Functional Analysis

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This is a preliminary guide for an introductory functional analysis course from Part 1 of Robinson's book on Infinite-dimensional dynamical systems. It contains additional definitions, exercises and corrections. This guide will be adapted/modified during the course Functional Analysis in Blok 5.

Functional analysis is a toolkit for solving equations in which the unknowns are functions rather than numbers. For instance, we may want to find a function $f = f(x)$ such that, for every $x \in [0, 1]$,

$$f(x) - \int_0^x \sin(x-t) f(t) dt = \cos(x).$$

This is an example of a linear integral equation. The left hand side defines a function of a function, which we refer to as a (linear) functional acting on the variable function f. In the nonlinear integral equation

$$f(x) - \int_0^x \sin(x-t) f(t)^2 dt = \cos(x),$$

the left hand side defines a nonlinear functional.

Most of the equations we solved in analysis and linear algebra required finding a solution as a number or a finite set of numbers, which, substituted in some given function, would make it zero, or would maximise or minimise it. Consequently we learned in linear algebra and analysis all sorts of things about linear and nonlinear functions defined on subsets of $\mathbb{R}^m$ and $\mathbb{C}^m$, finite dimensional vector spaces over the real or complex numbers, equipped with a natural (inner product) norm. Our lifes were made easy by the fact that bounded closed sets in $\mathbb{R}^m$ and $\mathbb{C}^m$ are compact so that bounded sequences have convergent subsequences. Another fact taking completely for granted was the continuity of linear functions.

In the infinite-dimensional setting needed to solve problems such as the integral equations above, we first need good normed vector spaces in which our solutions are to be found. We will see many different possibilities to assign a norm to a function, leading to different spaces. There are many candidates for $\mathbb{R}^\infty$ so to speak. This will make the theory of even only linear functionals a subtle issue in which linear algebra and analysis (epsilons and delta's) merge.

1 Banach spaces

We begin with the concept of a (real) Banach space and related concepts in Section 1.1. in the book. Unless stated otherwise all references are to the book and not to this guide.

Definition 1 *Let X be a normed vector space. A sequence x_n in X, indexed by $n \in \mathbb{N}$, is called*

convergent in X if $\exists\, \bar{x} \in X \quad \forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n \geq N \quad ||x_n - x|| \leq \epsilon$;

Cauchy if $\forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m, n \geq N \quad ||x_n - x_m|| \leq \epsilon$.

If all Cauchy sequences in X are convergent, then X is called a complete normed space or a Banach space. A set $\mathcal{O} \subset X$ is called open if

$$\forall\, \bar{x} \in \mathcal{O} \quad \exists\, \epsilon > 0 \quad B(\bar{x}, \epsilon) = \{x \in X : ||x - \bar{x}|| < \epsilon\} \subset \mathcal{O}$$

A set $\mathcal{G}$ is called closed if $\mathcal{G}^c = \{x \in X : x \notin \mathcal{G}\}$ is open.

The open sets thus defined form a topology: the empty set $\emptyset$ is open, X is open, arbitrary unions of open sets are open, and finite intersections of open sets are open. Equivalent norms (1.3) give the same convergent and Cauchy sequences, and the same open sets.

If a normed space X is not complete, it can be made complete by "adding" to X all limits of Cauchy sequences which are not convergent in X. We will not need this construction. If X is a subspace of some larger Banach space Y then X is complete if and only if X is closed in Y.

Exercise 1 Let Y be a Banach space and let $X \subset Y$ be a linear subspace. Prove that X is closed if and only if X is Banach.

1.1 Finite-dimensional spaces

The first example of a Banach space is $\mathbb{R}^m$, discussed in Section 1.2, which has finite dimension m. Note that the dimension of a vector space X is the supremum (possibly $+\infty$) of all n for which there exist n linearly independent $x_1, \ldots, x_n \in X$, i.e.

$$\lambda_1 x_1 + \cdots \lambda_n x_n = 0 \quad \Rightarrow \quad \lambda_1 = \cdots = \lambda_n = 0$$

There are many norms on $\mathbb{R}^m$, but they are all equivalent (Theorem 1.1). This follows from the fact that in $\mathbb{R}^m$ bounded closed sets are compact (equivalent: bounded sequences have convergent subsequences). Theorem 1.1 holds for every finite-dimensional real vector space, as the following exercise shows.

