Higher homotopy operations and the cohomology of diagrams

1. Background

Algebraic topology tries to answer problems of homotopy theory by translating them into algebra, using invariants such as the homotopy or cohomology groups of a space. These invariants, even when endowed with extra algebraic structure such as Whitehead products or Steenrod operations, rarely reflect fully the original homotopical information, and the additional data needed for a full invariant is usually not purely algebraic in character: for example, the differentials of a DGA in rational homotopy, k-invariants (cohomology classes of the successive Postnikov sections), higher Massey products, action of a suitable operad, and so on.

Higher homotopy and cohomology operations have a long history, starting with Toda brackets, Massey products, and Adem's secondary cohomology operations. They have been used with great effect in various computations (e.g., [Ad, BJM, MP]), but little has been done to organize them systematically, or even define them in general (beyond Spanier's attempts in [S1, S2]).

In practice, higher operations often appear in an algebraic form: as differentials in spectral sequences (e.g., in [Ad, Ch. 2]), as Ext classes (see [Mar, Ch. 16, 3]), and so on. Such descriptions often serve as a good way to actually compute the operations. Here we consider the operation itself as the intrinsic homotopy-theoretic "fact," which may manifest itself in different (seemingly unrelated) algebraic guises.

All such higher homotopy or cohomology operations can be described geometrically as obstructions to rectifying homotopy-commutative diagrams $\mathcal{A} : \Gamma \to \mathrm{ho}\,\mathcal{T}op$ for various indexing categories Γ (that is, making $\mathcal{A}$ strictly commute by finding a lift to $\hat{\mathcal{A}} : \Gamma \to \mathcal{T}op$). However, making this precise − including the exact conditions when the higher operations are *defined* in the first place − is not easy, and we know of no fully satisfactory general definition. However, the approach described in [BMa] appears sufficient for our purposes.

So far, there have been two significant attempts to categorize complete collections of (secondary) operations:

1. Adams, in [Ad, Ch. 3], first produced a general definition and classification of (stable) secondary cohomology operations. The definition was later generalized by Maunder to n-th order cohomology operations (see [Mau]). A reformulation of Adams' approach from a more categorical point of view appears in [Ba2].
2. A more ambitious project of H.-J. Baues and his collaborators associates a "quadratic" algebraic structure called the *secondary homotopy groups* to each topological space X; these encode, inter alia, the secondary homotopy operations acting on X (see, e.g., [BMu]). This approach has also been fruitful in dealing with the secondary cohomology structure, and has been used to calculate the Adams E_3-term for the sphere (cf. [BJ]).

Neither of these approaches is directly related to the route we propose to take here, though one of our goals is, of course, to explore how it is related to these and other existing work on higher operations.

2. Objectives and significance of research

Our overall objectives in this project are:

(a) To define geometrically (as general as possible) a class $\mathcal{C}_n$ of n-th order homotopy operations for which we have precise inductive conditions for the existence in terms of vanishing of lower order operations. Such a geometric definition should specify in particular the shape of the associated indexing categories Γ.

(b) To find a sub-class $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}_n)_{n=2}^{\infty}$ of such operations which suffices to determine the homotopy type of any space, as well as the homotopy class of all maps;

We may view this as a partial answer Quillen's question in [Q, I, 4.9]: what homotopy-invariant structure must we add to the homotopy category $\operatorname{ho}\mathcal{T}op$ in order to recover the full "homotopy theory" of $\mathcal{T}op$?

(c) To describe these operations, including the inductive conditions for existence, in terms of the (André-Quillen) cohomology of appropriate diagrams of Π-algebras (i.e., graded groups equipped with an action of compositions and Whitehead products).

We may view this as an algebraicization of the higher operations – in as much as we are willing to consider Π-algebras to be "algebraic".

(d) To understand what general structure exists on the set of all secondary operations (and eventually, on the set of all operations in $\tilde{\mathcal{C}}$).

(e) To show how these can be computed in favorable cases (say, for rational spaces, or in a small range of dimensions).

(f) To do all of the above also in the dual case of higher order (mod p) cohomology operations.

Note that a good part of this project may be carried out in an arbitrary model category, as long as it is equipped with a suitable set of "models" to play the role of the spheres in $\mathcal{T}op$ (see [DKSt], [BJT, §3] and [Bl6, §4]).

