An Introduction to Fourier Theory

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Introduction

Linear transforms, especially Fourier and Laplace transforms, are widely used in solving problems in science and engineering. The Fourier transform is used in linear systems analysis, antenna studies, optics, random process modeling, probability theory, quantum physics, and boundary-value problems (Brigham, 2–3) and has been very successfully applied to restoration of astronomical data (Brault and White). The Fourier transform, a pervasive and versatile tool, is used in many fields of science as a mathematical or physical tool to alter a problem into one that can be more easily solved. Some scientists understand Fourier theory as a physical phenomenon, not simply as a mathematical tool. In some branches of science, the Fourier transform of one function may yield another physical function (Bracewell, 1–2).

The Fourier Transform

The *Fourier transform*, in essence, decomposes or separates a waveform or function into sinusoids of different frequency which sum to the original waveform. It identifies or distinguishes the different frequency sinusoids and their respective amplitudes (Brigham, 4). The Fourier transform of $f(x)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx.$$

Applying the same transform to $F(s)$ gives

$$f(w) = \int_{-\infty}^{\infty} F(s) e^{-i2\pi xs} dx.$$

If $f(x)$ is an even function of x, that is $f(x) = f(-x)$, then $f(w) = f(x)$. If $f(x)$ is an odd function of x, that is $f(x) = -f(-x)$, then $f(w) = f(-x)$. When $f(x)$ is neither even nor odd, it can often be split into even or odd parts.

To avoid confusion, it is customary to write the Fourier transform and its inverse so that they exhibit reversibility:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx$$
$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs} dx$$

so that

$$f(x) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx \right] e^{i2\pi xs} ds$$

as long as the integral exists and any discontinuities, usually represented by multiple integrals of the form $\frac{1}{2}[f(x_+) + f(x_-)]$, are finite. The transform quantity $F(s)$ is often represented as $\tilde{f}(s)$ and the Fourier transform is often represented by the operator $\mathscr{F}$ (Bracewell, 6–8).

There are functions for which the Fourier transform does not exist; however, most physical functions have a Fourier transform, especially if the transform represents a physical quantity. Other functions can be treated with Fourier theory as limiting cases. Many of the common theoretical functions are actually limiting cases in Fourier theory.

Usually functions or waveforms can be split into even and odd parts as follows

$$f(x) = E(x) + O(x)$$

where

$$E(x) = \frac{1}{2}[f(x) + f(-x)]$$
$$O(x) = \frac{1}{2}[f(x) - f(-x)]$$

and $E(x)$, $O(x)$ are, in general, complex. In this representation, the Fourier transform of $f(x)$ reduces to

$$2\int_0^\infty E(x)\cos(2\pi xs)\,dx - 2i\int_0^\infty O(x)\sin(2\pi xs)\,dx$$

It follows then that an even function has an even transform and that an odd function has an odd transform. Additional symmetry properties are shown in Table 1 (Bracewell, 14).

Table 1: Symmetry Properties of the Fourier Transform

function	transform
real and even	real and even
real and odd	imaginary and odd
imaginary and even	imaginary and even
complex and even	complex and even
complex and odd	complex and odd
real and asymmetrical	complex and asymmetrical
imaginary and asymmetrical	complex and asymmetrical
real even plus imaginary odd	real
real odd plus imaginary even	imaginary
even	even
odd	odd

An important case from Table 1 is that of an *Hermitian function*, one in which the real part is even and the imaginary part is odd, *i.e.*, $f(x) = f^*(-x)$. The Fourier transform of an Hermitian function is even. In addition, the Fourier transform of the complex conjugate of a function $f(x)$ is $F^*(-s)$, the *reflection* of the conjugate of the transform.

