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Boundary differentiability of infinity harmonic functions

Guanghao Hong *

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, PR China

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We prove that an infinity harmonic function $u(x) \in C(\bar{\Omega})$ is differentiable at a boundary point $x_0 \in \partial\Omega$ if both $\partial\Omega$ and the boundary data $g(x)$ are differentiable at x_0. This work improved a former result of Changyou Wang and Yifeng Yu (2012) [10]. They obtained the boundary differentiability of $u(x)$ with the C^1 assumption on $\partial\Omega$ and $g(x)$.

1. Introduction

We study the boundary regularity of infinity harmonic functions in this paper. Let $\Omega \subset R^n$ be a connected open set; then an infinity harmonic function $u(x) \in C(\Omega)$ is a viscosity solution of the infinity Laplace equation:

$$\Delta_\infty u(x) := \sum_{1\le i,j\le n} u_{x_i} u_{x_j} u_{x_i x_j} = 0, \quad x \in \Omega.$$

The ∞-Laplace equation arises as the Euler–Lagrange equation of the sup-*norm variational problem* of $|\triangledown u|$. It can also be interpreted as the limit equation of the p-Laplace equations $\Delta_p u := \mathrm{div}(|\triangledown u|^{p-2}\triangledown u) = 0$ as $p \to \infty$. This problem was firstly studied by G. Aronsson [1,2]. We refer the readers to [3] for a full background and historical development on this theory. Here we only mention some most important facts.

In 1993, Jensen [4] proved the existence and uniqueness of the viscosity solution $u(x) \in C(\bar{\Omega})$ of the Dirichlet problem:

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1}$$

with any bounded domain $\Omega \subset R^n$ and $g \in C(\partial\Omega)$. He also showed that a function $u(x) \in C(\Omega)$ is an infinity harmonic function if and only if u is an *absolutely minimizing Lipschitz extension* that means u satisfies the following property: for any open set $V \subset\subset \Omega$,

$$\sup_{x\neq y\in\partial V} \frac{|u(x)-u(y)|}{|x-y|} = \sup_{x\neq y\in\bar{V}} \frac{|u(x)-u(y)|}{|x-y|}.$$

Or equivalently, for any $V \subset\subset \Omega$, and $v = u$ on ∂V,

$$\|Du\|_{L^\infty} \le \|Dv\|_{L^\infty}.$$

* Tel.: +86 29 82660954.
E-mail addresses: ghhongmath@mail.xjtu.edu.cn, ghhongmath@gmail.com.

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In 2001, Crandall–Evans–Gariepy [5] proved that $u(x) \in C(\Omega)$ is an infinity harmonic function if and only if u enjoys the *comparison with cones property*: for any $V \subset\subset \Omega$ and any cone function $C(x) = a + b|x - z|$ with $a, b \in R$,

$$u(x) \le C(x) \quad \text{on } \partial(V \setminus \{z\}) \Rightarrow u(x) \le C(x) \text{ in } V;$$
$$u(x) \ge C(x) \quad \text{on } \partial(V \setminus \{z\}) \Rightarrow u(x) \ge C(x) \text{ in } V.$$

The comparison with cones property turns out to be very useful in the study of regularity and other properties of infinity harmonic functions.

The locally Lipschitz continuity of infinity harmonic functions was obtained from the comparison with cones property immediately. In 2001, Crandall–Evans [6] proved that any blow-up limit around an arbitrary point must be a linear function. Ten years later, Evans–Smart [7] established the uniqueness for the blow-up limit around any point from which the everywhere differentiability follows. In 2-dimension, Savin [8] and Evans–Savin [9] proved the C^1 and $C^{1,\alpha}$ regularity earlier. For $n \ge 3$, the C^1 regularity remains the most prominent open problem in this field.

Recently, Wang–Yu [10] published the first boundary regularity paper of infinity harmonic functions. One of their two results is the following theorem (Theorem 1.2 in [10]).

Theorem 1. *For $n \ge 2$, let $\Omega \subset R^n$ be a bounded domain with $\partial\Omega \in C^1$ and $g \in C^1(R^n)$. Assume that $u \in C(\bar{\Omega})$ is the viscosity solution of the infinity Laplacian equation (1). Then u is differentiable on the boundary, i.e., for any $x_0 \in \partial\Omega$, there exists $Du(x_0) \in R^n$ such that*

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + o(|x - x_0|), \quad \forall x \in \bar{\Omega}.$$

They also ask a question (Remark 1.1 in [10]): whether the C^1 assumption of g and $\partial\Omega$ in Theorem 1 can be relaxed to be everywhere differentiable. In this paper, we give a positive answer (even stronger) to their question.