2.1. *Remark.* Higher order operations have always played an important practical role in homotopy theory – in fact, they are one of the main computational tools in Toda's seminal work on the homotopy groups of spheres (see [To]) – but little has been done to place them in a unifying conceptual framework. One rarely finds in the literature general results on such operations, such as Joel Cohen's result on the stable homotopy groups of spheres (cf. [Co, Thm. 5.12]), or the folk theorem on formality and the vanishing of Massey products (cf. [DGMS, P. 247]. Thus we feel that providing some overall framework for discussing such operations, however abstract, may serve a useful purpose.

Moreover, we also believe that despite the difficulties in carrying out calculations with the cohomology of Π-algebras in general (see, e.g., [BJT, §6]), our approach may actually be computationally feasible for rational spaces, where it reduces to the usual Chevalley-Eilenberg cohomology of (graded) Lie algebras over $\mathbb{Q}$.

2.2. **Higher categories.** Another approach to bridging the gap between the homotopy category and the full model category $\mathcal{T}op$ of topological spaces – or more precisely, the totality of homotopy-invariant information of $\mathcal{T}op$ – is through the concept of higher categories. These are intended to encode homotopies, homotopies between homotopies, and so on, constituting the basic ingredients used in defining the higher homotopy (and cohomology) operations. See, e.g., [Bae, Bat, L, Ta], and the various papers in [GK].

In some sense, our goal is to distill the data contained in an n-category $\mathcal{C}$ into "algebraic" invariants – if possible, enough of them to enable us to recover *all* the homotopy-invariant information in $\mathcal{C}$.

3. Methodology and plan of operation

Some start has already been made on the initial part of the program described above:

3.1. The class of higher operations. As noted above, we use the definition of [BMa] for higher homotopy (and cohomology) operations. The indexing category Γ for the diagrams in question must be a *lattice* in the sense of [BMa, §2.1] – that is, a finite directed category with weak initial and terminal objects – but this is no restriction for our purposes: namely, to define a full collection $(\mathcal{C}_n)_{n=2}^{\infty}$ as in §2 (a).

Note, however, that we require the pointed version (cf. [BMa, §3.10], which is not given there in detail, so some care is needed in using the results of [BMa] in our context.

In [Bl1] a certain set of higher operations were described which may serve as the complete invariants $\tilde{\mathcal{C}}$, since they suffice to determine the realizablity of any Π-algebra, and thus to distinguish all homotopy types of spaces (together with the homotopy Π-algebra $\pi_* X$ itself, of course). The indexing categories in this case are all of the same shape: final segments of the restricted simplicial category Δ. The collection $(\tilde{\mathcal{C}}_n)_{n=2}^{\infty}$ which serves as the complete invariants for homotopy types of spaces are given by indexing categories Γ of this type, and diagrams $\mathcal{A} : \Gamma \to \Pi$ (the homotopy category of wedges of spheres). to obtain complete invariants for maps, too, we must allow the terminal space of $\mathcal{A}$ to be an arbitrary space (see [Bl2]).

3.2. The obstructions. In [DKSm], Dwyer, Kan, and Jeff Smith define a series of obstructions to rectifying homotopy-commutative diagrams, taking value in the cohomology of a Postnikov system of the diagram in question.

Hopefully, one can show that the "algebraic" obstructions they define are actually in one-to-one correspondence with the above geometric definition; doing this in detail would be the first real step in the project.

3.3. Diagrams of Π-algebras. A less direct approach, also inspired by the work of Dwyer and Kan, is to first apply the functor π_* to $\mathcal{A}$ to produce a diagram of Π-algebras $\Lambda := \pi_* \mathcal{A} : \Gamma \to \Pi\text{-}\mathcal{A}lg$, and then try to realize Λ by a diagram in $\mathcal{T}op_*$. In general we lose a great deal of information in this way: the whole point of [BDG] is to explain how to recover a space X from its Π-algebra $\pi_* X$. Nevertheless, when dealing with $(\tilde{\mathcal{C}}_n)_{n=2}^{\infty}$ of §3.1, where all spaces in the diagram are (wedges of) spheres, no loss is entailed by considering Λ, since $\pi_* : \Pi \to \Pi\text{-}\mathcal{A}lg$ is an equivalence of categories.

In [BJT], the authors showed that all obstructions to realizing any diagram $\Lambda : \Gamma \to \Pi\text{-}\mathcal{A}lg$ (and for distinguishing between two such realizations) lie in the cohomology of the diagram Λ (with suitable coefficients). The next step would then be to identify the higher operations of $\tilde{\mathcal{C}}$ with classes in the cohomology of the corresponding diagrams of Π-algebras.

