How analysis and topology interact in bivariant K-theory

Ralf Meyer

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The goal

- We will review some basic properties of bivariant K-theory for C^*-algebras, focussing on its universal property and universal coefficient theorems.
- We will see a general scheme for the study of *Assembly Maps* and *Universal Coefficient Theorems* in the context of C^*-algebra K-theory:
  - the original problem is replaced by a *localisation* that is purely topological;
  - the original problem and its localisation are related using analysis.

1 Triangulated categories of C^*-algebras

1.1 Kasparov Theory and E-theory

- We want to view Kasparov Theory as a C^*-algebraic analogue of the derived category of *motives*.
- A functor F from the category C^* of (separable) C^*-algebras to an additive category $\mathcal{A}$ is called

  stable if $F(A) \xrightarrow{\cong} F(\mathbb{K}(\ell^2\mathbb{N}) \otimes A)$;

  homotopy invariant if $F(A) \xrightarrow{\cong} F\big(C([0,1], A)\big)$;

  split-exact if it is exact on *split* extensions;

  exact if it is *half*-exact on all extensions.

Definition 1. Let $C^* \to \mathrm{KK}$ and $C^* \to \mathrm{E}$ be the *universal* functors that are stable and split-exact or exact.

Theorem 2 (Higson)**.** *stable & split-exact* $\Longrightarrow$ *homotopy invariant*

Question 3. Do KK and E *exist*? — Yes, obviously.Can we *describe* KK and E explicitly?

Some historic comments

Atiyah tried to construct K-*homology via elliptic differential operators.*

Brown-Douglas-Fillmore related K-*homology* to C^*-algebra extensions.

Kasparov *defined* KK *via generalised elliptic operators* and related it to extensions.

Cuntz described $\mathrm{KK}(A, B)$ as the set of homotopy classes of $*$-homomorphisms $qA \to B \otimes \mathbb{K}$ or $q(A \otimes \mathbb{K}) \to q(B \otimes \mathbb{K})$, where $qA = \ker\big(A \sqcup A \xrightarrow{(\mathrm{id},\mathrm{id})} A\big)$, and almost stated its universal property.

Higson stated the *universal property* of KK and defined E via its universal property.

Connes-Higson realised $\mathrm{E}(A, B)$ using *asymptotic morphisms*

$$A \otimes C_0(\mathbb{R}, \mathbb{K}) \to B \otimes C_0(\mathbb{R}, \mathbb{K}).$$

Equivariant generalisations

Let G be a locally compact group.

Definition 4. A functor F on G-C^*-algebras is called *stable* if

$$A \otimes \mathbb{K}(\mathcal{H}_1) \to A \otimes \mathbb{K}(\mathcal{H}_1 \oplus \mathcal{H}_2) \leftarrow A \otimes \mathbb{K}(\mathcal{H}_2)$$

are isomorphisms for all G-Hilbert spaces.

Definition 5. KK^G and E^G are the universal stable (split) exact functors on the category of G-C^*-algebras.

References

[1] R. Meyer. Equivariant Kasparov theory and generalized homomorphisms. K-theory 21, 2000.

Properties of KK and E

- *Bott periodicity*
- KK^G is exact for *certain* extensions.
- $\mathrm{KK}_*(\mathbb{C}, A) \cong \mathrm{K}_*(A)$ and $\mathrm{E}_*(\mathbb{C}, A) \cong \mathrm{K}_*(A)$.
- KK^G and E^G are *triangulated* categories, with triangles defined by *mapping cone* triangles.
- KK^G and E^G are *tensor* triangulated categories.
- The universal property yields a natural transformation $\mathrm{KK}^G_*(A, B) \to \mathrm{E}^G_*(A, B)$, which is an isomorphism if G is trivial and A is nuclear.

- *Universal Coefficient Theorem*: If A is KK-equivalent to a commutative C^*-algebra, then there is a natural exact sequence

$$\mathrm{Ext}\big(\mathrm{K}_*(A), \mathrm{K}_{1-*}(B)\big) \rightarrowtail \mathrm{KK}_*(A, B) \twoheadrightarrow \mathrm{Hom}\big(\mathrm{K}_*(A), \mathrm{K}_*(B)\big).$$

- Is there a nuclear C^*-algebra for which this fails?

Commutative versus non-commutative topology

- Separable *commutative* C^*-algebras are equivalent to pointed *compact* metrisable spaces.
- The thick subcategory of KK where the UCT holds is equivalent to a full subcategory of a similar localisation of the *stable homotopy category.*
- Its right-orthogonal complement is the thick subcategory $\mathcal{N}$ of C^*-algebras with $\mathrm{K}_*(A) = 0$.
- This subcategory is *not* tractable by topological methods.

