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Some applications of Miller–Mocanu lemma on certain classes of meromorphic functions

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ABSTRACT

In this paper, making use of a linear operator we introduce and study certain new classes of meromorphic functions. We derive some inclusion results. These classes contain many known classes as a special case.

1. Introduction

Let $\sum$ denotes the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic in the punctured open unit disc $D=\{z\colon 0<|z|<1\}$. Further, let $P_k(\gamma)$ be the class of functions $p(z)$, analytic in $E=D\cup\{0\}$ satisfying $p(0)=1$ and

$$\int_0^{2\pi}\left|\frac{\operatorname{Re}p(z)-\gamma}{1-\gamma}\right|d\theta\leqslant k\pi, \tag{1.2}$$

where $z=re^{i\theta}$, $k\geqslant 2$, $0\leqslant\gamma<1$. This class was introduced by Padmanbhan and Paravatham [9]. For $\gamma=0$ we obtain the class P_k defined by Pinchuk [11] and $P_2(\gamma)=P(\gamma)$ is the class with positive real part greater than γ.

Also note that for $p\in P_k(\gamma)$ if and only if

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z),$$

where $p_1,p_2\in P(\gamma)$ for $z\in E$. By $MC(\gamma)$, $MS^*(\gamma)$ and $MK(\gamma)$, we mean the subclasses of meromorphic convex, meromorphic starlike and meromorphic close-to-convex functions of order γ respectively. The class $\sum$ is closed under the convolution or Hadamard product denoted and defined by

$$(f*g)(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kb_kz^k, \tag{1.3}$$

where

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^kg(z)=\frac{1}{z}+\sum_{k=0}^{\infty}b_kz^k.$$

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The study of operators plays a vital role in Geometric Function Theory. In 1999, using the technique of convolution Noor [6] defined an integral operator (see also [7]). Many author's [1–4,10] studied the properties of Noor integral operator and generalize it in many directions. Motivated, from Noor works, in [12] Yuan et al. defined an operator $I_{n,\mu}: \sum \to \sum$ as follows:

$$I_{n,\mu}f(z) = f_{n,\mu}(z) * f(z), \tag{1.4}$$

where

$$f_{n,\mu}(z) * \frac{1}{z(1-z)^{n+1}} = \frac{1}{z(1-z)^{\mu}} \quad n > -1,\ \mu > 0,\ z \in D. \tag{1.5}$$

Using (1.4) and (1.5), one can easily have

$$I_{n,\mu}f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k z^k, \tag{1.6}$$

where $(a)_k$ is the Pochhammer symbol given by

$$(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1), \quad k \in N.$$

From (1.4) and (1.6), it can be easily verified that

$$z(I_{n+1,\mu}f(z))' = (n+1)I_{n,\mu}f(z) - (n+2)I_{n+1,\mu}f(z) \tag{1.7}$$

and

$$z(I_{n,\mu}f(z))' = \mu I_{n,\mu+1}f(z) - (\mu+1)I_{n,\mu}f(z). \tag{1.8}$$

Furthermore, for $c > 0$ the Generalized Bernardi Operator is defined as

$$J_c f(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt. \tag{1.9}$$

Using the operator $I_{n,\mu}$, we define the following new classes of meromorphic functions

Definition 1.1. Let $f \in \sum,\ n > -1,\ \mu > 0,\ 0 \leqslant \gamma < 1,\ z \in D$, then $f \in MV_k(n,\mu,\gamma)$ if and only if

$$-\frac{(z(I_{n,\mu}f(z))')'}{(I_{n,\mu}f(z))'} \in P_k(\gamma).$$

Definition 1.2. Let $f \in \sum,\ n > -1,\ \mu > 0,\ 0 \leqslant \gamma < 1,\ z \in D$, then $f \in MR_k(n,\mu,\gamma)$ if and only if

$$-\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}f(z)} \in P_k(\gamma).$$

It can be easily observed that

$$f \in MV_k(n,\mu,\gamma) \text{ if and only if } zf' \in MR_k(n,\mu,\gamma).$$

Definition 1.3. Let $f \in \sum,\ n > -1,\ \mu > 0,\ 0 \leqslant \alpha,\ \beta < 1,\ z \in D$, then $f \in MT_k^*(n,\mu,\alpha,\beta)$ if and only if there exists $g \in MR_2(n,\mu,\alpha)$ such that

$$-\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}g(z)} \in P_k(\beta).$$

Remark 1.1. For special values of parameters n, μ, γ and k, we have many known classes of meromorphic functions, see [8,12].