The *cosine transform* of a function $f(x)$ is defined as

$$F_c(s) = 2\int_0^\infty f(x)\cos\ 2\pi sx\ dx.$$

This is equivalent to the Fourier transform if $f(x)$ is an even function. In general, the even part of the Fourier transform of $f(x)$ is the cosine transform of the even part of $f(x)$. The cosine transform has a reverse transform given by

$$f(x) = 2\int_0^\infty F_c(s)\cos\ 2\pi sx\ ds.$$

Likewise, the *sine transform* of $f(x)$ is defined by

$$F_s(s) = 2\int_0^\infty f(x)\sin\ 2\pi sx\ dx.$$

As a result, i times the odd part of the Fourier transform of $f(x)$ is the sine transform of the odd part of $f(x)$.

Combining the sine and cosine transforms of the even and odd parts of $f(x)$ leads to the Fourier transform of the whole of $f(x)$:

$$\mathscr{F} f(x) = \mathscr{F}_c E(x) - i\mathscr{F}_s O(x)$$

where $\mathscr{F}$, $\mathscr{F}_c$, and $\mathscr{F}_s$ stand for $-i$ times the Fourier transform, the cosine transform, and the sine transform respectively, or

$$F(s) = \frac{1}{2}F_c(s) - \frac{1}{2}iF_s(s)$$

(Bracewell, 17–18).

Since the Fourier transform $F(s)$ is a frequency domain representation of a function $f(x)$, the s characterizes the frequency of the decomposed cosinusoids and sinusoids and is equal to the number of cycles per unit of x (Bracewell, 18–21). If a function or waveform is not periodic, then the Fourier transform of the function will be a continuous function of frequency (Brigham, 4).

The Two Domains

It is often useful to think of functions and their transforms as occupying two domains. These domains are referred to as the upper and the lower domains in older texts, "as if functions circulated at ground level and their transforms in the underworld" (Bracewell, 135). They are also referred to as the function and transform domains, but in most physics applications they are called the time and frequency domains respectively. Operations performed in one domain have corresponding operations in the other. For example, as will be shown below, the convolution operation in the time domain becomes a multiplication operation in the frequency domain, that is, $f(x) \otimes g(x) \leftrightarrow F(s)\ G(s)$. The reverse is also true, $F(s) \otimes G(s) \leftrightarrow f(x)\ g(x)$. Such theorems allow one to move between domains so that operations can be performed where they are easiest or most advantageous.

Fourier Transform Properties

Scaling Property

If $\mathscr{F}\{f(x)\} = F(s)$ and a is a real, nonzero constant, then

$$
\begin{aligned}
\mathscr{F}\{f(ax)\} &= \int_{-\infty}^{\infty} f(ax)e^{i2\pi sx}dx \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(\beta)e^{i2\pi\frac{s}{a}\beta}d\beta \\
&= \frac{1}{|a|}F\left(\frac{s}{a}\right).
\end{aligned}
$$

From this, the *time scaling property*, it is evident that if the width of a function is decreased while its height is kept constant, then its Fourier transform becomes wider and shorter. If its width is increased, its transform becomes narrower and taller.

A similar *frequency scaling property* is given by

$$
\mathscr{F}\left\{\frac{1}{|a|}f\left(\frac{x}{a}\right)\right\} = F(as).
$$

Shifting Property

If $\mathscr{F}\{f(x)\} = F(s)$ and x_0 is a real constant, then

$$
\begin{aligned}
\mathscr{F}\{f(x-x_0)\} &= \int_{-\infty}^{\infty} f(x-x_0)e^{i2\pi sx}dx \\
&= \int_{-\infty}^{\infty} f(\beta)e^{i2\pi s(\beta+x_0)}d\beta \\
&= e^{i2\pi x_0 s}\int_{-\infty}^{\infty} f(\beta)e^{i2\pi s\beta}d\beta \\
&= F(s)e^{i2\pi x_0 s}.
\end{aligned}
$$

This *time shifting property* states that the Fourier transform of a shifted function is just the transform of the unshifted function multiplied by an exponential factor having a linear phase.

