HILBERT-KUNZ MULTIPLICITY AND REDUCTION MOD p

V. TRIVEDI

In this paper, we show that the Hilbert-Kunz multiplicities of the reductions to positive characteristics of an irreducible projective curve in characteristic 0 have a well-defined limit as the characteristic tends to ∞.

1. An estimate for the HK multiplicity of a curve

Let X be a nonsingular projective curve over an algebraically closed field k of characteristic $p > 0$.

We fix the following notations for a vector bundle V on X. If

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_t \subset F_{t+1} = V$$

is the Harder-Narasimhan filtration (or HN filtration) then we denote

$$\underline{\mu}(F_i) = \mu(F_i/F_{i-1}) \text{ and } \overline{\mu}(F_i) = \mu(F_{i+1}/F_i).$$

Now throughout the section we fix a vector bundle V of rank r with its HN filtration

$$0 = \widetilde{E}_0 \subset \widetilde{E}_1 \subset \widetilde{E}_2 \subset \cdots \subset \widetilde{E}_l \subset \widetilde{E}_{l+1} = V.$$

Let $\mu_i = \mu(\widetilde{E}_{i+1}/\widetilde{E}_i)$ and let $\mu = \mu(V)$, then the definition of HN filtration implies that

$$\mu_1 > \mu_2 \cdots > \mu_{l+1}$$

and for some $1 \leq i \leq l$ we have $\mu_i \geq \mu \geq \mu_{i+1}$.

Lemma 1.1. *Suppose the characteristic p satisfies $p > 4(g-1)r!$. Then*

$$F^*\widetilde{E}_1 \subset F^*\widetilde{E}_2 \subset \cdots \subset F^*\widetilde{E}_l \subset F^*V$$

is a subfiltration of the HN filtration of F^*V.

Proof. For each $0 \leq i \leq l+1$, let

$$F^*\widetilde{E}_i \subset E_{i1} \subset \cdots \subset E_{it_i} \subset F^*\widetilde{E}_{i+1}$$

be a filtration of vector bundles on X such that

$$0 \subset E_{i1}/F^*\widetilde{E}_i \subset E_{i2}/F^*\widetilde{E}_i \subset \cdots \subset F^*\widetilde{E}_{i+1}/F^*\widetilde{E}_i$$

is the HN filtration of $F^*(\widetilde{E}_{i+1}/\widetilde{E}_i)$. Then by [SB], we have

$$0 \leq \mu_{max}F^*(\widetilde{E}_{i+1}/\widetilde{E}_i) - \mu_{min}F^*(\widetilde{E}_{i+1}/\widetilde{E}_i) \leq (2g-2)(r-1). \tag{1.1}$$

Since

$$\mu_{max}F^*(\widetilde{E}_{i+1}/\widetilde{E}_i) \geq \mu(F^*(\widetilde{E}_{i+1}/\widetilde{E}_i)) \geq \mu_{min}F^*(\widetilde{E}_{i+1}/\widetilde{E}_i),$$

we have

$$0 \leq \mu(E_{i1}/F^*\widetilde{E}_i) - \mu(F^*(\widetilde{E}_{i+1}/\widetilde{E}_i)) \leq (2g-2)(r-1) \tag{1.2}$$

$$0 \leq \mu(F^*(\widetilde{E}_i/\widetilde{E}_{i-1})) - \mu(F^*(\widetilde{E}_i)/E_{i-1t_{i-1}}) \leq (2g-2)(r-1). \tag{1.3}$$

This implies

$$-2(2g-2)(r-1) + p(\mu_i - \mu_{i+1}) \leq \mu(F^*(\widetilde{E}_i)/E_{i-1t_{i-1}}) - \mu(E_{i1}/F^*\widetilde{E}_i) \leq p(\mu_i - \mu_{i+1}). \tag{1.4}$$

Preliminary version.

Now, since $p > 4(g-1)r!$, this implies that

$$\mu(F^*(\widetilde{E}_i)/E_{i-1t_{i-1}}) > \mu(E_{i1}/F^*\widetilde{E}_i),$$

Moreover, by construction

$$\mu(E_{ij}/E_{ij-1}) > \mu(E_{ij+1}/E_{ij}), \text{ for all } 0 \leq i \leq l \text{ and for all } j.$$

Hence

$$0 \subset E_{01} \subset \cdots \subset E_{0t_0} \subset F^*\widetilde{E}_1 \subset \cdots \subset F^*\widetilde{E}_i \subset E_{i1} \subset \cdots \subset E_{it_i} \subset F^*\widetilde{E}_{i+1} \subset \cdots \subset F^*V$$

is the HN filtration of F^*V. This proves the lemma. □

The following corollary is easy, from the above lemma, the estimate (1.4), and induction on s.

