Some Applications of the Residue Theorem
Supplementary Lecture Notes
MATH 322, Complex Analysis
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1 Introduction

These notes supplement a freely downloadable book *Complex Analysis* by George Cain (henceforth referred to as Cain's notes), that I served as a primary text for an undergraduate level course in complex analysis. Throughout these notes I will make occasional references to results stated in these notes. The aim of my notes is to provide a few examples of applications of the residue theorem. The main goal is to illustrate how this theorem can be used to evaluate various types of integrals of real valued functions of real variable.

Following Sec. 10.1 of Cain's notes, let us recall that if C is a simple, closed contour and f is analytic within the region bounded by C except for finitely many points $z_0, z_1, \dots, z_k$ then

$$\int_C f(z)dz = 2\pi i \sum_{j=0}^{k} \mathrm{Res}_{z=z_j} f(z),$$

where $\mathrm{Res}_{z=a} f(z)$ is the residue of f at a.

2 Evaluation of Real-Valued Integrals.

2.1 Definite integrals involving trigonometric functions

We begin by briefly discussing integrals of the form

$$\int_0^{2\pi} F(\sin at, \cos bt)dt. \tag{1}$$

Our method is easily adaptable for integrals over a different range, for example between 0 and π or between $\pm\pi$.

Given the form of an integrand in (1) one can reasonably hope that the integral results from the usual parameterization of the unit circle $z = e^{it}$, $0 \leq t \leq 2\pi$. So, let's try $z = e^{it}$. Then (see Sec. 3.3 of Cain's notes),

$$\cos bt = \frac{e^{ibt} + e^{-ibt}}{2} = \frac{z^b + 1/z^b}{2}, \quad \sin at = \frac{e^{iat} - e^{-iat}}{2} = \frac{z^a - 1/z^a}{2}.$$

Moreover, $dz = ie^{it}dt$, so that

$$dt = \frac{dz}{iz}.$$

Putting all of this into (1) yields

$$\int_0^{2\pi} F(\sin at, \cos bt)dt = \int_C F\left(\frac{z^a - 1/z^a}{2}, \frac{z^b + 1/z^b}{2}\right)\frac{dz}{iz},$$

where C is the unit circle. This integral is well within what contour integrals are about and we might be able to evaluate it with the aid of the residue theorem.

It is a good moment to look at an example. We will show that

$$\int_0^{2\pi} \frac{\cos 3t}{5 - 4\cos t} dt = \frac{\pi}{12}. \tag{2}$$

Following our program, upon making all these substitutions, the integral in (1) becomes

$$\begin{aligned}
\int_C \frac{(z^3 + 1/z^3)/2}{5 - 4(z + 1/z)/2} \frac{dz}{iz} &= \frac{1}{i} \int_C \frac{z^6 + 1}{z^3(10z - 4z^2 - 4)} dz \\
&= -\frac{1}{2i} \int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)} dz \\
&= -\frac{1}{2i} \int_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)} dz.
\end{aligned}$$

The integrand has singularities at $z_0 = 0$, $z_1 = 1/2$, and $z_2 = 2$, but since the last one is outside the unit circle we only need to worry about the first two. Furthermore, it is clear that $z_0 = 0$ is a pole of order 3 and that $z_1 = 1/2$ is a simple pole. One way of seeing it, is to notice that within a small circle around $z_0 = 0$ (say with radius 1/4) the function

$$\frac{z^6 + 1}{(2z - 1)(z - 2)}$$

is analytic and so its Laurent series will have all coefficients corresponding to the negative powers of z zero. Moreover, since its value at $z_0 = 0$ is

$$\frac{0^6 + 1}{(2 \cdot 0 - 1)(0 - 2)} = \frac{1}{2},$$

the Laurent expansion of our integrand is

$$\frac{1}{z^3} \frac{z^6 + 1}{(2z - 1)(z - 2)} = \frac{1}{z^3} \left(\frac{1}{2} + a_1 z + \dots \right) = \frac{1}{2} \frac{1}{z^3} + \frac{a_1}{z^2} + \dots,$$

which implies that $z_0 = 0$ is a pole of order 3. By a similar argument (using a small circle centered at $z_1 = 1/2$) we see that $z_1 = 1/2$ is a simple pole. Hence, the value of integral in (2) is

$$2\pi i \left(\mathrm{Res}_{z=0} \left(\frac{z^6 + 1}{z^3(2z - 1)(z - 2)} \right) + \mathrm{Res}_{z=1/2} \left(\frac{z^6 + 1}{z^3(2z - 1)(z - 2)} \right) \right).$$

