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RESEARCH ARTICLE - MATHEMATICS

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On Analytic Functions Associated with the Dziok–Srivastava Linear Operator and Srivastava–Owa Fractional Integral Operator

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Abstract In this article, we consider some subordination and superordination results involving the Dziok–Srivastava linear operator and the Srivastava–Owa fractional integral operator for certain normalized analytic functions in the open unit disk. Moreover, we discuss the existence of univalent solutions for fractional differential equations involving the Dziok–Srivastava linear operators. Some well-known results are introduced as special cases.

Keywords Dziok–Srivastava linear operator · Srivastava–Owa fractional integral operator · Univalent solution · Subordination · Superordination

Mathematics Subject Classification (2010) 30C45 · 30C50 · 33C20

الملخص

نعتبر في هذا البحث بعض النتائج التابعة والرائدة المتعلقة بمؤثر دزيوك – سرفاستافا الخطي ومؤثر سرفاستافا – أوا التكاملي الكسري لدوال ناظمية تحليلية معينة في قرص الوحدة المفتوح. بالإضافة إلى ذلك، نناقش وجود حلول عالمية لمعادلات تفاضلية كسرية تحتوي على مؤثرات دزيوك – سرفاستافا خطية. نقدم بعض النتائج المعروفة جيداً كحالات خاصة.

1 Introduction

Let $\mathcal{H}(\mathrm{U})$ denote the class of analytic functions in the unit disk $U := \{z \in \mathbb{C}, |z| < 1\}$. For n positive integers and $a \in \mathbb{C}$, let $\mathcal{H}[a, n] := \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\}$, and $\mathcal{A}_n = \{f \in H(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{H}[a, n]$ is convex in U if it is univalent and $f(U)$ is convex. Recall that for two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z)$. Also, it is well known that f is convex if and only if $\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, z \in U$.
For the given two functions F and G in the unit disk U, the function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. Alternatively, given two functions F and G, which are analytic in U, the function F is said to be subordination to G in U if there exists a function h, analytic in U with $h(0) = 0$ and $|h(z)| < 1, \forall z \in U$ such that $F(z) = G(h(z))$. Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent

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in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$, then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

For $\alpha_j \in \mathbb{C}(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}(j = 1, 2, \ldots, m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots \beta_m; z)$ is defined by the infinite series

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} $$
$$ (l \leq m + 1 : l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}) $$

where $(a)_n$ is the Pochhammer symbol defined by

$$ (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N}). \end{cases} $$

Corresponding to the function

$$ h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots \beta_m; z) := z\,{}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots \beta_m; z), $$

the Dziok–Srivastava operator (see [5–8,18]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots \beta_m)$ is defined by the Hadamard product

$$ \begin{aligned} H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots \beta_m) f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!} \\ &:= H_m^l[\alpha_1] f(z). \end{aligned} $$

We can verify that

$$ [z(H_m^l[\alpha_1] f(z))' = \alpha_1 H_m^l[\alpha_1 + 1] f(z) - (\alpha_1 - 1) H_m^l[\alpha_1] f(z). $$

Special cases of the Dziok–Srivastava linear operator include the Hohlov linear operator [3], the Carlson–Shaffer linear operator [9], the Ruscheweyh derivative operator [21], the generalized Bernardi–Libera–Livingston linear integral operator [17] and the Srivastava–Owa fractional derivative operator [19].

Definition 1.1 The fractional derivative of order α is defined, for a function f, by

$$ D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta; \quad 0 \leq \alpha < 1, $$

where the function f is analytic in a simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2 The fractional integral of order α is defined, for a function f, by

$$ I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0, $$

where the function f is analytic in a simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 1.3 [19]

$$ D_z^{\alpha}\{z^{\mu}\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\{z^{\mu-\alpha}\}, \quad \mu > -1; 0 \leq \alpha < 1 $$

and

$$I_z^\alpha\{z^\mu\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\{z^{\mu+\alpha}\}, \quad \mu > -1; \alpha > 0.$$

The main objective of this paper is to find the sufficient conditions for certain normalized analytic functions f, g to satisfy

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_1(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_2(z)}{\sigma(z)}\right]^\mu$$

and

$$q_1(z) \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \prec q_2(z), \sigma(z) \neq 0, z \in U$$

where $\mu \geq 1$, q_1 and q_2 are given univalent functions in U. Also, we obtain the results as special cases. Further, in this paper, we study the existence of univalent solutions for the fractional differential equations

$$D_z^\alpha \rho(z)u(z) = H_m^l[\alpha_1]f(z), \tag{1}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$, $\rho : U \to \mathbb{C}$ is an analytic function in $z \in U$ and $f : U \to \mathbb{C}$ is a univalent function in U. The existence is obtained by applying the Schauder Fixed Point Theorem. Moreover, we discuss some properties of this solution involving fractional differential subordination.