Corollary 1.2. *Assume that the characteristic p satisfies $p > 4(g-1)r!$, and let*

$$0 \subset F_1 \subset \cdots \subset F_{t+1} = F^{s*}V$$

be the HN filtration of $F^{s*}V$.

(1) *If F_j is the pullback of one of the subbundles in the HN filtration of V, i.e. if $F_j = F^{s*}\widetilde{E}_i$, for some $1 \leq i \leq l$, then*

$$-2(2g-2)(r-1)(1+p+\cdots+p^{s-1}) + p^s(\mu_i - \mu_{i+1}) \leq \underline{\mu}(F_j) - \overline{\mu}(F_j) \leq p^s(\mu_i - \mu_{i+1}) \tag{1.5}$$

$$-(2g-2)(r-1)(1+p+\cdots+p^{s-1})+p^s(\mu_i+\mu_{i+1}) \leq \underline{\mu}(F_j)+\overline{\mu}(F_j) \leq (2g-2)(r-1)(1+p+\cdots+p^{s-1})+p^s(\mu_i+\mu_{i+1}). \tag{1.6}$$

(2) *If F_j is not the pullback of any $\widetilde{E}_i$ (i.e., F_j is not equal to $F^{s*}\widetilde{E}_i$ for any i), then for some $k < s$, we have*

$$0 \leq \underline{\mu}(F_j) - \overline{\mu}(F_j) \leq p^k(2g-2)(r-1) \tag{1.7}$$

$$-2(2g-2)(r-1)(1+p+\cdots+p^k)+2p^{k+1}\mu_{l+1} \leq \underline{\mu}(F_j)+\overline{\mu}(F_j) \leq 2(2g-2)(r-1)(1+p+\cdots+p^k)+2p^{k+1}\mu_1. \tag{1.8}$$

Recall that if R is a Noetherian commutative ring of prime characteristic p, and $I \subseteq R$ an ideal for which $\ell(R/I)$ is finite, then one defines the *Hilbert-Kunz multiplicity* of R with respect to I as

$$HKM(R,I) = \lim_{n\to\infty} \frac{\ell(R/I^{(p^n)})}{p^{nd}}$$

where

$$\begin{aligned} I^{(p^n)} &= n\text{-th Frobenius power of } I \\ &= \text{ideal generated by } p^n\text{-th powers of elements of } I. \end{aligned}$$

Let $\mathcal{L}$ be a base-point free line bundle on X of degree $\widetilde{d} > 0$, and let $W \subseteq H^0(X,\mathcal{L})$ which generates $\mathcal{L}$. Let $HKM(X,\mathcal{L},W)$ denote the HK multiplicity of the section ring $R = \oplus_{n\geq 0}H^0(X,\mathcal{L}^{\otimes n})$ with respect to the ideal generated by W, i.e., the ideal $W.R$. Consider the corresponding exact sequence

$$0 \to V \to W \otimes \mathcal{O}_X \to \mathcal{L} \to 0,$$

where V is a vector-bundle of rank $r = (\dim\ W - 1)$ and degree $-\widetilde{d} < 0$. Let

$$0 = \widetilde{E}_0 \subset \widetilde{E}_1 \subset \widetilde{E}_2 \subset \cdots \subset \widetilde{E}_l \subset \widetilde{E}_{l+1} = V,$$

be the HN filtration of V and $d_i = \deg\ \widetilde{E}_i$, $r_i = \text{rank}\ \widetilde{E}_i$, $\mu_i = \mu(\widetilde{E}_{i+1}/\widetilde{E}_i)$ and $d = \deg\ V = -\widetilde{d}$ and $\mu = \mu(V)$. Now by the theorem of Langer [L], there exists $s \geq 0$ such that $E = F^{s*}(V)$ has a strongly stable Harder-Narasimhan filtration, say,

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_t \subset F_{t+1} = F^{s*}V.$$