2. Preliminary results

Lemma 2.1 [5]. *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex valued function satisfying the conditions:*

(i) $\Psi(u,v)$ is continuous in $D \subset C^2$,
(ii) $(1,0) \in D$ and $\mathrm{Re}\,\Psi(1,0) > 0$,
(iii) $\mathrm{Re}\,\Psi(iu_2, v_1) \leqslant 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z)$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\mathrm{Re}\,\Psi\,(h(z), zh'(z)) > 0$ for $z \in E$, then $\mathrm{Re}\,h(z) > 0$ in E.

3. Main results

Theorem 3.1. *Let* $n > -1$, $\mu > 0$, $0 \leqslant \alpha, \beta, \gamma < 1$, $z \in D$, *then*

$$MR_k(n, \mu+1, \alpha) \subset MR_k(n, \mu, \beta) \subset MR_k(n+1, \mu, \gamma).$$

Proof. First we prove

$$MR_k(n, \mu+1, \alpha) \subset MR_k(n, \mu, \beta).$$

For this let $f(z) \in MR_k(n, \mu+1, \alpha)$ and set

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z) = -\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}f(z)}, \tag{3.1}$$

where $H(z)$ is analytic in E and $H(0) = 1$. Using (1.8) and (3.1) and after some simplification, we obtain

$$-\frac{z(I_{n,\mu+1}f(z))'}{I_{n,\mu+1}f(z)} = H(z) + \frac{zH'(z)}{-H(z)+(\mu+1)} \in P_k(\alpha). \tag{3.2}$$

Let

$$\Phi_\mu(z) = \frac{1}{\mu+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty} z^k\right] + \frac{\mu}{\mu+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty} kz^k\right],$$

then

$$\begin{aligned} H(z) * z\Phi_\mu(z) &= H(z) + \frac{zH'(z)}{-H(z)+(\mu+1)} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{-h_1(z)+(\mu+1)}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{zh_2'(z)}{-h_2(z)+(\mu+1)}\right). \end{aligned} \tag{3.3}$$

Since $f(z) \in MR_k(n, \mu+1, \gamma)$, it follows from (3.2) and (3.3) that $h_i(z) + \frac{zh_i'(z)}{-h_i(z)+(\mu+1)} \in P(\alpha)$, for $i = 1, 2$.

Let $h_i(z) = \beta + (1-\beta)p_i(z)$. Then

$$\left[(\beta-\alpha) + (1-\beta)p_i(z) + \frac{(1-\beta)zp_i'(z)}{-(1-\beta)p_i(z) + (-\beta+\mu+1)}\right] \in P, \quad z \in E.$$

We formulate a functional $\Psi(u, v)$ by taking $u = h_i(z)$ and $v = zh_i'(z)$, then

$$\Psi(u, v) = (\beta-\alpha) + (1-\beta)u + \frac{(1-\beta)v}{-(1-\beta)u + (-\beta+\mu+1)}.$$

Clearly, $\Psi(u, v)$ satisfies first two conditions of Lemma 2.1. For the third condition, we proceed as follows:

$$\operatorname{Re}\Psi(iu_2, v_1) = (\beta-\alpha) + \frac{(1-\beta)(1+\mu-\beta)v_1}{(1+\mu-\beta)^2 + (1-\beta)^2u_2^2}.$$

By putting $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$, we have

$$\operatorname{Re}\Psi(iu_2, v_1) \leqslant \frac{A+Bu_2^2}{2C},$$

where

$$\begin{aligned} A &= 2(\beta-\alpha)(1+\mu-\beta)^2 - (1-\beta)(1+\mu-\beta), \\ B &= 2(\beta-\alpha)(1-\beta)^2 - (1-\beta)(1+\mu-\beta), \\ C &= (1+\mu-\beta)^2 + (1-\beta)^2u_2^2. \end{aligned}$$

We note that $\operatorname{Re}\Psi(iu_2, v_1) \leqslant 0$ if $A \leqslant 0$ and $B \leqslant 0$. From $A \leqslant 0$, we obtain

$$\beta = \frac{1}{4}\left[(3+2\mu+2\alpha) - \sqrt{(3+2\mu+2\alpha)^2 - 8}\right]. \tag{3.4}$$

Now using Lemma 2.1, we see that $p_i \in P$, for $i = 1, 2$ and $z \in E$. Hence $H(z) \in P_k(\beta)$ and consequently $f(z) \in MR_k(n, \mu, \beta)$. Working in a similar way, we obtain the other inclusion. □