Likewise, the *frequency shifting property* states that if $F(s)$ is shifted by a constant s_0, its inverse transform is multiplied by $e^{i2\pi xs_0}$

$$
\mathscr{F}\left\{f(x)e^{i2\pi xs_0}\right\} = F(s-s_0).
$$

Convolution Theorem

We now derive the aforementioned *time convolution theorem*. Suppose that $g(x) = f(x) \otimes h(x)$. Then, given that $\mathscr{F}\{g(x)\} = G(s)$, $\mathscr{F}\{f(x)\} = F(s)$, and $\mathscr{F}\{h(x)\} = H(s)$,

$$
\begin{aligned}
G(s) &= \mathscr{F}\left\{f(x) \otimes h(x)\right\} \\
&= \mathscr{F}\left\{\int_{-\infty}^{\infty} f(\beta)h(x-\beta)\, d\beta\right\} \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\beta)h(x-\beta)\, d\beta\right] e^{-i2\pi sx} dx \\
&= \int_{-\infty}^{\infty} f(\beta)\left[\int_{-\infty}^{\infty} h(x-\beta)e^{-i2\pi sx} dx\right] d\beta \\
&= H(s)\int_{-\infty}^{\infty} f(\beta)e^{-i2\pi s\beta} d\beta \\
&= F(s)\ H(s).
\end{aligned}
$$

This extremely powerful result demonstrates that the Fourier transform of a convolution is simply given by the product of the individual transforms, that is

$$\mathscr{F}\left\{f(x) \otimes h(x)\right\} = F(s)\ H(s).$$

Using a similar derivation, it can be shown that the Fourier transform of a product is given by the convolution of the individual transforms, that is

$$\mathscr{F}\left\{f(x)\ h(x)\right\} = F(s) \otimes H(s).$$

This is the statement of the *frequency convolution theorem* (Gaskill, 194–197; Brigham, 60–65).

Correlation Theorem

The correlation integral, like the convolution integral, is important in theoretical and practical applications. The correction integral is defined as

$$h(x) = \int_{-\infty}^{\infty} f(u)g(x+u)du$$

and like the convolution integral, it forms a Fourier transform pair given by

$$\mathscr{F}\left\{h(x)\right\} = F(s)G^{*}(s).$$

This is the statement of the *correlation theorem.* If $f(x)$ and $g(x)$ are the same function, the integral above is normally called the *autocorrelation* function, and the *crosscorrelation* if they differ (Brigham, 65–69). The Fourier transform pair for the autocorrelation is simply

$$\mathscr{F}\left\{\int_{-\infty}^{\infty} f(u)f(x+u)du\right\} = |F|^{2}.$$

Parseval's Theorem

Parseval's Theorem states that the power of a signal represented by a function $h(t)$ is the same whether computed in signal space or frequency (transform) space; that is,

$$\int_{-\infty}^{\infty} h^2(t)dt = \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

(Brigham, 23). The *power spectrum*, $P(f)$, is given by

$$P(f) = |H(f)|^2, -\infty \leq f \leq +\infty.$$

Sampling Theorem

A *bandlimited signal* is a signal, $f(t)$, which has no spectral components beyond a frequency B Hz; that is, $F(s) = 0$ for $|s| > 2\pi B$. The *sampling theorem* states that a real signal, $f(t)$, which is bandlimited to B Hz can be reconstructed without error from samples taken uniformly at a rate $R > 2B$ samples per second. This minimum sampling frequency, $\mathcal{F}_s = 2B$ Hz, is called the *Nyquist rate* or the *Nyquist frequency.* The corresponding sampling interval, $T = \frac{1}{2B}$ (where $t = nT$), is called the *Nyquist interval.* A signal bandlimited to B Hz which is sampled at less than the Nyquist frequency of $2B$, *i.e.,* which was sampled at an interval $T > \frac{1}{2B}$, is said to be *undersampled.*

Aliasing

A number of practical difficulties are encountered in reconstructing a signal from its samples. The sampling theorem assumes that a signal is bandlimited. In practice, however, signals are timelimited, not bandlimited. As a result, determining an adequate sampling frequency which does not lose desired information can be difficult. When a signal is undersampled, its spectrum has overlapping tails; that is $F(s)$ no longer has complete information about the spectrum and it is no longer possible to recover $f(t)$ from the sampled signal. In this case, the tailing spectrum does not go to zero, but is folded back onto the apparent spectrum. This inversion of the tail is called *spectral folding* or *aliasing* (Lathi, 532-535).