For various investigations based upon the Dziok– Srivastava linear operator, readers may be referred to some recent papers (see [4,10,20]). We need the following preliminaries.

Definition 1.4 [14] *Denote by* $\boldsymbol{Q}$ *the set of all functions* $f(z)$ *that are analytic and injective on* $\overline{U} - E(f)$ *where* $E(f) := \{\zeta \in \partial U : lim_{z\to\zeta} f(z) = \infty\}$ *and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U - E(f)$.

Lemma 1.5 [15] *Let* q *be univalent in the unit disk* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. *Suppose that*

1. $Q(z)$ *is starlike univalent in* U, *and*
2. $\Re\frac{zh'(z)}{Q(z)} > 0$ *for* $z \in U$.

If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, *then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.6 [16] *Let* q *be convex univalent in the unit disk* U *and* $\psi, \gamma \in \mathbb{C}$ *with* $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$. *If* $p(z)$ *is analytic in* U *and* $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$, *then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.7 [2] *Let* q *be convex univalent in the unit disk* U *and* ϑ *and* φ *be analytic in a domain* D *containing* $q(U)$. *Suppose that*

1. $zq'(z)\varphi(q(z))$ *is starlike univalent in* U, *and*
2. $\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ *for* $z \in U$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap \boldsymbol{Q}$, *with* $p(U) \subseteq D$ *and* $\vartheta(p(z)) + zp'(z)\varphi(z)$ *is univalent in* U *and* $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$, *then* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Lemma 1.8 [14] *Let* q *be convex univalent in the unit disk* U *and* $\gamma \in \mathbb{C}$. *Further, assume that* $\Re\{\overline{\gamma}\} > 0$. *If* $p(z) \in \mathcal{H}[q(0), 1] \cap \boldsymbol{Q}$, *with* $p(z) + \gamma zp'(z)$ *univalent in* U, *then* $q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$ *implies* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Theorem 1.9 *Arzela–Ascoli (see* [13]*). Let* E *be a compact metric space and* $\mathcal{C}(E)$ *be the Banach space of real or complex valued continuous functions normed by* $\|f\| := sup_{t\in E}|f(t)|$. *If* $A = \{f_n\}$ *is a sequence in* $\mathcal{C}(E)$ *such that* f_n *is uniformly bounded and equicontinuous, then* $\overline{A}$ *is compact.*

Let M be a subset of Banach space X and $A : M \to M$ an operator. The operator A is called *compact* on the set M if it carries every bounded subset of M into a compact set. If A is continuous on M (that is, it maps bounded sets into bounded sets), then it is said to be *completelycontinuous* on M.

Theorem 1.10 *(Schauder) (see* [1]*) Let* X *be a Banach space,* $M \subset X$ *a nonempty closed bounded convex subset and* $P : M \to M$ *is compact. Then* P *has a fixed point.*

2 Subordination and Superordination

In this section, we study some important properties of the fractional differential and integral operators D_z^α, I_z^α, which are useful in the next results of the subordination and superordination.

Theorem 2.1 [11] *If* $\alpha \in [0, 1)$ *and* f *is a continuous function, then*

(i) $DI_z^\alpha f(z) = \frac{(z)^{\alpha-1}}{\Gamma(\alpha)} f(0) + I_z^\alpha Df(z);\ D = \frac{d}{dz}$.
(ii) $I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z)$.

The next result shows the subordination involving Dziok–Srivastava linear operator and fractional integral operator I_z^α.

Theorem 2.2 *Let* f, g *be analytic in* U *and* $q(z) := \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu$ *be univalent in* U *such that* zq' *is starlike in* U *and* $q(z) \neq 0$ *for* $\mu \geq 1$. *If the subordination*

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\}$$
$$\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\}$$

holds and $G(z) := \frac{(I_z^\alpha H_m^l[\alpha_1]g(z))'}{I_z^\alpha H_m^l[\alpha_1]g(z)} + \frac{1}{z} - \frac{\sigma'(z)}{\sigma(z)}$ *with*

$$\Re\left\{2 + \frac{zG'(z)}{G(z)} + \frac{zq'(z)}{q(z)}\right\} > 0,\ z \in U, \tag{2}$$

then

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu$$

and $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g}{\sigma}\right]^\mu$ *is the best dominant.*