Theorem 3.2. *Let* $n > -1$, $\mu > 0$, $0 \leqslant \alpha, \beta, \gamma < 1$, $z \in D$, *then*

$$MV_k(n,\mu+1,\alpha)\subset MV_k(n,\mu,\beta)\subset MV_k(n+1,\mu,\gamma).$$

Proof. Proof follows immediately, by using Remark 1.1 and Theorem 3.1. □

Theorem 3.3. *Let $n > -1$, $\mu > 0$, $0 \leqslant \alpha, \beta, \gamma, \delta < 1$, $z \in D$, then*

$$MT^*_k(n,\mu+1,\alpha,\beta)\subset MT^*_k(n,\mu,\alpha,\gamma)\subset MT^*_k(n+1,\mu,\alpha,\delta).$$

Proof. First we prove

$$MT^*_k(n,\mu+1,\gamma)\subset MT^*_k(n,\mu,\gamma)$$

Let $f(z)\in MT^*_k(n,\mu+1,\alpha,\beta)$ then there exists $g(z) \in MR_2(n, \mu+1, \alpha)$ such that

$$-\frac{z(I_{n,\mu+1}f(z))'}{I_{n,\mu+1}g(z)}\in P_k(\beta). \tag{3.5}$$

We set

$$\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}g(z)}=-H(z)=-\gamma-(1-\gamma)p(z), \tag{3.6}$$

where $g(z) \in MR_2(n, \mu, \alpha)$, because $MR_2(n, \mu+1, \alpha) \subset MR_2(n, \mu, \alpha)$. Then also

$$-\frac{z(I_{n,\mu}g(z))'}{I_{n,\mu}g(z)}\in P_2(\alpha)=P(\alpha), \tag{3.7}$$

this implies

$$\frac{z(I_{n,\mu}g(z))'}{I_{n,\mu}g(z)}=-Q(z)=-\alpha-(1-\alpha)q(z),\quad q(z)\in P. \tag{3.8}$$

Consider

$$\frac{z(I_{n,\mu+1}f(z))'}{I_{n,\mu+1}g(z)}=\frac{I_{n,\mu+1}(zf'(z))}{I_{n,\mu+1}g(z)}. \tag{3.9}$$

From (1.8) and (3.9), we have

$$\frac{z(I_{n,\mu+1}f(z))'}{I_{n,\mu+1}g(z)}=\frac{\frac{z(I_{n,\mu}(zf'(z))'}{I_{n,\mu}g(z)}+(\mu+1)\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}g(z)}}{\frac{z(I_{n,\mu}g(z))'}{I_{n,\mu}g(z)}+(\mu+1)}. \tag{3.10}$$

By using (3.6) and (3.8) and after some simplification, we have

$$\frac{z(I_{n,\mu}(zf'(z))'}{I_{n,\mu}g(z)}=H(z)Q(z)-zH(z). \tag{3.11}$$

From (3.7)–(3.11), and after some simplification, we obtain

$$-\frac{z(I_{n,\mu+1}f(z))'}{I_{n,\mu+1}g(z)}=H(z)+\frac{zH'(z)}{-Q(z)+(\mu+1)}\in P_k(\beta). \tag{3.12}$$

Let

$$H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\{\gamma+(1-\gamma)h_1(z)\}-\left(\frac{k}{4}-\frac{1}{2}\right)\{\gamma+(1-\gamma)h_2(z)\}.$$

Then (3.12) becomes

$$-\frac{z(I_{n,\mu+1}f(z))'}{I_{n,\mu+1}g(z)}=\left(\frac{k}{4}+\frac{1}{2}\right)\left\{\gamma+(1-\gamma)h_1(z)+\frac{(1-\gamma)zh_1'(z)}{-Q(z)+(\mu+1)}\right\}-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\gamma+(1-\gamma)h_2(z)+\frac{(1-\gamma)zh_2'(z)}{-Q(z)+(\mu+1)}\right\},$$

where

$$\left\{\gamma+(1-\gamma)h_i(z)+\frac{(1-\gamma)zh_i'(z)}{-Q(z)+(\mu+1)}\right\}\in P(\beta),\quad \text{for } i=1,2,\quad z\in E.$$

that is

$$\left\{(\gamma-\beta)+(1-\gamma)h_i(z)+\frac{(1-\gamma)zh_i'(z)}{-Q(z)+(\mu+1)}\right\}\in P(\beta),\quad \text{for } i=1,2,\quad z\in E.$$