Exercise 2 (i) Show that in every normed space the function $x \to ||x||$ is continuous. (ii) Assume that $x_1, \ldots, x_n \in X$ are linearly independent. Define the map $L : \mathbb{R}^n \to X$ by $L(\xi) = L(\xi_1, \ldots, \xi_n) = ||\xi_1 x_1 + \cdots \xi_n x_n||$. Show that L is continuous and that there exists $0 < m \leq M < \infty$ such that $m \leq L(\xi) \leq M$

for all $\xi = (\xi_1, \dots, \xi_n) \in I\!R^n$ with $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2 = 1$. (iii) Show by scaling that $m|\xi| \le L(\xi) \le M|\xi|$ for all $\xi = (\xi_1, \dots, \xi_n) \in I\!R^n$. (iv) Show that on a finite-dimensional vector space all norms are equivalent. (v) Show that every finite-dimensional normed space is complete.

Exercise 3 For a sequence x_n in a Banach space X let $s_n = x_1 + \cdots + x_n$. Show that the sequence s_n is convergent if

$$\sum_{N=1}^{\infty} ||x_n|| < \infty$$

In view of Exercise 2 above every finite-dimensional normed space X has the Heine-Borel property: closed bounded subsets are compact. The Heine-Borel property characterises finite-dimensional spaces. In any infinite-dimensional normed space it is possible, given a $0 < \delta < 1$, by using what is known as Riesz' lemma, to find a sequence x_n with $||x_n|| = 1$ and $||x_n - x_m|| \ge \delta$ if $m \ne n$. Such a sequence is bounded, but cannot have a convergent subsequence.

1.2 Spaces of continuous functions

Functional analysis is (linear and nonlinear) analysis in infinite-dimensional complete normed spaces (Banach spaces). Complete because we can hardly prove anything if Cauchy sequences are not convergent. Functions on such spaces are often called functionals, so as to distinguish them from the familiar functions defined on subsets of $I\!R^m$. Many Banach spaces consists of such ordinary functions, for instance the space $C([0,T])$ consisting of all real valued continuous functions on a closed bounded interval $[0,T]$. Denoted as $C^0([0,T])$ this is a special case of the function spaces discussed in Section 1.3, and also of the spaces $C([0,T], I\!R^n)$ consisting of all continuous functions $x : [0,T] \to I\!R^n$ equipped with the supremum (or maximum) norm

$$||x||_\infty = \sup_{0 \le t \le T} |x(t)| = \max_{0 \le t \le T} |x(t)|$$

Here $|x(t)|$ is the Euclidian norm of $|x(t)|$ in $I\!R^n$. These spaces are used in Chapter 2 for the construction of solutions of (systems of) ordinary differential equations,

Exercise 4 Prove that $C([0,T]) = C([0,T], I\!R)$ equipped with the maximum norm is a Banach space. Construct a bounded sequence which does not have a convergent subsequence. Show also that

$$||x||_2 = \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}}$$

defines a norm on $C([0,T])$, but that with this norm the space is not complete.

$C([0,T])$ is of course infinite-dimensional, but not too infinite-dimensional. It is a separable normed space: there exists a sequence x_n in $X = C([0,T])$ such that every element in X is the limit of some subsequence of x_n (equivalent: there exists a countable dense subset). Non-separable spaces are to large for man to handle and should be avoided.

Exercise 5 Let $x \in C([0,T])$. Use the uniform continuity of $x(t)$ on $[0,T]$ to show that the (continuous) piecewise linear functions are dense in $C([0,T])$ and prove that $C([0,T])$ is separable.

Exercise 6 Do Exercise 1.1 and use it to show that $C([0,T], I\!R^n)$, being the product of n copies of $C([0,T])$, is separable. Show that the product norm is equivalent to the maximum norm defined above on $C([0,T], I\!R^n)$.

Exercise 7 Do Exercise 1.3. Use $C(\bar{\Omega}) = C^0(\bar{\Omega})$ as the larger Banach space in which $C_c^0(\Omega)$ is contained.

Next we jump to Chapter 2, existence and uniqueness results for (systems of) differential equations

$$\frac{dx}{dt} = f(x)$$

using the Banach's fixed point theorem in $C([0,T])$ (systems: $C([0,T], I\!R^n)$). We skip Sections 2.2, 2.4, 2.5 which further develop the theory of differential equations (ODE's), but Section 2.3 is important, also because of the Arzelà-Ascoli theorem.