Let $\widetilde{d}_i = \deg\ F_i$, $\widetilde{r}_i = \text{rank}\ F_i$, $\widetilde{\mu}_i = \mu(F_{i+1}/F_i)$ and $\bar{d} = \deg\ F^{s*}V = -p^s\widetilde{d}$ and $\widetilde{\mu} = \mu(F^{s*}V)$. Then we have shown in [T] (see also [B]) that

$$HKM(X,\mathcal{L},W) = \frac{\widetilde{d}(r+1)}{2r} + \frac{1}{p^{2s}\widetilde{d}}\left[(\widetilde{\mu}_{t+1}-\widetilde{\mu})(\bar{d} - \frac{r(\widetilde{\mu}_{t+1}+\widetilde{\mu})}{2}) + \sum_{i=1}^{t}(\widetilde{\mu}_i - \widetilde{\mu}_{i+1})(\widetilde{d}_i - \frac{\widetilde{r}_i(\widetilde{\mu}_i+\widetilde{\mu}_{i+1})}{2})\right].$$

We remark here that, provided the characteristic p satisfies $p > 4(g-1)r!$, the existence of the strong HN filtration (Langer's theorem) follows easily from lemma 1.

Now we prove the following theorem. Here p is as in lemma 1.1.

Theorem 1.3. *With the notation as above, if $p > 4(g-1)r!$, then*

$$HKM(X,\mathcal{L},W) = \frac{\widetilde{d}(r+1)}{2r} + \frac{1}{\widetilde{d}}\left[(\mu_{l+1}-\mu)(d - \frac{r(\mu_{l+1}+\mu)}{2}) + \sum_{i=1}^{l}(\mu_i - \mu_{i+1})(d_i - \frac{r_i(\mu_i+\mu_{i+1})}{2})\right] + \frac{C}{p},$$

where $|C| \leq A(\max\{r^2, g^2, \widetilde{d}, |\mu_i|\})^2$, for some absolute constant A.

Proof. Let $M = \max\{r^2, g^2, \widetilde{d}, |\mu_i|\}$. We write $O(1)$ to denote any number bounded in absolute value by BM, where B is an absolute constant.

Case (1) Suppose $\widetilde{\mu}_j = \mu(F_j)$, such that F_j descends to some E_i, that is, $F_j = F^{s*}(E_i)$. Then by corollary 1.2 above

$$\widetilde{\mu}_j - \widetilde{\mu}_{j+1} = \underline{\mu}(F_j) - \overline{\mu}(F_j) = p^s(\mu_i - \mu_{i+1}) + C_1,$$

$$\widetilde{\mu}_j + \widetilde{\mu}_{j+1} = \underline{\mu}(F_j) + \overline{\mu}(F_j) = p^s(\mu_i + \mu_{i+1}) + C_2,$$

where

$$(C_1/p^{s-1}) = O(1),\ (C_2/p^{s-1}) = O(1).$$

Moreover $\widetilde{d}_j/\widetilde{r}_j = p^s(d_i/r_i)$. Hence

$$\frac{1}{p^{2s}}(\widetilde{\mu}_j - \widetilde{\mu}_{j+1})(\widetilde{d}_j - \frac{\widetilde{r}_j(\widetilde{\mu}_j+\widetilde{\mu}_{j+1})}{2}) = (\mu_i - \mu_{i+1})(d_i - \frac{r_i(\mu_i+\mu_{i+1})}{2}) + \frac{C_3}{p},$$

where $|C_3| < B_1M^2$ for an absolute constant B_1.

Case (2) If F_j does not descend to any E_i then, by corollary 1.2 above

$$0 \leq \widetilde{\mu}_j - \widetilde{\mu}_{j+1} = \underline{\mu}(F_j) - \overline{\mu}(F_j) = p^k(2g-2)(r-1) \text{ and}$$

$$-C_4 + 2p^{k+1}\mu_{l+1} \leq \widetilde{\mu}_j + \widetilde{\mu}_{j+1} \leq 2p^{k+1}\mu_1 + C_4,$$

where

$$p^s\mu_{l+1} \leq \widetilde{d}_j/\widetilde{r}_j \leq \widetilde{\mu}_1 \leq (2g-2)(r-1)(1+p+\cdots+p^{s-1}) + p^s\mu_1 = p^s\mu_1 + C_5$$

where $k+1 \leq s$, and

$$(C_4/p^{s-1}) = O(1),\ (C_5/p^{s-1}) = O(1).$$

This implies

$$\left|\frac{1}{p^{2s}}(\widetilde{\mu}_j - \widetilde{\mu}_{j+1})(\widetilde{d}_j - \frac{\widetilde{r}_j(\widetilde{\mu}_j+\widetilde{\mu}_{j+1})}{2})\right| < \frac{B_2M^2}{p}$$

for some absolute constant B_2. Similarly

$$\frac{1}{p^{2s}}(\widetilde{\mu}_{t+1} - \widetilde{\mu})(\widetilde{d}_j - \frac{\widetilde{r}(\widetilde{\mu}_{t+1}+\widetilde{\mu})}{2}) = (\mu_{l+1} - \mu)(d - \frac{r(\mu_{l+1}+\mu)}{2}) + \frac{D}{p},$$

for some D with $|D| < B_3M^2$ for an absolute constant B_3, as $0 \leq \widetilde{\mu}_{t+1} - p^s\mu_{l+1} \leq (2g-2)r$, by equation (1.2) and $\widetilde{\mu} = p^s\mu$.