As an example, Figure 1 shows a unit gaussian curve sampled at three different rates. The FFT (or Fast Fourier Transform) of the undersampled gaussian appears flattened and its tails do not reach zero. This is a result of aliasing. Additional spectral components have been folded back onto the ends of the spectrum or added to the edges to produce this curve.

The FFT of the oversampled gaussian reaches zero very quickly. Much of its spectrum is zero and is not needed to reconstruct the original gaussian.

Finally, the FFT of the critically-sampled gaussian has tails which reach zero at their ends. The data in the spectrum of the critically-sampled gaussian is just sufficient to reconstruct the original. This gaussian was sampled at the Nyquist frequency.

Figure 1 was generated using IDL with the following code:

```
!P.Multi=[0,3,2]
a=gauss(256,2.0,2) ; undersampled
fa=fft(a,-1)
b=gauss(256,2.0,0.1) ; oversampled
fb=fft(b,-1)
c=gauss(256,2.0,0.8) ; critically sampled
fc=fft(c,-1)
plot,a,title='!6Undersampled Gaussian'
plot,b,title='!6Oversampled Gaussian'
plot,c,title='!6Critically-Sampled Gaussian'
plot,shift(abs(fa),128),title='!6FFT of Undersampled Gaussian'
plot,shift(abs(fb),128),title='!6FFT of Oversampled Gaussian'
plot,shift(abs(fc),128),title='!6FFT of Critically-Sampled Gaussian'
```

The `gauss` function is as follows:

```
function gauss,dim,fwhm,interval
;
; gauss - produce a normalized gaussian curve centered in dim data
;         points with a full width at half maximum of fwhm sampled
;         with a periodicity of interval
;
; dim      = the number of points
; fwhm     = full width half max of gaussian
; interval = sampling interval
;
center=dim/2.0 ; automatically center gaussian in dim points
x=findgen(dim)-center
sigma=fwhm/sqrt(8.0 * alog(2.0))  ; fwhm is in data points
coeff=1.0 / ( sqrt(2.0*!Pi) * (sigma/interval) )
data=coeff * exp( -(interval * x)^2.0 / (2.0*sigma^2.0) )
return,data
end
```