After the ODE theory we return to Chapter 1. We mention the spaces $C^r(\bar{\Omega})$ with $r \in I\!N$ and $C^{r,\gamma}(\bar{\Omega})$, but we skip the mollification theme for now.

Exercise 8 Let $0 < \gamma \leq 1$. Do Exercise 1.5 with $r = 0$ and $\bar{\Omega} = [0,T]$. Define the function x_a by $x_a(t) = |t-a|^\gamma$. What can you say about the distance between x_a and x_b in $C^{0,\gamma}([0,T])$ if $a, b \in [0,T]$? Conclude that $C^{0,\gamma}([0,T])$ is not separable.

Exercise 9 Show that $C^{0,\gamma}([0,T])$ is a space of dimension 1 if $\gamma > 0$.

1.3 Lebesgue spaces

The next important class of function spaces are the Lebesque spaces. Roughly speaking $L^p(\Omega)$, where $1 \leq p < \infty$, is the space of all functions $u : \Omega \to I\!R$ for which the p-norm

$$||u||_p = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}$$

is well-defined. We stress that the estimates in Section 1.4 are in some sense more essential than the abstract nonsense. This view is shared by the author of the book, who clearly does not like measure theory and has come up with a quick way to introduce the Lebesgue spaces, without refering to the concept of a

Lebesgue measurable set. Unfortunately his approach is fundamentally wrong. There is no way in which the characteristic function of a Lebesgue measurable set is included as an element of L^1 as he defines it.

Indeed, the Lebesgue spaces may be introduced starting from (1.10) and (1.11), however not with blocks I_j, but with Lebesgue measurable sets E_j. The Lebesgue measure extends the definition of

$$\mu(I) = (b_1 - a_1)\cdots(b_n - a_n) \quad \text{for blocks} \quad I = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

to a collection Λ_m of so-called Lebesgue measurable subsets E of $I\!R^m$. We briefly recall the essential step. The starting point is that for measurable subsets $F \subset I\!R^m$ it should certainly be true that

$$F \subset \cup_{n=1}^{\infty} I_n \quad \Rightarrow \quad \mu(F) \leq \sum_{n=1}^{\infty} \mu(I_n),$$

so that an obvious definition of $\mu(F)$ would be

$$\mu(F) = \inf_{F \subset \cup_{n=1}^{\infty} I_n} \sum_{n=1}^{\infty} \mu(I_n) \in [0, \infty],$$

being the best we can do using our blocks. This definition makes sense for every subset $F \subset I\!R^m$, and obviously one has

$$\forall\, E, F \subset I\!R^m \quad \mu(F) \leq \mu(F \cap E) + \mu(F \cap E^c),$$

but it is impossible to prove that this statement remains true when $\leq$ is replaced by $=$, as one might have expected.

The Lebesgue measurable subsets $E \subset I\!R^m$ are precisely those subsets E with

$$\forall\, F \subset I\!R^m \quad \mu(F) = \mu(F \cap E) + \mu(F \cap E^c).$$

(cutting a loaf of bread F with E does not miraculously increase the amount of bread....) In fact it is sufficient to check that equality holds for all blocks F. The collection Λ_m and the map $\mu : \Lambda_m \to [0, \infty]$ allow all countable operations one could reasonably expect to be allowed, see Klaas van Harn's thorough exposition on the subject.

For the construction of nonmeasurable subsets one needs Zorn's lemma ($\Leftrightarrow$ Axiom of choice, uncountable version), which goes beyond the ability of mortal man. Thus for all practical purposes, all subsets of $I\!R^m$ we encounter in daily life of applied math are measurable, at least as long as we resist the temptation of Zorn.

Note that the author's approach is essentially based on the obvious but erroneous try to define measurable sets as being those sets for which $\nu(E) = \mu(E)$, where

$$\nu(F) = \sup_{\cup_{n=1}^{\infty} I_n \subset F} \sum_{n=1}^{\infty} \mu(I_n) \in [0, \infty].$$

But this would exclude all sets with empty interior such as for instance Cantor type sets, and certainly at the VU we would not want that at all.

With the above correction everything seems OK until Section 1.4.3, where the definition of $L^\infty(\Omega)$ makes no sense without the assumption that its elements f have the property that the inverse image of every measurable set is measurable.