The residue at a simple pole $z_1 = 1/2$ is easy to compute by following a discussion preceding the second example in Sec. 10.2 in Cain's notes:

$$\frac{z^6 + 1}{z^3(2z - 1)(z - 2)} = \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)} = \frac{z^6 + 1}{2z^5 - 5z^4 + 2z^3}$$

is of the form $p(z)/q(z)$ with $p(1/2) = 2^{-6} + 1 \neq 0$ and $q(1/2) = 0$. Now, $q'(z) = 10z^4 - 20z^3 + 6z^2$, so that $q'(1/2) = 10/2^4 - 20/2^3 + 6/2^2 = -3/2^3$. Hence, the residue at $z_1 = 1/2$ is

$$\frac{p(1/2)}{q'(1/2)} = -\frac{(2^6+1)\cdot 2^3}{2^6 \cdot 3} = -\frac{65}{24}.$$

The residue at a pole of degree 3, $z_0 = 0$, can be obtained in various ways. First, we can take a one step further a method we used to determine the degree of that pole: since on a small circle around 0,

$$\frac{z^6+1}{(2z-1)(z-2)} = \frac{z^6}{(2z-1)(z-2)} + \frac{1}{(2z-1)(z-2)}. \tag{3}$$

is analytic, the residue of our function will be the coefficient corresponding to z^2 of Taylor expansion of the function given in (3). The first term will not contribute as its smallest non-zero coefficient is in front of z^6 so we need to worry about the second term only. Expand each of the terms $1/(2z-1)$ and $1/(z-2)$ into its Taylor series, and multiply out. As long as $|2z| < 1$ we get

$$\begin{aligned}
\frac{1}{(2z-1)(z-2)} &= \frac{1}{2}\cdot\frac{1}{1-2z}\cdot\frac{1}{1-z/2} \\
&= \frac{1}{2}\left(1+2z+2^2z^2+\dots\right)\cdot\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\dots\right) \\
&= \frac{1}{2}\left(1+A_1z+2\cdot\frac{1}{2}z^2+\frac{1}{4}z^2+4z^2+\dots\right) \\
&= \frac{1}{2}\left(1+A_1z+\left(5+\frac{1}{4}\right)z^2+\dots\right) \\
&= \frac{1}{2}+\frac{a_1}{2}z+\frac{21}{8}z^2+\dots
\end{aligned}$$

so that the residue at $z_0 = 0$ is $21/8$ (from the calculations we see that $A_1 = 2 + 1/2$, but since our interest is the coefficient in front of z^2 we chose to leave A_1 unspecified). The same result can be obtained by computing the second derivative (see Sec. 10.2 of Cain's notes) of