3.4. A local-to-global spectral sequence. Unfortunately the cohomology of a diagram is often difficult to compute – see [BJT, §4] for the case of a single map – despite the fact that it has applications in various areas of mathematics: in the context of deformation theory (see [GS2, GS1, GGS]), and in classifying diagrams of groups (see [Ce]). See [BG], [DS], [FW], [O], [Pa], and [BC] for further applications.

One of our first objectives is to provide to a computational tool for calculating the cohomology of a diagram Γ through an appropriate "local-to-global spectral sequence", whose E^2-term is expressed in terms of an appropriate cover of Γ by smaller diagrams. From a preliminary study, we do not expect such a spectral sequence to be defined for arbitrary covers of a general diagram, but in our context it appears feasible. Moreover, the only difficulty is with the shape of the indexing category Γ, so the result should apply both to the Dwyer-Kan-Smith cohomology and to the Π-algebra version. In fact, such a spectral sequence would work for any generalized Quillen cohomology (see [Bl7], and compare [Q, II, §5]).

Note that the local-to-global spectral sequence of Jibladze and Pirashvili (cf. [JP]) uses a different definition of cohomology, based on the Baues-Wirsching and Hochschild-Mitchell cohomologies of categories (cf. [BW, Mit]). This is to related Quillen homology by a spectral sequence due to Schwede (see [Sc, §5.5]).

3.5. **Conditions for higher operations to be defined.** One serious lacuna in the approach of [BMa] is that there is no explicit treatment of the conditions for a higher operation to be *defined* – i.e., the "coherent vanishing of all lower-order operations". (The theory of [DKSm] is not directly relevant, since the induction there is on the (simplicial) dimension, rather than the complexity of the diagram.)

Assume now that we have a working identification of a higher operation with a class χ_Γ in the cohomology of the corresponding diagram Γ , as in §3.3. Once the spectral sequence of §3.4 has been set up, we would hope that the conditions for the higher order operation corresponding to χ_Γ to be defined may be expressed in terms of the vanishing of differentials on a set of lower-dimensional classes. In this way we hope to "algebraicize" the coherent vanishing condition; of course this approach is speculative at this point.

3.6. **Structure of the algebra of higher operations.** It seems unrealistic to expect the collection $\mathcal{C}$ of all higher homotopy operations to have any reasonable algebraic structure; however, the obvious operations of concatenating or grafting diagrams to one another yield certain operations on $\mathcal{C}$, in the spirit of [To, §1], so that one can at least make sense of Joel Cohen's theorem that the stable homotopy groups of spheres are generated under higher operations by certain elements.

3.7. **Special cases.** Even without any further structure on $\mathcal{C}$, there are a number of cases which might be more accessible to global calculation, in the sense that one might be able to say something meaningful about the totality of higher order (or at least, secondary) operations:

1. Since the primary rational homotopy operations consist only of Lie brackets, it makes sense to ask whether higher Whitehead products (see [Al, Po]) generate all higher order rational operations.
2. The dual question of whether all rational higher cohomology operations are generated by Massey products seems easier to approach, (inter alia, because generalized Eilenberg-Mac Lane spaces are abelian group objects). Perhaps this might be related to the A_∞-models for rational homotopy types (see, e.g., [MP]).

3. Baues, in [Ba2], has analyzed the set of all stable mod p secondary cohomology operations (see also [H, ch. 5]). A similar analysis of the unstable secondary operations might be possible, too. Again, this might be related to Mandell's E_∞-models for p-adic homotopy types (cf. [Man]).
4. Because truncated versions of $\Pi\text{-}\mathcal{A}lg$ (in the stable range) are well understood algebraically (cf. [BJT, §4, §6]), one could study the set of all stable secondary (or higher) homotopy operations in a restricted range as a useful test case.

3.8**. A more general version.** As noted in [BJT, Thm. 3.16]), the obstruction theory given there for diagrams of spaces is actually valid for diagrams over more general E^2-model categories (cf. [BJT, §3.12]). These include spectra, but also algebraic categories such as chain complexes, simplicial or differential graded algebras, and more. Thus our approach to higher homotopy operations should carry over to these contexts, where they have been studied in various connections by several authors (see, e.g., [BKS, FI, GV, R]).

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