Theorem 2. *For $n \ge 2$, let $\Omega \subset R^n$ be a domain and $u \in C(\bar{\Omega})$ is an infinity harmonic function in Ω. Assume that for $x_0 \in \partial\Omega$, $\partial\Omega$ and $g := u|_{\partial\Omega}$ are differentiable at x_0. Then u is differentiable at x_0.*

Without lost of generality, we may assume that $x_0 = 0$ and the tangential plane of $\partial\Omega$ at 0 is $\{x = (x', x_n) \in R^n : x_n = 0\}$. Denote $B(x, r) := \{y \in R^n : |y - x| < r\}$ for $x \in R^n$, $B(r) := B(0, r)$, $\hat{B}(x', r) := \{y' \in R^{n-1} : |y' - x'| < r\}$ for $x' \in R^{n-1}$ and $\hat{B}(r) := \hat{B}(0, r)$. We assume for some $r_0 > 0$,

$$\Omega \cap B(r_0) = \{x \in B(r_0) : x_n > f(x')\},$$

where $f \in C(\hat{B}(r_0))$ is differentiable at 0 with $f(0) = Df(0) = 0$. Denote $\hat{g}(x') = g(x', f(x'))$ for $x' \in \hat{B}(r_0)$, then $\hat{g}(x') \in C(\hat{B}(r_0))$ is differentiable at 0.

In Section 2 we prove the following easier theorem first.

Theorem 3. *Assume that u, Ω and g satisfy the conditions in Theorem 2. We assume additionally $\hat{g}(x') \in C^1(\hat{B}(r_0))$. Then u is differentiable at* 0.

In Section 3 we will apply Theorem 3 to prove Theorem 2 by a barrier argument. The method that we use to prove Theorems 2 and 3 is the same as that in [10]. However, we have to do more works on the upper-semicontinuity of $S(x)$ (see the next section for definition) at 0 because of the weaker boundary conditions.

2. Proof of Theorem 3

For $x \in \bar{\Omega} \cap B_{r_0/2}$ and $0 < r < r_0$, we define (as in [10])

$$S_r^+(x) = \sup_{y \in \partial(B(x,r) \cap \Omega) \setminus \{x\}} \frac{u(y) - u(x)}{|y - x|}$$

and

$$S_r^-(x) = \sup_{y \in \partial(B(x,r) \cap \Omega) \setminus \{x\}} \frac{u(x) - u(y)}{|y - x|}.$$

From the comparison with cones property, it can be verified (see Lemma 2.4 in [5]) that $S_r^+(x)$ and $S_r^-(x)$ are nonnegative and monotone decreasing functions of $r > 0$. Hence

$$S^+(x) = \lim_{r \to 0} S_r^+(x) \quad \text{and} \quad S^-(x) = \lim_{r \to 0} S_r^-(x)$$

exist. Let

$$S_r(x) := \max\{S_r^+(x), S_r^-(x)\}$$

and

$$S(x) := \max\{S^+(x), S^-(x)\}.$$

Lemma 2.7 in [5] tells us that for $x \in \Omega$, $S^+(x) = S^-(x) = S(x)$, and from the recent result of Evans–Smart [7] we know $S(x) = |Du(x)|$.

For $x \in \Omega$, it is easy to verify that

$$\limsup_{y \to x} S(y) \le S(x).$$

Now we prove that $S(x)$ is upper-semicontinuous at 0 under the conditions of Theorem 2.

Lemma 1. *For any $\epsilon > 0$, there exists $r(\epsilon, u) > 0$, such that*

$$\sup_{x \in \bar{\Omega} \cap B(r)} S(x) \le S(0) + \epsilon.$$

Proof. For $\epsilon > 0$, since $\hat{g}(x') \in C^1(\hat{B}(r_0))$ and $|D\hat{g}(0)| \le S(0)$, there exists $r_1 > 0$ such that

$$\sup_{x \neq y \in \partial\Omega \cap B(r_1)} \frac{|u(x) - u(y)|}{|x - y|} \le \sup_{x \neq y \in \partial\Omega \cap B(r_1)} \frac{|\hat{g}(x') - \hat{g}(y')|}{|x' - y'|} \le S(0) + \epsilon. \tag{2}$$

Since $\lim_{r \to 0} S_r(0) = S(0)$, there exists $0 < r_2 \le r_1/2$, such that

$$S_{r_2}(0) \le S(0) + \frac{\epsilon}{2}.$$

From the locally Lipschitz continuity of u, there exists $0 < r_3 \ll r_2$, such that

$$\sup_{y \in \partial B(x, r_2) \cap \Omega} \frac{|u(y) - u(x)|}{r_2} \le S(0) + \epsilon \quad \text{for } x \in \bar{\Omega} \cap B(r_3). \tag{3}$$