$$\frac{1}{2!}z^3\frac{z^6+1}{z^3(2z-1)(z-2)},$$

and evaluating at $z = 0$. Alternatively, one can open Maple session and type:

```
residue((z^6+1)/z^3/(2*z-1)/(z-2),z=0);
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to get the same answer again.

Combining all of this we get that the integral in (2) is

$$-\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z-1)(z-2)}dz = -\frac{1}{2i}(2\pi i)\left(\frac{21}{8}-\frac{65}{24}\right) = \pi\frac{65-63}{24} = \frac{\pi}{12},$$

as required.

2.2 Evaluation of improper integrals involving rational functions

Recall that improper integral

$$\int_0^\infty f(x)dx$$

is defined as a limit

$$\lim_{R\to\infty} \int_0^R f(x)dx,$$

provided that this limit exists. When the function $f(x)$ is even (i.e. $f(x) = f(-x)$, for $x \in \mathbf{R}$) one has

$$\int_0^R f(x)dx = \frac{1}{2}\int_{-R}^R f(x)dx,$$

and the above integral can be thought of as an integral over a part of a contour C_R consisting of a line segment along the real axis between $-R$ and R. The general idea is to "close"the contour (often by using one of the semi-circles with radius R centered at the origin), evaluate the resulting integral by means of residue theorem, and show that the integral over the "added"part of C_R asymptotically vanishes as $R \to 0$. As an example we will show that

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}. \tag{4}$$

Consider a function $f(z) = 1/(z^2+1)^2$. This function is not analytic at $z_0 = i$ (and that is the only singularity of $f(z)$), so its integral over any contour encircling i can be evaluated by residue theorem. Consider C_R consisting of the line segment along the real axis between $-R \le x \le R$ and the upper semi-circle $A_R := \{z = Re^{it},\, 0 \le t \le \pi\}$. By the residue theorem

$$\int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i \mathrm{Res}_{z=i}\left(\frac{1}{(z^2+1)^2}\right).$$

The integral on the left can be written as

$$\int_{-R}^R \frac{dz}{(z^2+1)^2} + \int_{A_R} \frac{dz}{(z^2+1)^2}.$$

Parameterization of the line segment is $\gamma(t) = t + i\cdot 0$, so that the first integral is just

$$\int_{-R}^R \frac{dx}{(x^2+1)^2} = 2\int_0^R \frac{dx}{(x^2+1)^2}.$$

Hence,

$$\int_0^R \frac{dx}{(x^2+1)^2} = \pi i \mathrm{Res}_{z=i}\left(\frac{1}{(z^2+1)^2}\right) - \frac{1}{2}\int_{A_R} \frac{dz}{(z^2+1)^2}. \tag{5}$$

Since

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2},$$

and $1/(z+i)^2$ is analytic on the upper half-plane, $z = i$ is a pole of order 2. Thus (see Sec. 10.2 of Cain's notes), the residue is

$$\frac{d}{dz}\left((z-i)^2 \frac{1}{(z-i)^2(z+i)^2}\right)_{z=i} = \left(\frac{-2}{(z+i)^3}\right)_{z=i} = -\frac{2}{(2i)^3} = -\frac{1}{4i^3} = \frac{1}{4i}$$

which implies that the first term on the right-hand side of (5) is

$$\frac{\pi i}{4i} = \frac{\pi}{4}.$$

Thus the evaluation of (4) will be complete once we show that

$$\lim_{R\to\infty} \int_{A_R} \frac{dz}{(z^2+1)^2} = 0. \tag{6}$$

But this is straightforward; for $z \in A_R$ we have

$$|z^2+1| \geq |z|^2 - 1 = R^2 - 1,$$

so that for $R > 2$

$$\left|\frac{1}{(z^2+1)^2}\right| \leq \frac{1}{(R^2-1)^2}.$$

Using our favorite inequality

$$\left|\int_C g(z)dz\right| \leq M \cdot \text{length}(C), \tag{7}$$

where $|g(z)| \leq M$ for $z \in C$, and observing that $\text{length}(A_R) = \pi R$ we obtain

$$\left|\int_{A_R} \frac{dz}{(z^2+1)^2}\right| \leq \frac{\pi R}{(R^2-1)^2} \longrightarrow 0,$$

as $R \to \infty$. This proves (6) and thus also (4).

2.3 Improper integrals involving trigonometric and rational functions.

