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Counterexample to C^1 boundary regularity of infinity harmonic functions[☆]

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ABSTRACT

In this short note, we construct a planar infinity harmonic function u in a C^1 domain Ω with smooth boundary data g; however Du is not continuous on the boundary. Changyou Wang and Yifeng Yu (2012) proved the C^1 boundary regularity of planar infinity harmonic functions provided that $\partial\Omega$ and g are C^2. They asked the question whether the result holds when $\partial\Omega$ and g are assumed to be C^1. Our counterexample answered this question negatively. Our construction of the counterexample is strongly inspired by the counterexample of Dongsheng Li and Lihe Wang (2006) for uniformly elliptic equations.

1. Introduction

Infinity harmonic functions are the viscosity solutions of the infinity Laplace equation

$$\triangle_\infty u := \sum_{i,j=1}^{n} u_i u_j u_{ij} = 0.$$

This highly degenerate nonlinear elliptic equation was introduced by G. Aronsson in 1960s [1] as the Euler equation of the absolutely minimizing Lipschitz extension variational problem. R. Jensen [2] proved the existence and uniqueness of viscosity solution to the Dirichlet problem:

$$\triangle_\infty u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{1}$$

with any bounded domain Ω and the continuous function g.

In 2001, Crandall, Evans and Gariepy [3] proved the extremely important characteristic *comparison with cones property*. That is a function $u \in C(\Omega)$ is an infinity harmonic function if and only if for any $V \subset\subset \Omega$ and $c(x) = a + b|x - x_0|$,

$$u(x) \le c(x) \text{ on } \partial\{V \setminus \{x_0\}\} \Rightarrow u(x) \le c(x) \text{ in } V,$$
$$u(x) \ge c(x) \text{ on } \partial\{V \setminus \{x_0\}\} \Rightarrow u(x) \ge c(x) \text{ in } V.$$

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From the comparison with the cone property, one can deduce [4] that at any interior point x_0 the blow-up limit $\lim_{k\to\infty} \frac{u(x_0+r_k x)-u(x_0)}{r_k}$ (if exists) is a linear function. But it is very hard to prove the uniqueness of the blow-up limits for different sequences $r_k \to 0$. This result was achieved recently by Evans and Smart [5] using deep pde techniques. So we know that the infinity harmonic functions are interior differentiable. However, the C^1 interior regularity is still a profound open problem. Earlier, Savin [6] proved the C^1 interior regularity in the 2 dimensional case. His argument heavily relied on the special topological property of the plane.

In 2012, Changyou Wang and Yifeng Yu published the first boundary regularity paper [7]. They proved two results: (1) if $\partial\Omega$ and g are C^1 then u is differentiable on the boundary; (2) for dimension $n = 2$, if $\partial\Omega$ and g are C^2 then $u \in C^1(\bar{\Omega})$. The first result was improved by the author [8] to that: if $\partial\Omega$ and g are differentiable at $x_0 \in \partial\Omega$ then u is differentiable at x_0. This is obviously optimal. For the second result, the C^2 assumption on the boundary condition is seemingly too strong for the C^1 conclusion. So they ask a question whether the result holds when $\partial\Omega$ and g are assumed to be C^1, a more natural assumption. Our counterexample in this note will show that this is not true.

Here we give a brief description of the counterexample. In the first step, we construct a convex planar domain Ω (the same as Counterexample 4.1 in [9]) with the properties that $0 \in \partial\Omega$ enjoys the interior ball condition and there is a sequence of boundary corner points $Z^k \to 0$. Let the nonnegative infinity harmonic function $u \equiv 0$ on $\partial\Omega \cap B_{\frac{1}{2}}$ and $u(0, \frac{1}{4}) = 1$. From the Hopf lemma (which still holds for the infinity harmonic function), we know $|Du(0)| > 0$. On the other hand, we can show that at boundary corner points $|Du(Z^k)| = 0$. So Du is not continuous at 0. However $\partial\Omega$ is not C^1 obviously. In the second step, we smooth the corners to modify Ω to a C^1 domain $\tilde{\Omega}$ and u to a new infinity harmonic function $\tilde{u}$. We can manage the modification to such that $|D\tilde{u}(0)| \geq \frac{1}{2}|Du(0)|$ and $|D\tilde{u}(\tilde{Z}^k)| \leq \frac{1}{4}|Du(0)|$ for all k, where $\tilde{Z}^k$ are boundary points of $\tilde{\Omega}$ corresponding to Z^k of $\partial\Omega$ and $\tilde{Z}^k \to 0$. The function $\tilde{u}$ serves as the counterexample.