From (2) and (3), we have

$$S_{r_2}(x) = \sup_{y \in \partial(B(x, r_2) \cap \Omega) \setminus \{x\}} \frac{|u(y) - u(x)|}{|y - x|} \le S(0) + \epsilon \quad \text{for } x \in \partial\Omega \cap B(r_3).$$

From comparison with cones property, we have

$$\frac{|u(y) - u(x)|}{|y - x|} \le S(0) + \epsilon \quad \text{for } x \in \partial\Omega \cap B(r_3) \text{ and } y \in \Omega \cap B(r_3). \tag{4}$$

From the continuity of u again, there exists $0 < r_4 \le r_3/2$, such that

$$\sup_{y \in \partial B(x, r_3/2) \cap \Omega} \frac{|u(y) - u(x)|}{r_3/2} \le S(0) + \epsilon \quad \text{for } x \in \bar{\Omega} \cap B(r_4). \tag{5}$$

From (4) and (5), we have

$$S(x) \le S_{r_3/2}(x) = \sup_{y \in \partial(B(x, r_3/2) \cap \Omega) \setminus \{x\}} \frac{|u(y) - u(x)|}{|y - x|} \le S(0) + \epsilon$$

for $x \in \bar{\Omega} \cap B(r_4)$. Take $r(\epsilon, u) = r_4$ and we complete the proof. □

The rest of the Proof of Theorem 3 is the same as that in [10]; we only describe it briefly and refer the readers to [10] for full details.

If $S(0) = 0$, then u is differentiable at 0 with $Du(0) = 0$ from the definition of $S(0)$. We only need to consider the case $S(0) > 0$. For any sequence of $\lambda_m \to 0$, set $\Omega_m = \lambda_m^{-1}\Omega$ and define

$$u_m(x) = \frac{u(\lambda_m x) - g(0)}{\lambda_m}, \quad x \in \Omega_m.$$

Then $\lim_{m \to \infty} \Omega_m = R^n_+$ and a subsequence of $\{u_m\}$ converge locally uniformly to a function $w \in C(R^n_+)$.

From the upper-semicontinuity of $S(x)$ at 0, we have $Lip(w, R^n_+) \le S(0)$. Then from the *tightness on a ray* argument (see [10]), w is a plane with $|Dw| = S(0)$ and $w(x', 0) = D\hat{g}(0) \cdot x'$. Then $w = e \cdot x$ with $e = (D\hat{g}(0), \sqrt{S(0)^2 - |D\hat{g}(0)|^2})$ if $S^+ \ge S^-(0)$ or $e = (D\hat{g}(0), -\sqrt{S(0)^2 - |D\hat{g}(0)|^2})$ if $S^+ < S^-(0)$. So the linear blow-up limit of $\{u_m\}$ at 0 is unique and independent of the sequence $\{\lambda_m\}$; thus u is differentiable at 0 with $Du(0) = e$.

3. Proof of Theorem 2

Under the conditions of Theorem 2, it is not necessary that $S(x)$ is upper-semicontinuous at 0. But this is true if $x \to 0$ in a nontangential way and that is enough to get the conclusion.

Lemma 2. *Given any* $0 < \theta \ll 1$, *we have* $\forall 0 < \epsilon < \frac{1}{8}$, $\exists r(\epsilon, \theta, u) > 0$, *such that*

$$\sup_{x \in \bar{\Omega} \cap B(r) \cap \{x_n \geq \theta |x'|\}} S(x) \leq S(0) + \epsilon.$$

Proof. 1. Since $\hat{g}(x') \in C(\hat{B}(r_0))$ is differentiable at 0, it is easy to verify that there exist $h^{\pm}(x') \in C^1(\hat{B}(r_0))$ satisfying

$$h^{\pm}(0) = 0, \qquad Dh^{\pm}(0) = D\hat{g}(0) \quad \text{and} \quad h^{-}(x') \leq \hat{g}(x') \leq h^{+}(x'). \tag{6}$$

In fact, we can firstly consider the functions

$$\tilde{h}^{\pm}(x') := D\hat{g}(0) \cdot x' \pm \sigma_{\hat{g}}(|x'|)|x'|,$$

where

$$\sigma_{\hat{g}}(r) := \sup_{x' \in \hat{B}(r) \setminus \{0\}} \frac{|\hat{g}(x') - D\hat{g}(0) \cdot x'|}{|x'|}.$$

It is clear that $\tilde{h}^{\pm}(x')$ satisfy (6), but they maybe not be C^1 functions in general. However, it is not very hard to modify $D\hat{g}(0) \cdot x' \pm 2\sigma_{\hat{g}}(|x'|)|x'|$ to be C^1 functions $h^{\pm}(x')$ satisfying (6).