Integrals like one we just considered may be "spiced up"to allow us to handle an apparently more complicated integrals with very little extra effort. We will illustrate it by showing that

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3}.$$

We keep the same function $1/(x^2+1)^2$, just to illustrate the main difference. This time we consider the function $e^{3iz}f(z)$, where $f(z)$ is, as before $1/(z^2+1)^2$. By following the same route we are led to

$$\int_{-R}^{R} \frac{e^{3xi}}{(x^2+1)^2}dx = 2\pi i \mathrm{Res}_{z=i}(f(z)e^{3zi} - \int_{A_R} f(z)e^{3zi}dz = \frac{2\pi}{e^3} - \int_{A_R} f(z)e^{3zi}dz.$$

At this point it only remains to use the fact that the real parts of both sides must the same, write $e^{3xi} = \cos 3x + i \sin 3x$, and observe that the real part of the left-hand side is exactly the integral we are seeking. All we need to do now is to show that the real part of the integral over A_R vanishes as $R \to \infty$. But, since for a complex number w, $|\mathrm{Re}(w)| \le |w|$ we have

$$\left|\mathrm{Re}\left(\int_{A_R} f(z)e^{3iz}dz\right)\right| \le \left|\int_{A_R} f(z)e^{3iz}dz\right|,$$

and the same bound as in (7) can be established since on A_R we have

$$|e^{3iz}| = |e^{3i(x+iy)}| = |e^{-3y}e^{3xi}| = e^{-3y} \le 1,$$

since A_R is in the upper half-plane so that $y \ge 0$.

Remark.

(i) This approach generally works for many integrals of the form

$$\int_{-\infty}^{\infty} f(z) \cos az dz \quad \text{and} \quad \int_{-\infty}^{\infty} f(z) \sin az dz,$$

where $f(z)$ is a rational function (ratio of two polynomials), where degree of a polynomial in the denominator is larger than that of a polynomial in the numerator (although in some cases working out the bound on the integral over A_R may be more tricky than in the above example). The following inequality, known as Jordan's inequality, is often helpful (see Sec. 2.4 for an illustration as well as a proof)

$$\int_0^{\pi} e^{-R\sin t}dt < \frac{\pi}{R}. \tag{8}$$

(ii) The integrals involving sine rather than cosine are generally handled by comparing the imaginary parts. The example we considered would give

$$\int_{-\infty}^{\infty} \frac{\sin 3x}{(x^2+1)^2}dx = 0,$$

but that is trivially true since the integrand is an odd function, and, clearly, the improper integral

$$\int_0^{\infty} \frac{\sin 3x}{(x^2+1)^2}dx$$

converges. For a more meaningful examples see Exercises 4-6 at the end.

2.4 One more example of the same type.

Here we will show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \tag{9}$$

This integral is quite useful (e.g. in Fourier analysis and probability) and has an interesting twist, namely a choice of a contour (that aspect is, by the way, one more thing to keep in mind; clever choice of a contour may make wonders).

Based on our knowledge form the previous section we should consider $f(z) = e^{iz}/z$ and try to integrate along our usual contour C_R considered in Sections 2.2 and 2.3. The only problem is that our function $f(z)$ has a singularity on the contour, namely at $z = 0$. To avoid that problem we will make a small detour around $z = 0$. Specifically, consider pick a (small) $\rho > 0$ and consider a contour consisting of the arc A_R that we considered in the last two sections followed by a line segment along a real axis between $-R$ and $-\rho$, followed by an upper semi-circle centered at 0 with radius ρ and, finally, a line segment along the positive part of the real exit from ρ to R (draw a picture to see what happens). We will call this contour $B_{R,\rho}$ and we denote the line segments by L_- and L_+, respectively and the small semi-circle by A_ρ. Since our function is analytic with $B_{R,\rho}$ its integral along this contour is 0. That is