Note that if a measurable set E with finite measure is contained in a countable union of blocks I_n such that

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(I_n) < \mu(E) + \epsilon,$$

then

$$\chi_{_E} \leq \sum_{n=1}^{\infty} \chi_{_{I_n}} \quad \text{and} \quad \int \chi_{_E} \leq \int \sum_{n=1}^{\infty} \chi_{_{I_n}} < \int \chi_{_E} + \epsilon,$$

so a characteristic function may be approximated in the L^1-norm by step functions on blocks. Since each $\chi_{_{I_n}}$ may be approximated in L^1-norm by a compactly supported smooth function, we can choose compactly supported smooth functions ψ_n such that $||\psi_n - \chi_{_{I_n}}|| < \epsilon 2^{-n}$, thereby approximating each finite sum $\sum_{n=1}^{N} \chi_{_{I_n}}$ ϵ-close with $\sum_{n=1}^{N} \psi_n$.

This shows that $\sum_{n=1}^{N} \psi_n$ is 2ϵ-close in L^1-norm to $\chi_{_E}$. With a similar argument it then follows that each $f \in L^1(\mathbb{R}^N)$ may be approximated in L^1-norm with a compactly supported smooth function. In other words, the compactly supported smooth functions are dense in $L^1(\mathbb{R}^N)$. The same statement holds in $L^p(\Omega)$, where $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ is open.

We skip Section 1.4.4, but the sequence spaces in Section 1.4.5 are very instructive.

Exercise 10 Do Exercise 1.8, 1.9 and 1.10, the latter for characteristic functions $\chi_{_E}$ of measurable subsets with finite measure.

2 Hilbert space theory

In Section 1.5 we look at Banach spaces in which the norm comes from an inner product. Such spaces are called Hilbert spaces. A fundamental theorem for Hilbert spaces is:

Theorem 1 *Let H be a Hilbert space and $K \subset H$ a closed convex subset. For every point $x \in H$ there exists a unique point $u \in K$ which is closer to x than any other point of K. The point $u \in K$ is called the projection of x on K, denoted by $u = P_K x$. An important special case is that of K being a linear subspace of H.*

Exercise 11 Prove that $||P_K x_1 - P_K x_2|| \leq ||x_1 - x_2||$ for all $x_1, x_2 \in H$. Note that in general P_K is not a linear map.

Finite-dimensional Hilbert spaces can, as far as their Hilbert space structure is concerned, be identified with $I\!R^m$, whereas every separable infinite-dimensional Hilbert space can be identified with the space l^2 defined in Section 1.4.5. It is easy to see that the unit ball in l^p is not compact, because the unit basis vectors form a sequence which is bounded, while all mutual distances equal $2^{\frac{1}{p}}$.

There is another remarkable difference between l^2 and $I\!R^m$, as the following exercise shows.

Exercise 12 Show there exists a continuous map from the closed unit ball in l^2 to the closed unit sphere which leaves the sphere pointwise invariant (goodbye Brouwer).

2.1 Compact linear maps

We then turn our attention to continuous linear operators (operator is just another word for map) with a quick pick from Chapter 3. For the moment we only need the following theorem and the notion of compactness for an operator.

Theorem 2 *Let X and Y be normed spaces. For a linear map $A : X \to Y$ continuity in any point is equivalent to*

$$||A||_{op} = \sup\{||Ax||_Y : ||x||_X \leq 1\} < \infty.$$

This property of a linear map A is often called boundedness of A: bounded on the unit ball (on all balls in fact), but not on X of course. Note that then also $||Ax_1 - Ax_2||_Y \leq ||A||_{op}\, ||x_1 - x_2||_X$ for all $x_1, x_2 \in X$.

A special case is $Y = I\!R$. The Riesz representation theorem states that the continous linear real valued functions on a Hilbert space are precisely the functions $x \to (x, y)$. Thus the closed hyperplanes in H are the sets of the form

$$\{x \in H : (x, y) = c\} \qquad y \in H, \quad c \in I\!R.$$

The proof relies on the projection theorem above.

We read in Section 3.4 that $A : X \to Y$ is called compact if the image of any bounded sequence in X always has a convergent subsequence in Y.

Exercise 13 Show that A compact and linear implies A continuous. To avoid confusion assume that Y is Banach.

