)}+\frac{z\beta(\beta-1)(I_z^\alpha g')^2(z)(I_z^\alpha g)^{\beta-2}(z)}{\beta(I_z^\alpha g)^{\beta-1}(z)(I_z^\alpha g'(z))}+\frac{1}{\gamma}\right\}\\ &= \Re\left\{1+\frac{zg''(z)}{g'(z)}+\frac{z(\beta-1)g'(z)}{g(z)}+\frac{z}{g'(z)}+\frac{1}{\gamma}\right\}>0.\end{aligned}$$

Hence $q(z)$ is convex univalent function in U. Now we must show that

$$p(z)+\gamma zp'(z) \prec q(z)+\gamma zq'(z),$$

where $\Re\{\overline{\gamma}\}>0$ and $\psi=1$. By using Theorem 2.1, and the assumption of the theorem we have

$$p(z)+\gamma zp'(z) = \left(I_z^\alpha f\right)^\beta(z)+\gamma z\left[\left(I_z^\alpha f\right)^\beta\right]'(z) \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta g^\beta(z)\left[1+\beta\gamma\frac{zg'(z)}{g(z)}\right]=q(z)+\gamma zq'(z).$$

Thus in view of Lemma 3.2, $p(z) \prec q(z)$ and q is the best dominant. □

Theorem 3.2. *Let f, g be analytic functions in U such that $f(0)=g(0)=0$, $g'(0)=1$. Denotes $G(z) := \frac{z^{\alpha-1}}{\Gamma(\alpha)}[\frac{g'(z)}{z}-\frac{g(z)}{z^2}]$ such that*

$$\Re\left\{2+\frac{zG'(z)}{G(z)}+\frac{(\mu-1)z^{3-\alpha}\Gamma(\alpha)G(z)}{g(z)}\right\}>0.$$

Assume $z[(\frac{I_z^\alpha g}{z})^\mu(z)]'$ is starlike univalent function in U. If $(\frac{I_z^\alpha f}{z})^\mu(z) \in \mathcal{A}$ and the subordination

$$\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu\left(\frac{f}{z}\right)^\mu(z)\left\{1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right\} \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu\left(\frac{g}{z}\right)^\mu(z)\left\{1+\mu\left(\frac{zg'(z)}{g(z)}-1\right)\right\}$$

holds then

$$\left(\frac{I_z^\alpha f}{z}\right)^\mu(z) \prec \left(\frac{I_z^\alpha g}{z}\right)^\mu(z), \quad \mu \geqslant 1,$$

and $(\frac{I_z^\alpha g}{z})^\mu(z)$ *is the best dominant.*

Proof. Our aim is to apply Lemma 3.1. Setting $p(z) := (\frac{I_z^\alpha f}{z})^\mu(z)$ and $q(z) := (\frac{I_z^\alpha g}{z})^\mu(z)$. First we show that $\Re\{2 + \frac{zq''(z)}{q'(z)}\} > 0$,

$$\Re\left\{2 + \frac{zq''(z)}{q'(z)}\right\} = \Re\left\{2 + \frac{zG'(z)}{G(z)} + \frac{(\mu-1)z^{3-\alpha}\Gamma(\alpha)G(z)}{g(z)}\right\} > 0.$$

By setting

$$\theta(\omega) := \omega \quad \text{and} \quad \phi(\omega) := 1,$$

it can easily observed that $\theta(z)$, $\phi(z)$ are analytic in $\mathbb{C}$. Also, we let

$$Q(z) := zq'(z)\phi(z) = zq'(z),$$
$$h(z) := \theta\big(q(z)\big) + Q(z) = q(z) + zq'(z).$$

By the assumptions of the theorem we find that $Q(z)$ is starlike univalent in U and that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{2 + \frac{zq''(z)}{q'(z)}\right\} = \Re\left\{2 + \frac{zG'(z)}{G(z)} + \frac{(\mu-1)z^{3-\alpha}\Gamma(\alpha)G(z)}{g(z)}\right\} > 0.$$