  Question 6. Does $\mathcal{N}$ have any non-zero compact objects?

  What are the thick subcategories of $\mathcal{N}$?

What is the analogy to motives?

- KK^G and E^G are *universal* homology theories (for suitable notions of homology theory).
- We need and have a more *concrete description.*
- They form *triangulated tensor categories.*
- *Correspondences* from X to Y generate $\mathrm{KK}_*\big(C_0(X), C_0(Y)\big)$ (Connes-Skandalis).
  - $Y \xleftarrow[\text{K-oriented}]{f} Z \xrightarrow[\text{proper}]{g} X$
  - $f_! \in \mathrm{KK}_*\big(C_0(Z), C_0(Y)\big) \quad g^* \in \mathrm{KK}_*\big(C_0(X), C_0(Z)\big)$
  - $f_! \circ g^* \in \mathrm{KK}_*\big(C_0(X), C_0(Y)\big)$

Other triangulated categories

- Relaxing our requirements for homology theories further, we can define other universal triangulated categories of C^*-algebras.
- *Thom*: add homotopy invariance, replace stability by *matrix-stability*, get *connective* version of E-theory which is still functorial for *finite* correspondences.
- Other *alternative*: require stability for compact operators and suspensions, homotopy invariance, and Puppe exact sequences

2 Assembly maps with spaces and C^*-algebras

2.1 Classifying spaces and homotopy quotients

Idea: replace a badly behaved groupoid by a homotopy equivalent one with better properties (free, proper)

EG	universal free proper G-space	BG	$G\backslash EG$
$\mathcal{E}\mathcal{G}$	universal proper G-space	$\mathcal{B}\mathcal{G}$	$G\backslash\mathcal{E}\mathcal{G}$

Definition 7. Homotopy quotient: $G\backslash(X \times EG)$ Alternative: $G\backslash(X \times \mathcal{E}G)$

Example 8 (Homeomorphism $f\colon X \to X$).

- take $E\mathbb{Z} = \mathcal{E}\mathbb{Z} = \mathbb{R}$
- homotopy quotient = *mapping torus* $X \times [0,1] \,/\, (0,x) \sim \big(1, f(x)\big)$

Homotopy quotients and localisation

- $EG \to \star$ is *non-equivariant* homotopy equivalence
- $X_1 \xrightarrow{f} X_2$ *non-equivariant* homotopy equivalence $\iff$ $X_1 \times EG \xrightarrow{f_*} X_2 \times EG$ *G-equivariant* homotopy equivalence
- Passage to $X \times EG$ *localises* at non-equivariant homotopy equivalences.
- $\mathcal{E}G \to \star$ is *H-equivariant* homotopy equivalence $\forall\, H \subseteq G$ compact
- $X_1 \xrightarrow{f} X_2$ *H-equivariant* homotopy equivalence $\forall\, H \subseteq G$ compact $\iff$ $X_1 \times \mathcal{E}G \xrightarrow{f_*} X_2 \times \mathcal{E}G$ *G-equivariant* homotopy equivalence
- Passage to $X \times \mathcal{E}G$ *localises* at equivariant homotopy equivalences with respect to compact subgroups.

Range of the localisation

- $X \times EG \to X$ is a G-homotopy equivalence $\iff$ X is free and proper G-space
- $X \times \mathcal{E}G \to X$ is a G-homotopy equivalence $\iff$ X is proper G-space
- Summing up, up to G-homotopy equivalence the functors $\_ \times EG$ and $\_ \times \mathcal{E}G$ retract the category of G-spaces onto the subcategory of (free and) proper G-spaces.

2.2 Transition to C^*-algebras

Transition to C^*-algebras

- we have no maps $C_0(X \times EG) \to C_0(X)$ or $C_0(X \times \mathcal{E}G) \to C_0(X)$ because $X \mapsto C_0(X)$ is only functorial for proper maps.
- If we use the pro-C^*-algebra $C(\mathcal{E}G) = \varprojlim C(K)$, where K runs through the compact subsets of $\mathcal{E}G$, we do get a map $C(X \times \mathcal{E}G) \to C(X)$, but there is no crossed product $G \ltimes C(X \times \mathcal{E}G)$.
- *Solution*: replace spaces by *duals* in KK^G:

  Definition 9. A G-equivariant *dual* for a G-space X is a G-C^*-algebra P_X for which there exists a natural isomorphism

  $$\mathrm{RKK}^G_*(X; A, B) \cong \mathrm{KK}^G_*(P_X \otimes A, B)$$

  compatible with tensor products.

What is RKK^G?