Sections 3.5-6 contain the strikingly easily proved Hilbert-Schmidt theorem which states that in a Hilbert space H every compact symmetric (i.e. $(Ax, y) = (x, Ay)$ for all $x, y \in H$) operator $A : H \to H$ comes with a basis of eigenvectors corresponding to a sequence of real eigenvalues which converge to zero. This statement generalises the corresponding theorem for symmetric matrices.

Many linear operators in ODE and PDE applications are not continuous from a normed space X into itself. Such operators are called unbounded. They are discussed in the rest of Chapter 3 in the Hilbert case context under the assumption that $(Ax, y) = (x, Ay)$ whenever both sides of the equality are defined.

2.2 Exercises

Let H and V be Hilbert spaces such that $V \subset H$. The inner product on H is denoted by single brackets, the inner product on V by double brackets. We shall write

$$(u,u) = |u|^2 \quad \text{for} \quad u \in H \qquad \text{and} \qquad ((u,u)) = ||u||^2 \quad \text{for} \quad u \in V$$

Throughout this subsection we assume that V is dense in H, and that V is compactly embedded in H, meaning that the inclusion map $i : V \to H$ defined by $i(x) = x$ is compact (and thus also bounded).

(i) Let $f \in H$. Prove that there exists a unique $u \in V$ such that $((u,v)) = (f,v)$ for all $v \in V$. Denote $u = Af$.

(ii) Prove that $A : H \to V$ is injective.

(iii) Prove that $A : H \to H$ is linear, symmetric and compact.

(iv) Prove that $A : V \to V$ is linear, symmetric and compact.

(v) Prove that $A : H \to H$ is positive, i.e. $(Af, f) > 0$ if $f \neq 0$.

(vi) Prove that $A : V \to V$ is positive, i.e. $((Af, f)) > 0$ if $f \neq 0$.

(vii) Prove that H has an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of eigenvectors of A corresponding to positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, with $\lambda_n \to 0$ as $n \to \infty$, where

$$\lambda_1 = \max_{f \in H} \frac{(Af, f)}{(f, f)}$$

and, more generally, for $n > 1$,

$$\lambda_n = \max_{f \in H, (f,\phi_1) = \cdots = (f, \phi_{n-1}) = 0} \frac{(Af, f)}{(f, f)}$$

(viii) Prove that V also has an orthonormal basis $\{\psi_1, \psi_2, \ldots\}$ of eigenvectors of A, which are multiples of $\{\phi_1, \phi_2, \ldots\}$. Determine these multiples. What are the corresponding eigenvalue formula's for $A : V \to V$? Evaluate these formula's in terms of norms only, i.e. without A appearing in the formula's. Hint: use the definition of A.

Let

$$H = L^2(0,1) = \{f : (0,1) \to I\!R \,|\, f \text{ is measurable}, \int_0^1 f^2 < \infty\},$$

equipped with the inner product $(f, g) = \int_0^1 fg$. We use the convention that $f = g$ in $L^2(0,1)$ means that the set $\{x \in (0,1) \,|\, f(x) \neq g(x)\}$ has zero measure. We say that $g \in L^1_{loc}(0,1)$ is a weak derivative of f if

$$\int_0^1 gv = -\int_0^1 fv'$$

for all $v \in C^1([0,1])$ with $v(0) = v(1) = 0$. One can show that g is unique if it exists, and that $f(x) - f(y) = \int_y^x g$ for all $0 < y < x < 1$. We write $g = f'$. On

$$V = \{f \in C([0,1]) \,|\, f(0) = f(1) = 0,\ f' \text{ exists},\ f' \in L^2(0,1)\}$$

we take the inner product $((f,g)) = \int_0^1 f'g'$.

(ix) Let $f \in C([0,1])$ and suppose that we look for $u \in C^2([0,1])$ with $u'' = f$ and $u(0) = u(1) = 0$. This boundary value problem can be solved by means of direct integration and the appropriate choice of integration constants. Show that $u = Af$, with A as above. In other words, with the operator A we solve this boundary value problem. What are the eigenfunctions and eigenvalues of A?

(x) Now let

$$V = \{f \in C^1([0,1]) \,|\, f(0) = f'(0) = f(1) = f'(1) = 0,\ f'' \text{ exists},\ f'' \in L^2(0,1)\},$$

with inner product $((f,g)) = \int_0^1 f''g''$. Define A as before. Which boundary value problem does this A solve? Can you compute the eigenvalues and eigenfunctions?