We formulate a functional $\Psi(u,v)$ by choosing $u = h_i(z) = u_1 + iu_2$ and $v = zh_i'(z) = v_1 + iv_2$

$$\Psi(u,v)=(\gamma-\beta)+(1-\gamma)u+\frac{(1-\gamma)v}{-(q_1+iq_2)+(\mu+1)}.$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the third condition with $v_1\leqslant -\frac{1}{2}(1+u_2^2)$ as follows:

$$\begin{aligned}\operatorname{Re}\Psi(iu_2,v_1)&=(\gamma-\beta)+\operatorname{Re}\left[\frac{(1-\gamma)v_1\{(-q_1+\mu+1)+iq_2\}}{(-q_1+\mu+1)^2+q_2^2}\right]\\&\leqslant\frac{2(\gamma-\beta)\left\{(-q_1+\mu+1)^2+q_2^2\right\}-(1-\gamma)(-q_1+\mu+1)(1+u_2^2)}{2\left\{(-q_1+\mu+1)^2+q_2^2\right\}}=\frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A&=2(\gamma-\beta)\{(-q_1+\mu+1)^2+q_2^2\}-(1-\gamma)(-q_1+\mu+1),\\B&=-(1-\gamma)(-q_1+\mu+1),\\C&=(-q_1+\mu+1)^2+q_2^2>0.\end{aligned}$$

Clearly, $\operatorname{Re}\Psi(iu_2,v_1)\leqslant 0$ if and only if $A\leqslant 0$ and $B\leqslant 0$. From $A\leqslant 0$, we obtain

$$\gamma=\frac{2\beta\left\{(-q_1+\mu+1)^2+q_2^2\right\}+(-q_1+\mu+1)}{2\left\{(-q_1+\mu+1)^2+q_2^2\right\}+(-q_1+\mu+1)}. \tag{3.13}$$

Hence it follows from Lemma 2.1 that $h_i\in P$ and consequently $H(z)\in P_k(\gamma)$, where γ is given by (3.13). This completes the proof. □

Next, we show that all the classes defined above are closed under Bernardi's operator.

Theorem 3.4. *For $n > -1$, $\mu > 0$, $0\leqslant\gamma < 1$, $z\in D$, $c > 0$, if $f(z)\in R_k(n,\mu,\gamma)$ then*

$$J_cf(z)\in R_k(n,\mu,\gamma).$$

Proof. We set

$$\frac{z(I_{n,\mu}J_cf(z))'}{I_{n,\mu}J_cf(z)}=-H(z)=-\gamma-(1-\gamma)p(z). \tag{3.14}$$

Using (1.9) and (3.14) and by logarithmic differentiation we have

$$-\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}f(z)}=H(z)+\frac{zH'(z)}{-H(z)+c+1}.$$

Now working in same way as in Theorem 3.1, we obtain the required result. □

Theorem 3.5. *For $n > -1$, $\mu > 0$, $0\leqslant\gamma < 1$, $z\in D$, $c > 0$, if $f(z)\in V_k(n,\mu,\gamma)$ then*

$$J_cf(z)\in MV_k(n,\mu,\gamma).$$

Proof. The proof follows immediately by using Remark 1.1 and Theorem 3.4. □

Theorem 3.6. *For $n > -1$, $\mu > 0$, $0\leqslant\alpha$, $\beta < 1$, $z\in D$, $c > 0$, if $f(z)\in T_k^*(n,\mu,\alpha,\beta)$ then*

$$J_cf(z)\in T_k^*(n,\mu,\alpha,\beta).$$

Proof. We set

$$\frac{z(I_{n,\mu}J_cf(z))'}{I_{n,\mu}J_cf(z)}=-H(z)=-\beta-(1-\beta)p(z), \tag{3.15}$$

where $g(z) \in R_2(n,\mu,\gamma)$, so $J_c g(z) \in R_2(n,\mu,\gamma)$. Then also

$$-\frac{z(I_{n,\mu}J_c g(z))'}{I_{n,\mu}J_c g(z)} \in P_2(\beta) = P(\beta). \tag{3.16}$$

Let

$$-\frac{z(I_{n,\mu}J_c g(z))'}{I_{n,\mu}J_c g(z)} = Q(z). \tag{3.17}$$

Now

$$\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}g(z)} = \frac{I_{n,\mu}(zf'(z))}{I_{n,\mu}g(z)}. \tag{3.18}$$

From (1.11) and (3.18), we have

$$\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}g(z)} = \frac{\frac{z(z(I_{n,\mu}J_c f'(z))')'}{I_{n,\mu}J_c g(z)} + (c+1)\frac{z(I_{n,\mu}J_c f(z))'}{I_{n,\mu}J_c g(z)}}{\frac{z(I_{n,\mu}J_c g(z))'}{I_{n,\mu}J_c g(z)} + (c+1)} \tag{3.19}$$

Now, working in similar way as in Theorem 3.3, we obtain the required result. □

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