Proof Set $p(z) := \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu$. We apply Lemma 1.1. Assume that $\theta(\omega) := \omega$ *and* $\phi(\omega) := 1$. It can easily be observed that θ, ϕ are analytic in $\mathbb{C}$. Also, we let

$$Q(z) := zq'(z)\phi(z) = zq'(z),$$
$$h(z) := \theta(q(z)) + Q(z) = q(z) + zq'(z).$$

By our assumptions, we find that Q is starlike univalent in U and that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{2 + \frac{zq''(z)}{q'(z)}\right\} = \Re\left\{2 + \frac{zG'(z)}{G(z)} + \frac{zq'(z)}{q(z)}\right\} > 0.$$

Applying Theorem 2.1 and using the assumptions of the theorem, computations show that

$$p(z) + zp'(z) = \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\}$$
$$\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\}$$
$$= q(z) + zq'(z).$$

Thus, in view of Lemma 1.1, $p(z) \prec q(z)$ and q is the best dominant. □

Setting $l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.2, we obtain the following result.

Corollary 2.3 *Let the assumptions of Theorem 2.2 hold. Then*

$$\left[\frac{zI_z^\alpha L(a,c)f(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha L(a,c)g(z)}{\sigma(z)}\right]^\mu$$

and $\left[\frac{zI_z^\alpha L(a,c)g}{\sigma}\right]^\mu$ *is the best dominant, where* $L(a,c)$ *is the Carlson–Shaffer linear operator.*

Theorem 2.4 *Let* f, g *be analytic in* U, q *be convex univalent in* U *with* $\Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\gamma}\right\}, \gamma \in \mathbb{C}$ *and* $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f}{\sigma}\right]^\mu$ *be analytic in* U*. If the subordination*

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \prec q(z) + \gamma z q'(z)$$

holds, then

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \prec q(z)$$

and q *is the best dominant.*

Proof Set

$$p(z) := \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu.$$

Our aim is to apply Lemma 1.2. Let $\psi := 1$. Since

$$\begin{aligned} p(z) + \gamma z p'(z) &= \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu + \gamma z\left(\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu\right)' \\ &= \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &\prec q(z) + \gamma z q'(z), \end{aligned}$$

it follows, in view of Lemma 1.2, that $p(z) \prec q(z)$ and q is the best dominant. □

Next, apply Lemmas 1.3 and 1.4, respectively, to obtain the following theorems.

Theorem 2.5 *Let* f, g *be analytic in* U, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g}{\sigma}\right]^\mu$ *be convex univalent in* U *such that* $\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)} \neq 0$, $z\left(\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g}{\sigma}\right]^\mu\right)'$ *be starlike univalent in* U *and* $\left(z\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f}{\sigma}\right]^\mu\right)'$ *be univalent in* U*. If the subordination*

$$\begin{aligned} &\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{zI_z^\alpha[H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \end{aligned}$$

holds and $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \in \mathcal{H}[0,1] \cap \boldsymbol{Q}$, *then*

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu$$

and $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g}{\sigma}\right]^\mu$ *is the best subordinant.*

Proof Set

$$p(z) := \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \textit{ and } \quad q(z) := \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu.$$

We apply Lemma 1.3. By taking

$$\vartheta(\omega) := \omega \quad \textit{and} \quad \varphi(\omega) := 1,$$

it can easily be observed that ϑ, φ are analytic in $\mathbb{C}$. Thus,

$$\Re\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} = 1 > 0.$$

Now, we must show that

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

Computations show that

$$\begin{aligned} q(z) + zq'(z) &= \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &= p(z) + zp'(z). \end{aligned}$$

Thus, in view of Lemma 1.3, $q(z) \prec p(z)$ and p is the best subordinant. □

Theorem 2.6 *Let* f, g *be analytic in* U, q *be convex univalent in* U, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \in \mathcal{H}[0,1] \cap \boldsymbol{Q}$ *and*

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\}, \Re\{\overline{\gamma}\} > 0,$$

be univalent in U. *If the subordination*

$$q(z) + \gamma zq'(z) \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\}$$

holds, then

$$q(z) \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu$$

and q *is the best subordinant.*

Proof Set

$$p(z) := \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu .$$

Our aim is to apply Lemma 1.4. Since

$$\begin{aligned} q(z) + \gamma z q'(z) &= \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]g(z)]'}{I_z^\alpha H_m^l[\alpha_1]g(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &= p(z) + \gamma z p'(z), \end{aligned}$$

it follows, in view of Lemma 1.4, that $q(z) \prec p(z)$ and q is the best subordinant. □

Combining the results of differential subordination and superordination, we state the following sandwich results.