3 The dual space and finite rank operators

We discuss Section 4.1 and then continue with a more thorough discussion of a special class of operators mentioned in Chapter 3, namely those with finite-dimensional range, the so-called finite rank operators. Below we concentrate on some of the more linear algebra aspects of functional analysis, in relation to what we (should) know about matrices. We want to solve equations of the form $(I - K)x = y$ in a Banach space X, where I the identity map and K is a finite rank operator.

The kernel of an arbitrary linear operator $A : X \to Y$ is denoted by

$$N(A) = \{x \in X : Ax = 0\}$$

(rather than by $\mathrm{Ker}(A)$), and the range of A is

$$R(A) = \{Ax : x \in X\}.$$

Exercise 14 Show that any finite rank operator $K : X \to X$ can be written as

$$Kx = \sum_{j=1}^{n} f_j(x) x_j,$$

with $x_j \in X$ and $f_j \in X^*$ $(j = 1, \ldots, n)$. Moreover, show that we can always take both the $x_j \in X$ and the $f_j \in X^*$ linearly independent. If you get stuck, go on with the exercises below, and assume that K is already in this form.

Exercise 15 To solve $x - Kx = y$ for given $y \in X$ it is of course sufficient to find Kx, which will be completely determined by the unknowns $\xi_1 = f_1(x), \ldots, \xi_n = f_n(x)$. Derive a system of linear equations for $\xi_1, \ldots, \xi_n$ by applying $f_1, \ldots, f_n$ to

$$x = \sum_{j=1}^{n} \xi_j x_j + y.$$

Write this system in the form $A\xi = b$ for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, where $b \in \mathbb{R}^n$ is expressed in y, and A is an n by n matrix. Show that x defined by the above equality is a solution of $x - Kx = y$ if and only if ξ is a solution of the (matrix) equation $A\xi = b$. Considering the case $y = 0$: show that there is a one-to-one correspondence between $N(A)$ and $N(I - K)$. What is the relation between $R(A)$ and $R(I - K)$?

Definition 2 *Let X and Y be Banach spaces spaces and let $A : X \to Y$ be linear and bounded. For a functional $f \in Y^*$ the functional $A^*f \in X^*$ is by definition the functional $x \to f(Ax)$. So $A^*f = f \circ A$ is just the composition of f and A. This defines a linear operator $A^* : Y^* \to X^*$, the adjoint of A.*

Exercise 16 In the case that $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, the dual spaces of X and Y may be identified with respectively X and Y, in view of the Riesz representation theorem. If $A : \mathbb{R}^m \to \mathbb{R}^n$ is given in matrix form, what is the matrix form of A^*?

Exercise 17 Let $K^* : X^* \to X^*$ be the adjoint of $K : X \to X$ above. Show directly from the definition that

$$K^*f = \sum_{j=1}^{n} f(x_j) f_j.$$

Show that every solution of the equation $f - K^*f = g$, where $g \in X^*$, is of the form

$$f = \sum_{j=1}^{n} \eta_j f_j + g.$$

Writing $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$ and $c = (g(x_1), \ldots, g(x_n))$, show that f is a solution of $f - K^*f = g$ if and only if η is a solution of $A^*\eta = c$, where A^* is the transposed matrix of A. What is the relation between $N(A^*)$ and $N((I - K)^*)$?

Definition 3 *Let X be a normed space and $M \subset X$. We define*

$$M^\circ = \{f \in X^* : f(x) = 0 \quad \forall\, x \in M\}.$$

For $Q \subset X^*$ we define

$$^\circ Q = \{x \in X : f(x) = 0 \quad \forall\, f \in Q\}.$$

Exercise 18 In the case that $X = H$ is a Hilbert space, the dual space X^* may be identified with H, in view of the Riesz representation theorem. Rewrite both M° and ${}^\circ Q$ using the definition

$$M^\perp = \{x \in H : x \cdot y = 0 \quad \forall\, y \in M\}$$

Show that $\{M^\perp\}^\perp = M$ if $M \subset H$ is a closed subspace.

From your knowledge of matrices you should know how $R(A)$, $N(A^*)$, $R(A^*)$, $N(A)$ are related in terms of this definition in the case that $A : I\!R^m \to I\!R^n$. Rederive these relations.

Exercise 19 Derive similar relations for $I - K$ and show that $N(I - K)$ and $N((I - K)^*)$ have the same finite dimension.