Now we must show that

$$p(z) + zp'(z) \prec q(z) + zq'(z).$$

A computation shows that

$$p(z) + zp'(z) = \left(\frac{I_z^\alpha f}{z}\right)^\mu(z) + z\left[\left(\frac{I_z^\alpha f}{z}\right)^\mu(z)\right]' \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu\left(\frac{g}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zg'(z)}{g(z)} - 1\right)\right\} = q(z) + zq'(z).$$

Thus in view of Lemma 3.1, $p(z) \prec q(z)$ and q is the best dominant. □

Theorem 3.3. *Let* f, g *be analytic functions in* U *such that* $f(0) = g(0) = 0$, $(I_z^\alpha g)^\beta(z)$ *be convex univalent in* U *and* $(I_z^\alpha f)^\beta(z) \in \mathcal{H}[0,1] \cap Q$. *Assume that* $[\frac{z^{\alpha-1}}{\Gamma(\alpha)}]^\beta f^\beta(z)[1 + \beta\gamma\frac{zf'(z)}{f(z)}]$ *is univalent in* U *where* $\Re\{\overline{\gamma}\} > 0$. *If* $(I_z^\alpha f)^\beta(z) \in \mathcal{A}$ *and the subordination*

$$\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta g^\beta(z)\left[1 + \beta\gamma\frac{zg'(z)}{g(z)}\right] \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta f^\beta(z)\left[1 + \beta\gamma\frac{zf'(z)}{f(z)}\right], \quad \beta \geqslant 1,$$

holds then $(I_z^\alpha g)^\beta(z) \prec (I_z^\alpha f)^\beta(z)$ *and* $(I_z^\alpha g)^\beta(z)$ *is the best subordinant.*

Proof. Our aim is to applied Lemma 3.4. Setting $p(z) := (I_z^\alpha f)^\beta$ and $q(z) := (I_z^\alpha g)^\beta$,

$$q(z) + \gamma zq'(z) = \big(I_z^\alpha g\big)^\beta(z) + \gamma z\big[\big(I_z^\alpha g\big)^\beta\big]'(z) \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta f^\beta(z)\left[1 + \beta\gamma\frac{zf'(z)}{f(z)}\right] = p(z) + \gamma zp'(z).$$

Hence in view of Lemma 3.4, $q(z) \prec p(z)$ and $q(z)$ is the best subordinant. □

Theorem 3.4. *Let* f, g *be analytic functions in* U *such that* $f(0) = g(0) = 0$, $(\frac{I_z^\alpha g}{z})^\mu(z)$ *be convex univalent in* U. *Let the following assumptions satisfy*: $z[(\frac{I_z^\alpha g}{z})^\mu(z)]'$ *is starlike univalent function in* U, $[\frac{z^{\alpha-1}}{\Gamma(\alpha)}]^\mu(\frac{f}{z})^\mu(z)\{1 + \mu(\frac{zf'(z)}{f(z)} - 1)\}$ *is univalent in* U *and* $(\frac{I_z^\alpha f}{z})^\mu(z) \in \mathcal{H}[0,1] \cap Q$ *with* $\mu(\alpha - 2) > 0$. *If* $(\frac{I_z^\alpha f}{z})^\mu(z) \in \mathcal{A}$ *and the subordination*

$$\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu\left(\frac{g}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zg'(z)}{g(z)} - 1\right)\right\} \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu\left(\frac{f}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zf'(z)}{f(z)} - 1\right)\right\}$$

holds then

$$\left(\frac{I_z^\alpha g}{z}\right)^\mu(z) \prec \left(\frac{I_z^\alpha f}{z}\right)^\mu(z), \quad \mu \geqslant 1,$$

and $(\frac{I_z^\alpha f}{z})^\mu(z)$ *is the best subordinant.*

Proof. Our aim is to apply Lemma 3.3. Setting $p(z) := (\frac{I_z^\alpha f}{z})^\mu(z)$ and $q(z) := (\frac{I_z^\alpha g}{z})^\mu(z)$. By taking

$$\vartheta(\omega) := \omega \quad \text{and} \quad \varphi(\omega) := 1,$$

it can easily observed that $\vartheta(z)$, $\varphi(z)$ are analytic in $\mathbb{C}$. Thus

$$\Re\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} = 1 > 0.$$

Now we must show that

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

A small computation shows that

$$q(z) + zq'(z) = \left(\frac{I_z^\alpha g}{z}\right)^\mu(z) + z\left[\left(\frac{I_z^\alpha g}{z}\right)^\mu(z)\right]' \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu \left(\frac{f}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} = p(z) + zp'(z).$$