$$\int_{A_R} \frac{e^{iz}}{z} dz = 0 + \int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{A_\rho} \frac{e^{iz}}{z} dz + \int_\rho^R \frac{e^{ix}}{x} dx,$$

Since

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx = -\int_\rho^R \frac{e^{-ix}}{x} dx,$$

by combining the second and the fourth integral we obtain

$$\int_\rho^R \frac{e^{ix} - e^{-ix}}{x} dx + \int_{A_R} \frac{e^{iz}}{z} dz + \int_{A_\rho} \frac{e^{iz}}{z} dz = 0.$$

Since the integrand in the leftmost integral is

$$2i \frac{e^{ix} - e^{-ix}}{2ix} = 2i \frac{\sin x}{x},$$

we obtain

$$2i \int_\rho^R \frac{\sin x}{x} dx = -\int_{A_R} \frac{e^{iz}}{z} dz - \int_{A_\rho} \frac{e^{iz}}{z} dz = 0,$$

or upon dividing by $2i$

$$\int_\rho^R \frac{\sin x}{x} dx = \frac{i}{2} \left(\int_{A_R} \frac{e^{iz}}{z} dz + \int_{A_\rho} \frac{e^{iz}}{z} dz \right).$$

The plan now is to show that the integral over A_R vanishes as $R \to \infty$ and that

$$\lim_{\rho \to 0} \int_{A_\rho} \frac{e^{iz}}{z} dz = -\pi i. \tag{10}$$

To justify this last statement use the usual parameterization of A_ρ: $z = \rho e^{it}$, $0 \le t \le \pi$. Then $dz = i\rho e^{it} dt$ and notice (look at your picture) that the integral over A_ρ is supposed to be clockwise (i.e. in the *negative* direction. Hence,

$$-\int_{A_\rho} \frac{e^{iz}}{z} dz = \int_0^\pi \frac{e^{i\rho e^{it}} i\rho e^{it}}{\rho e^{it}} dt = i \int_0^\pi e^{i\rho e^{it}} dt.$$

Thus to show (10) it suffices to show that

$$\lim_{\rho \to 0} \int_0^\pi e^{i\rho e^{it}} dt = \pi.$$

To this end let us look at

$$\left| \int_0^\pi e^{i\rho e^{it}} dt - \pi \right| = \left| \int_0^\pi e^{i\rho e^{it}} dt - \int_0^\pi dt \right| = \left| \int_0^\pi \left(e^{i\rho e^{it}} - 1 \right) dt \right|.$$

We will want to use once again inequality (7). Since the length of the curve is π we only need to show that

$$|e^{i\rho e^{it}} - 1| \longrightarrow 0, \quad \text{as} \quad \rho \to 0. \tag{11}$$

But we have

$$\begin{aligned} |e^{i\rho e^{it}} - 1| &= |e^{i\rho(\cos t + i \sin t)} - 1| = |e^{-\rho \sin t} e^{i\rho \cos t} - 1| \\ &= |e^{-\rho \sin t} e^{i\rho \cos t} - e^{-\rho \sin t} + e^{-\rho \sin t} - 1| \\ &\le e^{-\rho \sin t} |e^{i\rho \cos t} - 1| + |e^{-\rho \sin t} - 1| \\ &\le |e^{i\rho \cos t} - 1| + |e^{-\rho \sin t} - 1|. \end{aligned}$$

For $0 \le t \le \pi$, $\sin t \ge 0$ so that $e^{-\rho \sin t} \le 1$ and thus the second absolute value is actually equal to $1 - e^{-\rho \sin t}$ which is less than $\rho \sin t$, since for any real u, $1 - u \le e^{-u}$ (just draw the graphs of these two functions).

We will now bound $|e^{i\rho \cos t} - 1|$. For any real number v we have

$$|e^{iv} - 1|^2 = (\cos v - 1)^2 + \sin^2 v = \cos^2 v + \sin^2 v - 2\cos v + 1 = 2(1 - \cos v).$$

We will show that

$$1 - \cos v \le \frac{v^2}{2} \tag{12}$$

If we knew that, then (11) would follow since we would have

$$|e^{i\rho e^{it}} - 1| \le |e^{i\rho \cos t} - 1| + |e^{-\rho \sin t} - 1| \le \rho(|\cos t| + |\sin t|) \le 2\rho \to 0.$$

But the proof of (12) is an easy exercise in calculus: let

$$h(v) := \frac{v^2}{2} + \cos v - 1.$$

Than (12) is equivalent to

$$h(v) \geq 0 \quad \text{for} \quad v \in \mathbf{R}.$$

Since $h(0) = 0$, it suffices to show that $h(v)$ has a global minimum at $v = 0$. But

$$h'(v) = v - \sin v \quad \text{so that} \quad h'(0) = 0, \quad \text{and} \quad h''(v) = 1 - \cos v \geq 0.$$

That means that $v = 0$ is a critical point and $h(v)$ is convex. Thus, it has to be the mininimum of $h(v)$.

All that remains to show is that