2. We define $g^{\pm}(x', f(x')) := h^{\pm}(x')$ for $x' \in \hat{B}(r_0)$, then

$$g^{-}(x) \leq g(x) \leq g^{+}(x) \quad \text{for } x \in \partial\Omega \cap B(r_0).$$

For $0 < r < r_0$, choose $\tilde{g}^{\pm}(x) \in C(\partial\Omega \cap \bar{B}(r))$ such that

$$\begin{cases} \tilde{g}^{\pm}(x) = g^{\pm}(x) & \text{for } x \in \partial\Omega \cap B(0.9r), \\ \tilde{g}^{-}(x) \leq g(x) \leq \tilde{g}^{+}(x) & \text{for } x \in \partial\Omega \cap B(r), \\ \tilde{g}^{\pm}(x) = g(x) & \text{on } x \in \partial\Omega \cap \partial B(r) \end{cases}$$

and let $u_r^{\pm}(x)$ to be the viscosity solutions of the Dirichlet problems:

$$\begin{cases} \Delta_\infty u_r^{\pm}(x) = 0 & \text{in } \Omega \cap B(r), \\ u_r^{\pm} = u & \text{on } \partial B(r) \cap \Omega, \\ u_r^{\pm} = \tilde{g}^{\pm} & \text{on } \partial\Omega \cap \bar{B}(r). \end{cases}$$

Then $u_r^{\pm}$ satisfy the conditions in Theorem 3 and

$$u_r^{-} \leq u \leq u_r^{+} \quad \text{in } \overline{\Omega \cap B(r)}$$

from the comparison principle.

3. Since $h^{+}(x') \in C^1(\hat{B}(r_0))$ and $|Dh^{+}(0)| = |D\hat{g}(0)| \leq S(0)$ and $\lim_{r\to 0} S_r(0) = S(0)$, there exists $r_1 > 0$, such that

$$S^{+}_{r_1, u^{+}_{r_1}}(0) := \sup_{y \in \partial(\Omega \cap B(r_1)) \setminus \{0\}} \frac{u^{+}_{r_1}(y) - g(0)}{|y - x|} \leq S(0) + \frac{\epsilon}{4}$$

and

$$S^{-}_{r_1, u^{+}_{r_1}}(0) := \sup_{y \in \partial(\Omega \cap B(r_1)) \setminus \{0\}} \frac{g(0) - u^{+}_{r_1}(y)}{|y - x|} \leq S^{-}_{r_1}(0) \leq S(0) + \frac{\epsilon}{4}.$$

Then

$$|Du^{+}_{r_1}(0)| = S_{u^{+}_{r_1}}(0) := \lim_{r \to 0} S^{\pm}_{r, u^{+}_{r_1}}(0) \leq S(0) + \frac{\epsilon}{4}.$$

Denote $P^{+}_{r_1}(x) := Du^{+}_{r_1}(0) \cdot x$.

4. We define

$$\sigma_f(r) := \sup_{x' \in \hat{B}(r) \setminus \{0\}} \frac{|f(x')|}{|x|}$$

and

$$\sigma_{u^+_{r_1}}(r) := \sup_{x\in\bar{\Omega}\cap B(r)\setminus\{0\}} \frac{|u^+_{r_1}(x) - P^+_{r_1}(x)|}{|x|}.$$

Choose $0 < r_2 \ll r_1$ such that $\sigma_f(r_2) \le \tau\theta\epsilon$ with

$$\tau := \begin{cases} 1 & \text{if } S(0) \le \dfrac{\epsilon}{3}, \\ \dfrac{\epsilon}{3S(0)} & \text{if } S(0) > \dfrac{\epsilon}{3} \end{cases}$$

and $\sigma_{u^+_{r_1}}(r_2) \le \frac{\theta\epsilon}{16}$, $\sigma_{\hat{g}}(r_2) \le \frac{\theta\epsilon}{24}$. Then for any $x \in \Omega \cap B(r_2) \cap \{x_n \ge \theta|x'|\}$ and $y \in \partial\Omega \cap B(r_2)$, we have

(i)

$$|x - (y', 0)| \ge x_n \ge \frac{\theta}{\sqrt{1+\theta^2}}|x| \ge \frac{\theta}{2}|x|$$

and

$$|y'| \le |x| + |x - (y', 0)| \le \left(\frac{2}{\theta} + 1\right)|x - (y', 0)| \le \frac{3}{\theta}|x - (y', 0)|.$$