Thus in view of Lemma 3.3, $q(z) \prec p(z)$ and p is the best subordinant. □

Combining Theorems 3.1 and 3.3 we get the following sandwich theorem.

Theorem 3.5. *Let f, g_1, g_2 be analytic functions in U such that $f(0) = g_1(0) = g_2(0) = 0$, $g_2'(0) = 1$, and $(I_z^\alpha g_1)^\beta(z)$, $(I_z^\alpha g_2)^\beta(z)$ be convex univalent functions in U satisfy*

$$\Re\left\{1 + \frac{zg_2''(z)}{g_2'(z)} + \frac{z(\beta-1)g_2'(z)}{g_2(z)} + \frac{z}{g_2'(z)} + \frac{1}{\gamma}\right\} > 0, \quad \beta \geqslant 1.$$

If $(I_z^\alpha f)^\beta(z) \in \mathcal{H}[0,1] \cap Q$, $[\frac{z^{\alpha-1}}{\Gamma(\alpha)}]^\beta f^\beta(z)[1 + \beta\gamma\frac{zf'(z)}{f(z)}]$ is univalent in U and satisfies the subordination

$$\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta g_1^\beta(z)\left[1 + \beta\gamma\frac{zg_1'(z)}{g_1(z)}\right] \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta f^\beta(z)\left[1 + \beta\gamma\frac{zf'(z)}{f(z)}\right] \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\beta g_2^\beta(z)\left[1 + \beta\gamma\frac{zg_2'(z)}{g_2(z)}\right], \quad \beta \geqslant 1,\ \gamma \in \mathbb{C},$$

with $\Re\{\overline{\gamma}\} > 0$. Then

$$\left(I_z^\alpha g_1\right)^\beta(z) \prec \left(I_z^\alpha f\right)^\beta(z) \prec \left(I_z^\alpha g_2\right)^\beta(z), \quad \beta \geqslant 1,$$

such that $(I_z^\alpha g_1)^\beta(z)$ is the best subordinant and $(I_z^\alpha g_2)^\beta(z)$ is the best dominant.

Combining Theorems 3.2 and 3.4 we get the following sandwich theorem.

Theorem 3.6. *Let f, g_1, g_2 be analytic functions in U such that $f(0) = g_1(0) = g_2(0) = 0$, $g_2'(0) = 1$, $(\frac{I_z^\alpha g_1}{z})^\mu(z)$ be convex univalent functions in U. Denotes $G_2(z) := \frac{z^{\alpha-1}}{\Gamma(\alpha)}[\frac{g_2'(z)}{z} - \frac{g_2(z)}{z^2}]$ such that*

$$\Re\left\{2 + \frac{zG_2'(z)}{G_2(z)} + \frac{(\mu-1)z^{3-\alpha}\Gamma(\alpha)G_2(z)}{g_2(z)}\right\} > 0,$$

and let the following assumptions be satisfy: $z[(\frac{I_z^\alpha g_2}{z})^\mu(z)]'$, $z[(\frac{I_z^\alpha g_1}{z})^\mu(z)]'$ are starlike univalent functions in U, $[\frac{z^{\alpha-1}}{\Gamma(\alpha)}]^\mu(\frac{f}{z})^\mu(z) \times \{1 + \mu(\frac{zf'(z)}{f(z)} - 1)\}$ is univalent in U and $(\frac{I_z^\alpha f}{z})^\mu(z) \in \mathcal{H}[0,1] \cap Q$ with $\mu(\alpha - 2) > 0$. If $(\frac{I_z^\alpha f}{z})^\mu(z) \in \mathcal{A}$ and the subordination