$$\lim_{R\to\infty} \int_{A_R} \frac{e^{iz}}{z} dz = 0. \tag{13}$$

This is the place where Jordan's inequality (8) comes into picture. Going once again through the routine of changing variables to $z = Re^{it}$, $0 \leq t \leq \pi$, we obtain

$$\begin{aligned}\left|\int_{A_R} \frac{e^{iz}}{z} dz\right| &= \left|i \int_0^\pi e^{iRe^{it}} dt\right| \leq \int_0^\pi \left|e^{iR(\cos t + i\sin t)}\right| dt \\ &= \int_0^\pi \left|e^{iR\cos t}\right| \cdot e^{-R\sin t} dt \leq \int_0^\pi e^{-R\sin t} dt \leq \frac{\pi}{R},\end{aligned}$$

where the last step follows from Jordan's inequality (8). It is now clear that (13) is true.

It remains to prove (8). To this end, first note that

$$\int_0^\pi e^{-R\sin t} dt = 2\int_0^{\pi/2} e^{-R\sin t} dt.$$

That's because the graph of $\sin t$ and (thus also of $e^{-R\sin t}$) is symmetric about the vertical line at $\pi/2$. Now, since the function $\sin t$ is concave between $0 \leq t \leq \pi$, its graph is above the graph of a line segment joining $(0, 0)$ and $(\pi/2, 1)$; in other words

$$\sin t \geq \frac{2t}{\pi}.$$

Hence

$$\int_0^{\pi/2} e^{-R\sin t} dt \leq \int_0^{\pi/2} e^{-\frac{2R}{\pi}t} dt = \frac{\pi}{2R}\left(1 - e^{-R}\right) \leq \frac{\pi}{2R}$$

which proves Jordan's inequality.

3 Summation of series.

As an example we will show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{14}$$

but as we will see the approach is fairly universal.

Let for a natural number N C_N be a positively oriented square with vertices at $(\pm 1 \pm i)(N+1/2)$ and consider the integral

$$\int_{C_N} \frac{\cos \pi z}{z^2 \sin \pi z} dz. \tag{15}$$

Since $\sin \pi z = 0$ for $z = 0, \pm 1, \pm 2, \ldots$ the integrand has simple poles at $\pm 1, \pm 2, \ldots$ and a pole of degree three at 0. The residues at the simple poles are

$$\lim_{z \to k} \frac{(z-k)\cos \pi z}{z^2 \sin \pi z} = \frac{\cos \pi k}{k^2} \lim_{z \to k} \frac{\pi (z-k)}{\pi \sin(\pi(z-k))} = \frac{1}{\pi k^2}.$$

The residue at the triple pole $z = 0$ is $-\pi/3$. To see that you can either ask Maple to find it for you by typing:

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residue(cos(Pi*z)/sin(Pi*z)/z^2,z=0);
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in your Maple session or compute the second derivative (see Sec. 10.2 of Cain's notes again) of

$$\frac{1}{2!} z^3 \frac{\cos \pi z}{z^2 \sin \pi z} = \frac{1}{2} \frac{z \cos \pi z}{\sin \pi z},$$

and evaluate at $z = 0$, or else use the series expansions for the sine and cosine, and figure out from there, what' s the coefficient in front of $1/z$ in the Laurent series for

$$\frac{\cos \pi z}{z^2 \sin \pi z}$$

around $z_0 = 0$. Applying residue theorem, and collecting the poles within the contour C_N we get

$$\int_{C_N} \frac{\cos \pi z}{z^2 \sin \pi z} dz = 2\pi i \left(-\frac{\pi}{3} + \sum_{k=1}^{N} \frac{1}{\pi k^2} + \sum_{k=-1}^{-N} \frac{1}{\pi k^2} \right) = 2\pi i \left(-\frac{\pi}{3} + 2 \sum_{k=1}^{N} \frac{1}{\pi k^2} \right).$$

The next step is to show that as $N \to \infty$ the integral on the left converges to 0. Once this is accomplished, we could pass to the limit on the right hand side as well, obtaining

$$\lim_{N \to \infty} \left(\frac{2}{\pi} \sum_{k=1}^{N} \frac{1}{k^2} \right) = \frac{\pi}{3},$$

which is equivalent to (14).

In order to show that the integral in (15) converges to 0 as $N \to \infty$, we will bound

$$\left| \int_{C_N} \frac{\cos \pi z}{z^2 \sin \pi z} dz \right|$$

using our indispensable inequality (7).

First, observe that each side of C_N has length $2N+1$ so that the length of the contour C_N is bounded by $8N+4 = O(N)$. On the other hand, for $z \in C_N$,