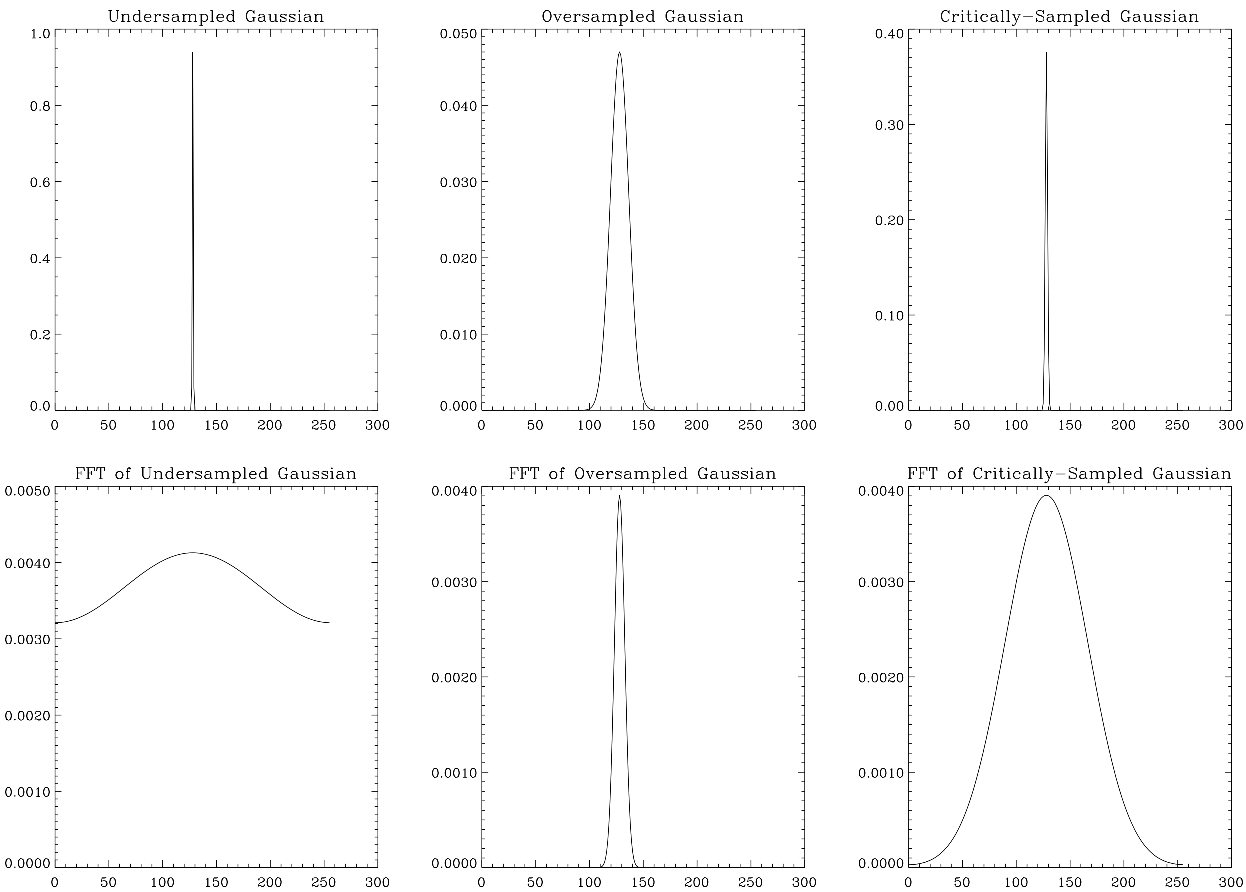

Figure 1: Undersampled, oversampled, and critically-sampled unit area gaussian curves.

Discrete Fourier Transform (DFT)

Because a digital computer works only with discrete data, numerical computation of the Fourier transform of $f(t)$ requires discrete sample values of $f(t)$, which we will call f_k. In addition, a computer can compute the transform $F(s)$ only at discrete values of s, that is, it can only provide discrete samples of the transform, F_r. If $f(kT)$ and $F(rs_0)$ are the kth and

rth samples of $f(t)$ and $F(s)$, respectively, and N_0 is the number of samples in the signal in one period T_0, then

$$f_k = Tf(kT) = \frac{T_0}{N_0} f(kT)$$

and

$$F_r = F(rs_0)$$

where

$$s_0 = 2\pi \mathcal{F}_0 = \frac{2\pi}{T_0}.$$

The *discrete Fourier transform (DFT)* is defined as

$$F_r = \sum_{k=0}^{N_0-1} f_k e^{-ir\Omega_0 k}$$

where $\Omega_0 = \frac{2\pi}{N_0}$. Its inverse is

$$f_k = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{ir\Omega_0 k}.$$

These equations can be used to compute transforms and inverse transforms of appropriately-sampled data. Proofs of these relationships are in Lathi (546-548).

Fast Fourier Transform (FFT)

The *Fast Fourier Transform (FFT)* is a DFT algorithm developed by Tukey and Cooley in 1965 which reduces the number of computations from something on the order of N_0^2 to $N_0 \log N_0$. There are basically two types of Tukey-Cooley FFT algorithms in use: decimation-in-time and decimation-in-frequency. The algorithm is simplified if N_0 is chosen to be a power of 2, but it is not a requirement.

Summary

The Fourier transform, an invaluable tool in science and engineering, has been introduced and defined. Its symmetry and computational properties have been described and the significance of the time or signal space (or domain) vs. the frequency or spectral domain has been mentioned. In addition, important concepts in sampling required for the understanding of the sampling theorem and the problem of aliasing have been discussed. An example of aliasing was provided along with a short description of the discrete Fourier transform (DFT) and its popular offspring, the Fast Fourier Transform (FFT) algorithm.

References

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