$$\left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu \left(\frac{g_1}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zg_1'(z)}{g_1(z)} - 1\right)\right\} \prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu \left(\frac{f}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zf'(z)}{f(z)} - 1\right)\right\}$$
$$\prec \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right]^\mu \left(\frac{g_2}{z}\right)^\mu(z)\left\{1 + \mu\left(\frac{zg_2'(z)}{g_2(z)} - 1\right)\right\}$$

holds then

$$\left(\frac{I_z^\alpha g_1}{z}\right)^\mu(z) \prec \left(\frac{I_z^\alpha f}{z}\right)^\mu(z) \prec \left(\frac{I_z^\alpha g_2}{z}\right)^\mu(z), \quad \mu \geqslant 1,$$

and $(\frac{I_z^\alpha g_1}{z})^\mu(z)$, $(\frac{I_z^\alpha g_2}{z})^\mu(z)$ are respectively the best dominant and the best subordinant.

4. Existence and uniqueness of univalent solution

Let $\mathcal{B} := \mathcal{C}[U, \mathbb{C}]$ be a Banach space of all continuous functions on U endowed with the sup norm

$$\|u\| := \sup_{z\in U}\big|u(z)\big|.$$

By using the properties in Theorem 2.1, we can easily obtain the following result.

Lemma 4.1. *If the function $f \in \mathcal{A}$, then the initial value problem* (1) *is equivalent to the nonlinear Volterra integral equation*

$$u(z) = \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta, \quad \alpha \in (0,1). \tag{2}$$

In other words, every solution of the Volterra equation (2) *is also a solution of the initial value problem* (1) *and vice versa.*
Therefore, we focus our attention on Eq. (2)*. This equation is called singular if $\alpha \in (0,1)$ and regular if $\alpha \geqslant 1$.*

Theorem 4.1 *(Existence). Let the function $f : U \times \mathbb{C} \to \mathbb{C}$ be a continuous analytic function such that $\|f\| \leqslant M$, $0 < M < \infty$. Then there exists a univalent function $u : U \to \mathbb{C}$ solving the problem* (1)*.*

Proof. Define an operator $P : \mathbb{C} \to \mathbb{C}$

$$(Pu)(z) := \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta, \quad \alpha \in (0,1). \tag{3}$$

Our aim is to applied Theorem 1.1. First we show that P is bounded operator:

$$\big|(Pu)(z)\big| \leqslant \int_0^z \left|\frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right| \big|f\big(\zeta, u(\zeta)\big)\big|\, d\zeta \leqslant M \int_0^z \left|\frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right| \leqslant \frac{M}{\Gamma(\alpha+1)} := r, \quad z \in U,$$

that is $\|Pu\|_{\mathcal{B}} = \sup_{z\in U} |(Pu)(z)|$. Define the set $S := \{u \in \mathcal{B}\colon\ \|u\| \leqslant r,\ r \geqslant 0\}$, thus $P : S \to S$. We proceed to prove that $P : S \to S$ is continuous operator. Since f is continuous function on $U \times S$, then it is uniformly continuous on a compact set $\widetilde{U} \times S$, where

$$\widetilde{U} := \big\{z \in U\colon\ |z| \leqslant \ell,\ 0 < \ell < 1\big\}.$$

Hence given $\epsilon > 0$, $\exists \mu > 0$ such that for all $u, v \in S$, we have

$$\big\|f(z,u) - f(z,v)\big\| < \frac{\epsilon\Gamma(\alpha+1)}{\ell^\alpha} \quad \text{for } \|u - v\| < \mu,$$

then

$$\begin{aligned}
\big|(Pu)(z) - (Pv)(z)\big| &= \left|\int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta - \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, v(\zeta)\big)\, d\zeta\right| \\
&\leqslant \int_0^z \frac{|(z-\zeta)^{\alpha-1}|}{\Gamma(\alpha)} \times \big|f\big(\zeta, u(\zeta)\big) - f\big(\zeta, v(\zeta)\big)\big|\, d\zeta \\
&\leqslant \frac{\epsilon\Gamma(\alpha+1)}{\ell^\alpha} \times \frac{\ell^\alpha}{\Gamma(\alpha+1)} = \epsilon.
\end{aligned}$$