- Consider the *transformation groupoid* $G \ltimes X$.
- $G \ltimes X$-C^*-algebras are G-equivariant bundles of C^*-algebras over X.
- Can define $\mathrm{KK}^{G \ltimes X}$ and $\mathrm{E}^{G \ltimes X}$ by universal properties, have similar properties as KK and E.
- A C^*-algebra A yields a constant bundle $C_0(X, A)$ over X.
- $\mathrm{RKK}^G_*(X; A, B) = \mathrm{KK}^{G \ltimes X}\big(C_0(X, A), C_0(X, B)\big)$

Properties of duals

- If X is compact, then $\mathrm{RKK}^G_*(X; A, B) \cong \mathrm{KK}^G_*\big(A, C(X, B)\big)$.
- $\mathrm{RKK}_*(X; \mathbb{C}, \mathbb{C})$ is the representable K-theory of X.
- A Cantor set has *no* dual.
- If X is a smooth spin manifold with isometric action of G preserving the spin structure, then $C_0(X)$ is a G-equivariant dual for X.
- Any locally finite, countable, finite-dimensional *simplicial complex* with simplicial action of G has a G-equivariant dual.
- $X \mapsto \mathrm{RKK}^G_*(X; A, B)$ is a homotopy invariant contravariant functor.
- $X \mapsto P_X$ is covariant functor
- Get $\mathrm{RKK}^G_*(X; A, B) \cong \mathrm{KK}^G_*(P_X \otimes A, B)$ from

  $$D \in \mathrm{KK}^G_*(P_X, \mathbb{C}), \qquad \text{Dirac morphism}$$
  $$\Theta \in \mathrm{RKK}^G_*(X; \mathbb{C}, P_X), \quad \text{local dual Dirac morphism.}$$

Dual of $\mathcal{E}G$

- $\forall$ locally compact groups G, $\mathcal{E}G$ has a dual.
- $\mathcal{E}G \to \star$ induces $D \in \mathrm{KK}^G(P_{\mathcal{E}G}, \mathbb{C})$ (*Dirac morphism*)
- D becomes invertible in $\mathrm{RKK}^G_*(\mathcal{E}G; P_{\mathcal{E}G}, \mathbb{C})$ and $\mathrm{KK}^H_*(P_{\mathcal{E}G}, \mathbb{C})$ for compact subgroups $H \subseteq G$
- The inverse is the *local dual Dirac* $\Theta \in \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, P_X)$.
- $D \otimes \mathrm{id}_A \in \mathrm{KK}^G(P_{\mathcal{E}G} \otimes A, A)$ localises KK^G at the weak equivalences.
- $D \otimes \mathrm{id}_A$ invertible $\iff$ A KK^G-equivalent to proper G-C^*-algebra
- $\mathrm{K}_*\big(G \ltimes_{\mathrm{r}} (P_{\mathcal{E}G} \otimes A)\big) \to \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ is the *Baum-Connes assembly map* with coefficients A.

2.3 How coarse geometry comes into play

Global dual Dirac

Definition 10. (Global) *dual Dirac*: $\eta \in \mathrm{KK}^G(\mathbb{C}, P_{\mathcal{E}G})$ with $p^*_{\mathcal{E}G}(\eta) = \Theta$

- equivalent: $D \otimes \eta = 1_{P_{\mathcal{E}G}}$ in $\mathrm{KK}^G(P_{\mathcal{E}G}, P_{\mathcal{E}G})$
- η exists $\Longrightarrow$ the assembly map is *split injective* with section induced by η.
- The existence of η is a *geometric* property of G:

  Theorem 11 (Emerson and Meyer)**.** *Let G_1 and G_2 be torsion-free discrete groups with finite-dimensional BG. If G_1 and G_2 are coarsely equivalent and G_1 has a dual Dirac, so has G_2.*
- *Idea of proof*: existence of dual Dirac is equivalent to invertibility of
$$p^*_{\mathcal{E}G}\colon\ \mathrm{KK}^G\big(\mathbb{C}, C_0(G)\big) \to \mathrm{RKK}^G\big(\mathcal{E}G; \mathbb{C}, C_0(G)\big).$$
  This map only depends on the coarse space underlying G.

Summary

Our treatment of assembly maps for group actions fits into a general scheme:

- We want to compute some homology theory for C^*-algebras.
- First we *localise* the homology theory at a suitable class of weak equivalences.
- This should replace the problem by another one that is tractable by methods from algebraic topology.
- The comparison of the localised and the original problem will probably involve some analysis and special geometric properties of the setup.
- This is how Kasparov theory allows us to prove statements that cannot be proven purely topologically.