2. Two boundary point lemmas

Lemma 1 (*Hopf Lemma*)**.** *Let $u \in C(\bar{\Omega})$ be a nonnegative infinity harmonic function on $\Omega \subset \mathbb{R}^n$. Assume that $B(x^0, r) \subset \Omega$, $x^1 \in \partial B(x^0, r) \cap \partial\Omega$ and $u(x^1) = 0$. Then $\liminf_{h\to 0} \frac{u(x^1+h(x^0-x^1))}{rh} \geq \frac{u(x^0)}{r}$.*

Proof. Let $c(x) := u(x^0) - \frac{u(x^0)}{r}|x - x^0|$, then $u(x) \geq c(x)$ on $\partial\{B(x^0, r) \setminus \{x^0\}\}$. So $u(x) \geq c(x)$ in $B(x^0, r)$. □

Lemma 2. *Let $u \in C(\bar{\Omega})$ be an infinity harmonic function on $\Omega \subset \mathbb{R}^2$. For $k > 0$, denote $\Gamma_k := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > k|x_1|\}$. Assume that for some $r_0 > 0$, $\Omega \cap B_{r_0} = \Gamma_k \cap B_{r_0}$ and $u(x) \equiv 0$ for $x \in \partial\Omega \cap B_{r_0}$. Then $u(x)$ is differentiable at 0 and $Du(0) = 0$.*

Proof. For $x \in \bar{\Omega} \cap B_{r_0}$, denote

$$S_r^+(x) = \sup_{y\in\partial(B_r(x)\cap\Omega)\setminus\{x\}} \frac{u(y) - u(x)}{|y - x|}$$

and

$$S_r^-(x) = \sup_{y\in\partial(B_r(x)\cap\Omega)\setminus\{x\}} \frac{u(x) - u(y)}{|y - x|}.$$

From the comparison with cone property, $S_r^+(x)$ and $S_r^-(x)$ are nonnegative and monotone nondecreasing (see Lemma 2.4 in [3]). So

$$S^+(x) = \lim_{r\to 0} S_r^+(x) \quad \text{and} \quad S^-(x) = \lim_{r\to 0} S_r^-(x)$$

exist. Let $S(x) := \max\{S^+(x), S^-(x)\}$. We claim that $S(x)$ is upper-semicontinuous at 0, that is

$$\limsup_{x\to 0} S(x) \leq S(0). \tag{2}$$

The proof of (2) is the same as the proof of Lemma 1 in [8], so we omit it here.

In order to prove the lemma, it is sufficient to show $S(0) = 0$. We do it by contradiction. Assume that $S(0) := \mu > 0$. Without loss of generality, we may assume $S^+(0) = \mu$. Then $S_r^+(0) \geq \mu$. Choose a sequence $r_j \to 0$ such that $r_j < \frac{r_0}{2}$; then there exist $x^j \in \partial B_{r_j} \cap \Omega$ such that $u(x^j) \geq \mu r_j$. Let $y^j \in \partial\Omega$ be such that $|x^j - y^j| = dist(x_j, \partial\Omega)$; then $|x^j - y^j| \leq \frac{r_j}{\sqrt{1+k^2}}$. From the differential mean value theorem, there exists z_j in each line segment $\overline{x^j y^j}$ such that $S(z_j) = |Du(z_j)| \geq \sqrt{1 + k^2}\mu > \mu$. This contradicts (2). □