Thus P is a continuous mapping on S. Now we show that P is an equicontinuous mapping on S. For $z_1, z_2 \in \widetilde{U}$ such that $z_1 \neq z_2$, then for all $u \in S$, we obtain

$$\begin{aligned}
\big|(Pu)(z_1) - (Pu)(z_2)\big| &= \left|\int_0^{z_1} \frac{(z_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta - \int_0^{z_2} \frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta\right| \\
&\leqslant \int_0^{z_1} \left|\frac{(z_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta\right| + \int_0^{z_2} \left|\frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta, u(\zeta)\big)\, d\zeta\right|
\end{aligned}$$

$$\leqslant M\int_0^{z_1}\left|\frac{(z_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right|d\zeta + M\int_0^{z_2}\left|\frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right|d\zeta$$
$$\leqslant M\left[\frac{(|z_1|)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(|z_2|^{\alpha})}{\Gamma(\alpha+1)}\right]\leqslant\frac{2M\ell^{\alpha}}{\Gamma(\alpha+1)},$$

which is independent on u. Hence P is an equicontinuous mapping on S. Further, under the assumption of Theorem 4.1, we can prove that P is a univalent function (see [1]). The Arzela–Ascoli theorem yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a univalent solution of the initial value problem (1). □

Theorem 4.2 *(Uniqueness). Let the function f be bounded and fulfill a Lipschitz condition with respect to the second variable: i.e.,*

$$\left\| f(z,u)-f(z,v)\right\| \leqslant L\|u-v\|$$

with some $L>0$ independent of u, v and z. If $L<\Gamma(\alpha+1)$, then there exists at most one univalent function $u: U\to\mathbb{C}$ solving the initial value problem (1)*.*

Proof. We need only to prove that the operator P in Eq. (3) has a unique fixed point.

$$\left|(Pu)(z)-(Pv)(z)\right| = \left|\int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta,u(\zeta)\big)\,d\zeta - \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f\big(\zeta,v(\zeta)\big)\,d\zeta\right|$$
$$\leqslant \left\| f(z,u)-f(z,v)\right\| \int_0^z \frac{|(z-\zeta)^{\alpha-1}|}{\Gamma(\alpha)}\,d\zeta$$
$$\leqslant \frac{L(|z|)^{\alpha}}{\Gamma(\alpha+1)}\|u-v\| < \frac{L}{\Gamma(\alpha+1)}\|u-v\|,$$

then for all $u, v\in U$ we obtain

$$\|Pu-Pv\|\leqslant\frac{L}{\Gamma(\alpha+1)}\|u-v\|.$$

Thus the operator P is a contraction mapping then in view of Banach fixed point theorem, P has a unique fixed point which is corresponding to the solution of the initial value problem (1). Moreover, we have found in the proof of Theorem 3.1, that P is an equicontinuous mapping and hence P is a univalent function, consequently the problem (1) has a unique univalent solution. □

In the next corollaries, we show the relation between univalent solutions of fractional differential problems.

Corollary 4.1. *Let the assumptions of Theorem* 4.1 *hold. If u and v are univalent solutions of the problem*

$$D_z^{\alpha} zu(z) = f\big(z,u(z)\big),\quad u(0)=0.$$

Then they satisfy the subordination $u(z)\prec v(z)$.

Proof. Setting

$$\mu := 1,\qquad f(z,u) := f(z),\qquad f(z,v) := g(z).\qquad \square$$

Corollary 4.2. *Let the assumptions of Theorem* 4.1 *hold. If u and v are univalent solutions of the problem*

$$D_z^{\alpha} u^{\frac{1}{\beta}}(z) = f\big(z,u(z)\big),\quad \beta\geqslant 1,\ u(0)=0.$$

Then they satisfy the subordination $u(z)\prec v(z)$.

Proof. Setting

$$f(z,u) := f(z),\qquad f(z,v) := g(z).\qquad \square$$

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