Combining the above estimates, this proves the theorem. □

2. Proof of the limit property

Let $(X, \mathcal{L}, W)$ be a base-point free linear system on a smooth projective curve X over a field k of characteristic 0, with $\mathcal{L}$ ample. Consider the associated exact sequence

$$0 \to V \to W \otimes \mathcal{O}_X \to \mathcal{L} \to 0,$$

which defines a vector bundle V on X. Here V is a vector-bundle of rank $r = (\dim\ W - 1)$ and degree $-\widetilde{d}$, where $\widetilde{d} = \deg\ \mathcal{L} > 0$. Let

$$0 = \widetilde{E}_0 \subset \widetilde{E}_1 \subset \widetilde{E}_2 \subset \cdots \subset \widetilde{E}_l \subset \widetilde{E}_{l+1} = V,$$

be the HN filtration of V and $d_i = \deg\ \widetilde{E}_i$, $r_i = \text{rank}\ \widetilde{E}_i$, $\mu_i = \mu(\widetilde{E}_{i+1}/\widetilde{E}_i)$ and $d = \deg\ V = -\widetilde{d}$ and $\mu = \mu(V)$.

We can choose a model of $(X, \mathcal{L}, W)$ over some finitely generated $\mathbb{Z}$-subsalgebra $A \subset k$, so that for all closed points $s \in S = \operatorname{Spec} A$, the corresponding curve X_s is a non-singular curve, with a line bundle $\mathcal{L}_s$ and base-point free linear system W_s. Then we have a corresponding model of V over the curve X_A, which restricts to the analogous bundle V_s on X_s. We may further assume that the HN filtration $\{\widetilde{E}_i\}$ of V on X is defined on the model V_A, and restricts to a filtration of V_s by subbundles, for each s. Under this reduction, the slopes of the respective quotients are preserved. Finally, by an openness property of semistable vector bundles ([Ma]), we may assume (after localizing A if necessary) that the resulting filtration of V_s on X_s is the HN filtration of V_s.

It is now clear that, if the residue field of s has sufficiently large characteristic p, then by Theorem 1.3, the HK multiplicty of $(X_s, \mathcal{L}_s, W_s)$ differs from

$$\frac{\widetilde{d}(r+1)}{2r} + \frac{1}{\widetilde{d}}\left[(\mu_{l+1} - \mu)(d - \frac{r(\mu_{l+1} + \mu)}{2}) + \sum_{i=1}^{l}(\mu_i - \mu_{i+1})(d_i - \frac{r_i(\mu_i + \mu_{i+1})}{2})\right]$$

by a quantity which is bounded by $\frac{1}{p}A$, where A depends only on the original data $(X, \mathcal{L}, W)$ in characteristic 0. This immediately implies the limit property.

We remark that, since the basic result underlying this is an estimate involving a vector bundle in characteristic $p >> 0$ (Corollary 1.2 above), this argument may be adapted to obtain a similar result for the reductions mod p of a 2-dimensional standard graded domain over a field of characteristic 0, with respect to the reductions mod p of any fixed homogeneous ideal of finite colength, using the formula for HK multiplicities in [B].

References

[B] Brenner, H., *The rationality of the Hilbert-Kunz multiplicity in graded dimension two*, preprint, (arXiv:math.AC/0402180v1).

[L] Langer, A., *Semistable sheaves in positive characteristic*, Ann. Math. 159 (2004).

[Ma] Maruyama, M., *Openness of a family of torsion free sheaves*, J. Math. Kyoto Univ. **16-3** (1976), 627-637.

[SB] Shepherd-Barron, N.I., *Semistability and reduction mod p*, Topology, 37 No.3, 659-664 (1998)

[T] Trivedi, V., *Semistabiltiy and HK multiplicities for curves*, preprint. (arXiv:math.AC/0402245v1)

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India

E-mail address: `vija@math.tifr.res.in`