3. Construction of the counterexample

Denote planar curves $\alpha_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^2\}$ and $\alpha_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 2x_1^2\}$. Let $Z^0 = (1, 1)$ and suppose we have chosen $Z^k \in \alpha_1$; choose $Z^{k+1} \in \alpha_1$ be such that $Z_1^{k+1} < Z_1^k$ and the straight line passing through Z^k and Z^{k+1} is a

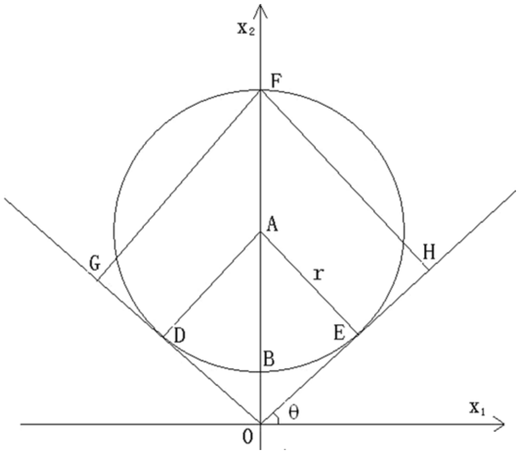

Fig. 1.

tangential line of α_2. Such a point Z^{k+1} exists and is unique. Denote the union of all line segments $\overline{Z^kZ^{k+1}}$ $(k = 0, 1, \ldots)$ by Γ^r. Define $\Gamma^l := \{(x_1, x_2) \in \mathbb{R}^2 : (-x_1, x_2) \in \Gamma^r\}$ and $\Gamma := \Gamma^l \cup \{0\} \cup \Gamma^r$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that $\partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 1\} = \Gamma$ and $\partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0.9\}$ is smooth. Choose a nonnegative smooth function g on $\partial\Omega$ such that $g \equiv 0$ on Γ and $g(P) > 0$ for some point $P \in \partial\Omega$. Let u be the viscosity solution of Dirichlet problem (1). Then $u(x) > 0\ \forall x \in \Omega$ from the maximum principle and Harnack inequality, especially, we may assume $u((0, \frac{1}{4})) = 1$. Note that Γ is between α_1 and α_2, so Γ is differentiable at 0. Thus $Du(0)$ exists and $|Du(0)| = \frac{\partial u}{\partial x_2}(0) \ge \frac{u((0,\frac{1}{4}))}{1/4} = 4$ from Lemma 1. $Du(Z^k)$ also exist and $Du(Z^k) = 0$ for $k = 1, 2, \ldots$ from Lemma 2. The construction of Ω is discovered by Dongsheng Li and Lihe Wang (Counterexample 4.1 in [9]) for uniformly elliptic equations. However our modification of Ω to a C^1 domain differs from the Counterexample 4.3 in [9].

Lemma 3. *Let $V \subset \mathbb{R}^2$ be a bounded domain such that for some $r_0 > 0$ and $0 < \theta < \frac{\pi}{4}$, $V \cap B_{r_0} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > \tan\theta|x_1|\} \cap B_{r_0}$. Assume that $v \in C(\bar{V})$ is a nonnegative infinity harmonic function with $v \equiv 0$ on $\partial V \cap B_{r_0}$. For $0 < r \ll r_0$, denote $A := (0, r\sec\theta)$, $B := (0, r\sec\theta - r)$, $D := (-r\sin\theta, r\sin\theta\tan\theta)$, $E := (r\sin\theta, r\sin\theta\tan\theta)$ (see Fig. 1). Take away the region enclosed by the arc DBE and the two straight line segments $D0$ and $0E$ from V and denote the new domain by W. Let $w \in C(\bar{W})$ be the infinity harmonic function with $w \equiv 0$ on the arc DBE and $w = v$ on the rest of the part of ∂W. Then for any given $\epsilon_1, \epsilon_2 > 0$, we can choose r small enough to make w to satisfy:*

$$\sup_{x\in W} |v(x) - w(x)| \le \epsilon_1$$

and

$$|Dw(B)| = \frac{\partial w}{\partial x_2}(B) \le \epsilon_2.$$

Proof. From Lemma 2, we know $Dv(0) = 0$. So there exists a monotone nondecreasing continuous function σ defined on $[0, r_0)$ with $\sigma(0) = 0$ such that $v(x) \le |x|\sigma(|x|)$ for any $x \in V \cap B_{r_0}$. Let r small enough to satisfy $2r\sigma(2r) \le \epsilon_1$; then $v \le \epsilon_1$ on the arc DBE. So

$$\sup_{x\in W} |v(x) - w(x)| = \sup_{x\in \partial W} |v(x) - w(x)| \le \epsilon_1.$$