Theorem 2.7 *Let f, g_1, g_2 be analytic in U, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_1}{\sigma}\right]^\mu$ be convex univalent in U such that $\frac{zI_z^\alpha H_m^l[\alpha_1]g(z)}{\sigma(z)} \neq 0$, $z\left(\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_1}{\sigma}\right]^\mu\right)'$, $z\left(\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_2}{\sigma}\right]^\mu\right)'$ be starlike univalent in U, $\left(z\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f}{\sigma}\right]^\mu\right)'$, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_2}{\sigma}\right]^\mu$ be univalent in U, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \in \mathcal{H}[0,1] \cap \boldsymbol{Q}$ and assume that* (2) *is satisfied. If the subordination*

$$\begin{aligned} &\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_1(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]g_1(z)]'}{I_z^\alpha H_m^l[\alpha_1]g_1(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \\ &\prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_2(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]g_2(z)]'}{I_z^\alpha H_m^l[\alpha_1]g_2(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \end{aligned}$$

holds, then

$$\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_1(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_2(z)}{\sigma(z)}\right]^\mu$$

and $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_1}{\sigma}\right]^\mu$, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]g_2}{\sigma}\right]^\mu$ are, respectively, the best subordinant and dominant.

For $l = m + 1$, $\frac{(\alpha_1)_{n-1}\cdots(\alpha_{l-1})_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_m)_{n-1}} = 1$ and $\alpha_l = 1$, the next result can be found in [12].

Corollary 2.8 *Let the assumptions of Theorem 2.6 hold. Then*

$$\left[\frac{zI_z^\alpha g_1(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha f(z)}{\sigma(z)}\right]^\mu \prec \left[\frac{zI_z^\alpha g_2(z)}{\sigma(z)}\right]^\mu$$

and $\left[\frac{zI_z^\alpha g_1}{\sigma}\right]^\mu$ and $\left[\frac{zI_z^\alpha g_2}{\sigma}\right]^\mu$ are, respectively, the best subordinant and dominant.

Theorem 2.9 *Let* $f, g_1, g_2 \in \mathcal{A}, q_1, q_2$ *be convex univalent in* U, *with* $\Re\left\{1 + \frac{zq_2''(z)}{q_2'(z)} + \frac{1}{\gamma}\right\}, \gamma \in \mathbb{C}$, $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f}{\sigma}\right]^\mu \in \mathcal{H}[0,1] \cap \boldsymbol{Q}$, *and analytic in* U *and* $\left[\frac{zI_z^\alpha H_m^l[\alpha_1]f}{\sigma}\right]^\mu \left\{1 + \mu\left(\frac{z[I_z^\alpha H_m^l[\alpha_1]f]'}{I_z^\alpha H_m^l[\alpha_1]f} - \frac{z\sigma'}{\sigma} + 1\right)\right\}$, $\Re\{\overline{\gamma}\} > 0$, *be univalent in* U. *If the subordination*

$$q_1(z) + \gamma z q_1'(z) \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \left\{1 + \mu\gamma\left(\frac{zI_z^\alpha[H_m^l[\alpha_1]f(z)]'}{I_z^\alpha H_m^l[\alpha_1]f(z)} - \frac{z\sigma'(z)}{\sigma(z)} + 1\right)\right\} \prec q_2(z) + \gamma z q_2'(z)$$

holds, then

$$q_1(z) \prec \left[\frac{zI_z^\alpha H_m^l[\alpha_1]f(z)}{\sigma(z)}\right]^\mu \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and the best dominant.*

3 Existence of Univalent Solution

Let $\mathcal{B} := \mathcal{C}[U, \mathbb{C}]$ be a Banach space of all continuous functions on U endowed with the sup. norm

$$\|u\| := sup_{z\in U}|u(z)|.$$

Applying Theorem 2.1, we easily obtain the following result.

Lemma 3.1 *If the function* $f \in \mathcal{A}$, *then the Initial Value Problem (1) is equivalent to the nonlinear integral equation*

$$u(z) = \frac{1}{\rho(z)} \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} H_m^l[\alpha_1]f(\zeta)d\zeta. \tag{3}$$

Theorem 3.2 *(Existence) Assume that* $\frac{1}{|\rho(z)|} \le M; M > 0$. *Then there exists a univalent function* $u : U \to \mathbb{C}$ *solving the Initial Value Problem (1).*

Proof Define an operator $P : \mathbb{C} \to \mathbb{C}$

$$(Pu)(z) := \frac{1}{\rho(z)} \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} H_m^l[\alpha_1]f(\zeta)d\zeta. \tag{4}$$