$$|z| \geq N + \frac{1}{2} \geq N,$$

so that

$$\left|\frac{1}{z^2}\right| \leq \frac{1}{N^2},$$

so that it would be (more than) enough to show that $\cos \pi z / \sin \pi z$ is bounded on C_N by a constant independent of N. By Exercise 9 from Cain's notes

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y,$$

where $z = x + iy$. Hence

$$\begin{aligned} \left|\frac{\cos \pi z}{\sin \pi z}\right|^2 &= \frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y} = \frac{\cos^2 \pi x}{\sin^2 \pi x + \sinh^2 \pi y} + \frac{\sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y} \\ &\leq \frac{\cos^2 \pi x}{\sin^2 \pi x + \sinh^2 \pi y} + 1. \end{aligned}$$

On the vertical lines of the contour C_N $x = \pm(N+1/2)$ so that $\cos(\pi x) = 0$ and $\sin(\pi x) = \pm 1$. Hence the first term above is 0. We will show that on the horizontal lines the absolute value of $\sinh \pi y$ is exponentially large, thus making the first term exponentially small since sine and cosine are bounded functions. For $t > 0$ we have

$$\sinh t = \frac{e^t - e^{-t}}{2} \geq \frac{e^t - 1}{2}.$$

Similarly, for $t < 0$

$$\sinh t = \frac{e^t - e^{-t}}{2} \leq \frac{1 - e^{-t}}{2} = -\frac{e^{-t} - 1}{2},$$

so that, in either case

$$|\sinh t| \geq \frac{e^{|t|} - 1}{2}.$$

Since on the upper horizontal line $|y| = N + 1/2$ we obtain

$$|\sinh \pi y| \geq \frac{e^{\pi(N+1/2)} - 1}{2} \to \infty,$$

as $N \to \infty$. All of this implies that there exists a positive constant K such that for all $N \geq 1$ and all for $z \in C_N$

$$\left|\frac{\cos \pi z}{\sin \pi z}\right| \leq K.$$

Hence, we conclude that

$$\left|\int_{C_N} \frac{\cos \pi z}{z^2 \sin \pi z} dz\right| \leq K \frac{8N+4}{N^2},$$

which converges to 0 as $N \to \infty$. This establishes (14).

The above argument is fairly universal and applies generally to the sums of the form

$$\sum_{n=-\infty}^{\infty} f(n).$$

Sums from 0 (or 1) to infinity are then often handled by using a symmetry of $f(n)$ or similarly simple observations. The crux of the argument is the following observation:

Proposition 3.1 *Under mild assumptions on $f(z)$,*

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum \operatorname{Res}\left(f(z)\frac{\cos \pi z}{\sin \pi z}\right),$$

where the sum extends over the poles of $f(z)$.

In our example we just took $f(z) = 1/z^2$.

Variations of the above argument allow us to handle other sums. For example, we have

Proposition 3.2 *Under mild assumptions on $f(z)$,*

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum \operatorname{Res}\left(\frac{f(z)}{\sin \pi z}\right),$$

where the sum extends over the poles of $f(z)$.

4 Exercises.

1. Evaluate $\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}dx$. Answer: $\pi/4$.

2. Evaluate $\int_{-\infty}^\infty \frac{x}{(x^2+1)(x^2+2x+2)}dx$. Answer: $-\pi/5$.

3. Evaluate $\int_0^\infty \frac{dx}{x^4+1}$. Answer: $\frac{\pi}{2\sqrt{2}}$.

4. Evaluate $\int_{-\infty}^\infty \frac{\sin x}{x^2+4x+5}dx$. Answer: $-\frac{\pi}{e}\sin 2$.

5. Evaluate $\int_{-\infty}^\infty \frac{x\sin x}{x^4+4}dx$. Answer: $\frac{\pi}{2e}\sin 1$.

6. Evaluate $\int_{-\infty}^\infty \frac{x\sin \pi x}{x^2+2x+5}dx$. Answer: $-\pi e^{-2\pi}$.

7. Evaluate $\int_{-\pi}^{\pi} \frac{dt}{1+\sin^2 t}$. Answer: $\sqrt{2}\pi$.

8. Evaluate $\int_0^{2\pi} \frac{\cos^2 3x}{5-4\cos 2x}dx$. Answer: $\frac{3}{8}\pi$.

9. Show that $\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$.

10. Use a suggestion of Proposition 3.2 to show that $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.