To prove the second inequality, we denote $F := (0, r\sec\theta + r)$ and consider the cone function $c(x) := \epsilon_2(2r - |x - F|)$. Denote

$$G := (-(r + r\cos\theta)\sin\theta, (r + r\cos\theta)\sin\theta\tan\theta)$$

and

$$H := ((r + r\cos\theta)\sin\theta, (r + r\cos\theta)\sin\theta\tan\theta).$$

Notice that they are the two nearest points from ∂W to F. Now consider the region Λ enclosed by the four straight line segments $\overline{FG}$, $\overline{FH}$, $\overline{GD}$, $\overline{EH}$ and the arc DBE. We want to prove that if r is small enough then the cone function $c(x) \ge w(x)$ on $\bar{\Lambda}$. The conclusion $\frac{\partial w}{\partial x_2}(B) \le \epsilon_2$ follows from this.

In fact, for x on the line segments $\overline{FG}$ or $\overline{FH}$, on the one hand

$$c(x) \ge c(G) = c(H) = \epsilon_2(2r - (r + r\cos\theta)) = \epsilon_2 r(1 - \cos\theta),$$

and on the other hand

$$w(x) \le v(x) \le 3r\sigma(3r) \le \epsilon_2 r(1-\cos\theta)$$

if we choose r small enough. On the other part of $\partial\Lambda$, $c(x) \ge 0 = w(x)$. So $c(x) \ge w(x)$ on $\partial\Lambda$ and hence $c(x) \ge w(x)$ on $\bar{\Lambda}$ because that $c(x)$ is also an infinity harmonic function in Λ. □

Now we are ready to finish the construction of the counterexample. Let Ω and u be defined as above. First, we modify Ω to Ω_1 at the corner Z^1 in the manner of V to W in Lemma 3 (Z^1 corresponds to 0), and modify u to u_1 in the manner of v to w. We choose r_1 (corresponding to r in Lemma 3) small enough to make u_1 to satisfy that $\sup_{x\in\Omega_1}|u(x)-u_1(x)| \le \frac{1}{4}$ and $|Du_1(\tilde{Z}^1)| \le 1$, where $\tilde{Z}^1 \in \Omega_1$ is the point that corresponds to B in Lemma 3. Suppose the domain Ω_k and the function u_k are defined; we modify Ω_k to Ω_{k+1} at the corner Z^{k+1} and u_k to u_{k+1} in the same manner. By choosing r_{k+1} (corresponding to r in Lemma 3) small enough we can make u_{k+1} to satisfy that $\sup_{x\in\Omega_{k+1}}|u_k(x)-u_{k+1}(x)| \le \frac{1}{2^{k+2}}$ and $|Du_{k+1}(\tilde{Z}^{k+1})| \le 1$. Define $\tilde{\Omega} := \cap_{k=1}^{\infty}\Omega_k$ and $\tilde{u}(x) := \lim_{k\to\infty} u_k(x)$ for $x \in \tilde{\Omega}$. The domain $\tilde{\Omega}$ is C^1 and the function $\tilde{u}$ is infinity harmonic. Moreover, $\tilde{u} \equiv 0$ on $\partial\tilde{\Omega} \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 1\}$ and $|D\tilde{u}(\tilde{Z}^k)| \le |Du_k(\tilde{Z}^k)| \le 1$ for all k since $\tilde{u} \le u_k$. On the other hand, notice that $\tilde{u}((0, \frac{1}{4})) \ge \frac{1}{2}$, so $|D\tilde{u}(0)| \ge 2$ from Lemma 1.

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