Denote by $B_n := \frac{(\alpha_1)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_m)_{n-1}} \frac{1}{(n-1)!}$. We apply Theorem 1.1. First, we show that P is a bounded operator:

$$\begin{aligned}
|(Pu)(z)| &= |\frac{1}{\rho(z)} \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} H_m^l[\alpha_1]f(\zeta)d\zeta| \\
&\le |\frac{1}{\rho(z)}|| \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} H_m^l[\alpha_1]f(\zeta)d\zeta| \\
&< M\left(1 + \sum_{n=2}^\infty B_n|a_n|\right)| \int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} d\zeta| \\
&= M\left(1 + \sum_{n=2}^\infty B_n|a_n|\right)\frac{|z^\alpha|}{\Gamma(\alpha+1)} \\
&\le M\left(1 + \sum_{n=2}^\infty B_n|a_n|\right)\frac{|z|^\alpha}{\Gamma(\alpha+1)} \\
&< \frac{M\left(1 + \sum_{n=2}^\infty B_n|a_n|\right)}{\Gamma(\alpha+1)}
\end{aligned}$$

Thus,

$$\|P\| < \frac{M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)}{\Gamma(\alpha+1)} := r.$$

That is, $P : B_r \to B_r$. Whence P maps B_r into itself. Now we proceed by proving that P is equicontinuous. Let $z_1, z_2 \in U$ be such that $z_1 \neq z_2$, $|z_2 - z_1| < \delta, \delta > 0$. Then for all $u \in S$, where

$$S := \left\{u \in \mathbb{C}, : |u| \leq \frac{M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)}{\Gamma(\alpha+1)} := r, r > 0\right\},$$

we obtain

$$\begin{aligned}
|(Pu)(z_1) - (Pu)(z_2)| &\leq M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)\left|\int_0^{z_1} \frac{(z_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)}d\zeta - \int_0^{z_2} \frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)}d\zeta\right| \\
&\leq M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)\left|\int_0^{z_1} \frac{[(z_1-\zeta)^{\alpha-1}-(z_2-\zeta)^{\alpha-1}]}{\Gamma(\alpha)}d\zeta + \int_{z_1}^{z_2} \frac{(z_2-\zeta)^{\alpha-1}}{\Gamma(\alpha)}d\zeta\right| \\
&= \frac{M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)}{\Gamma(\alpha+1)}|[2(z_2-z_1)^{\alpha} + z_2^{\alpha} - z_1^{\alpha}]| \\
&< \frac{2M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)}{\Gamma(\alpha+1)}|z_2-z_1|^{\alpha} \\
&< \frac{2M\left(1+\sum_{n=2}^{\infty} B_n|a_n|\right)}{\Gamma(\alpha+1)}\delta^{\alpha},
\end{aligned}$$

which is independent of u. Hence, P is an equicontinuous mapping on S. By the assumptions of the theorem, we can show that P is a univalent function. The Arzela–Ascoli Theorem implies that every sequence of functions from $P(S)$ has a uniformly convergent subsequence, and therefore, $P(S)$ is relatively compact. Schauder's Fixed Point Theorem asserts that P has a fixed point. By construction, a fixed point of P is a univalent solution of the Initial Value Problem (1). □

The next theorems clarify the relation between univalent solutions and the subordination for a class of fractional differential problem.

Theorem 3.3 *Let the assumptions of Theorem 2.6 be satisfied. The univalent solutions u_1, u, u_2, of the problem*

$$D_z^{\alpha}u(z) = F(z, u(z)), \tag{5}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$ and $F : U \times \mathbb{C} \to \mathbb{C}$, is an analytic function in $z \in U$, satisfy the subordination $u_1 \prec u \prec u_2$.

Proof Set $\mu = 1$ and let $F(z, u_1(z)) := H_m^l[\alpha_1]g_1(z)$, $F(z, u(z)) := H_m^l[\alpha_1]f(z)$, $F(z, u_2(z)) := H_m^l[\alpha_1]g_2(z)$ and $\sigma(z) := z$, $z \in U$ in Theorem 2.6. □

Theorem 3.4 *Let the assumptions of Theorem 2.7 be satisfied. Then every univalent solution u of the problem*

$$D_z^{\alpha}\rho(z)u(z) = F(z, u(z)), \tag{6}$$

subject to the initial condition $u(0) = 0$, satisfies the subordination $q_1(z) \prec u(z) \prec q_2(z)$, where q_1 and q_2 are univalent functions in U.

Proof Set $\mu = 1$, $F(z, u(z)) := H_m^l[\alpha_1]f(z)$ and $\rho(z) := \frac{\sigma(z)}{z}$ in Theorem 2.7. □

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