(ii)

$$|x - y| \ge |x - (y', 0)| - |y_n| \ge |x - (y', 0)| - \tau\theta\epsilon|y'| \ge (1 - 3\tau\epsilon)|x - (y', 0)|.$$

(iii)

$$u(x) \le u^+_{r_1}(x) \le P^+_{r_1}(x) + \frac{\theta\epsilon}{16}|x| \le P^+_{r_1}(x) + \frac{\epsilon}{8}|x - (y', 0)|$$

and

$$\begin{aligned} u(y) = \hat{g}(y') &\ge D\hat{g}(0)\cdot y' - \sigma_{\hat{g}}(|y'|)|y'| \\ &= P^+_{r_1}((y', 0)) - \sigma_{\hat{g}}(|y'|)|y'| \\ &\ge P^+_{r_1}((y', 0)) - \frac{\epsilon}{8}|x - (y', 0)|. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{u(x) - u(y)}{|x - y|} &\le \frac{P^+_{r_1}(x) - P^+_{r_1}((y', 0)) + \frac{\epsilon}{4}|x - (y', 0)|}{(1 - 3\tau\epsilon)|x - (y', 0)|} \\ &\le \frac{S(0) + \frac{\epsilon}{2}}{1 - 3\tau\epsilon} \le S(0) + \epsilon. \end{aligned}$$

By doing the same things with $u^-_{r_2}$, we can get

$$\frac{u(y) - u(x)}{|x - y|} \le S(0) + \epsilon.$$

So we have

$$\frac{|u(y) - u(x)|}{|x - y|} \le S(0) + \epsilon \tag{7}$$

for any $x \in \Omega \cap B(r_2) \cap \{x_n \ge \theta|x'|\}$ and $y \in \partial\Omega \cap B(r_2)$.

5. From the continuity of u, there exists $0 < r_3 \ll r_2$ such that

$$\sup_{y\in\partial B(x, r_2/2)\cap\Omega} \frac{|u(y) - u(x)|}{r_2/2} \le S(0) + \epsilon \quad \text{for } x \in \bar{\Omega} \cap B(r_3). \tag{8}$$

From (7) and (8), we have

$$S(x) \le S_{r_2/2}(x) = \sup_{y\in\partial(B(x, r_2/2)\cap\Omega)\setminus\{x\}} \frac{|u(y) - u(x)|}{|y - x|} \le S(0) + \epsilon$$

for $x \in \bar{\Omega} \cap B(r_3) \cap \{x_n \ge \theta|x'|\}$. Take $r(\epsilon, \theta, u) = r_3$ and we complete the proof. □

The rest of Proof of Theorem 2 follows in the same way as the Proof of Theorem 3 that we have described in the end of Section 2.

References

[1] G. Aronsson, Extension of functions satisfying Lipschitz conditions, Ark. Mat. 6 (1967) 551–561.
[2] G. Aronsson, On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, Ark. Mat. 7 (1968) 395–425.
[3] M.G. Crandall, A visit with the ∞-Laplace equation, in: Calculus of Variations and Nonlinear Partial Differential Equations, in: Lecture Notes in Mathematics, vol. 1927, Springer, Berlin, 2008, pp. 75–122.
[4] R. Jensen, Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient, Arch. Ration. Mech. Anal. 123 (1993) 51–74.
[5] M.G. Crandall, L.C. Evans, R.F. Gariepy, Optimal Lipschitz extensions and the infinity Laplacian, Calc. Var. Partial Differential Equations 13 (2) (2001) 123–139.
[6] M.G. Crandall, L.C. Evans, A remark on infinity harmonic functions, in: Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Vina del Mar-Valparaiso, 2000), in: Electron. J. Differ. Equ. Conf., vol. 6, 2001, pp. 123–129.
[7] L.C. Evans, C.K. Smart, Everywhere differentiability of infinity harmonic functions, Calc. Var. Partial Differential Equations 42 (1–2) (2011) 289–299.
[8] O. Savin, C^1 regularity for infinity harmonic functions in two dimensions, Arch. Ration. Mech. Anal. 176 (3) (2005) 351–361.
[9] L.C. Evans, O. Savin, $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions, Calc. Var. Partial Differential Equations 32 (3) (2008) 325–347.
[10] C.Y. Wang, Y.F. Yu, C^1-boundary regularity of planar infinity harmonic functions, Math. Res. Lett. 19 (4) (2012) 823–835.