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Estimates for Conformal Metric Ratios

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Abstract. We present uniform and pointwise estimates for various ratios of the hyperbolic, quasihyperbolic and Möbius metrics. We determine when these ratios are constant. We exhibit numerous illustrative examples.

Keywords. Möbius metric, Poincaré hyperbolic metric, quasihyperbolic.

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1. Introduction

Throughout this article Ω is a proper subdomain of the Riemann sphere $\hat{\mathbb{C}}$ possessing at least two boundary points. When such an Ω is a plane domain (i.e. $\Omega \subset \mathbb{C}$), one can study the domain constants

$$\sup \lambda\delta = \sup_{\Omega} \lambda\delta := \sup_{z\in\Omega} \lambda_\Omega(z)\delta_\Omega(z)$$

and

$$\inf \lambda\delta = \inf_{\Omega} \lambda\delta := \inf_{z\in\Omega} \lambda_\Omega(z)\delta_\Omega(z);$$

for example, see [FH99, HM92, Har90] and the references mentioned in these papers. Here $\lambda = \lambda_\Omega$ is the *Poincaré hyperbolic* metric-density in Ω (i.e. the scale factor or density for the maximal constant curvature -1 metric) and

$$\delta(z) = \delta_\Omega(z) := \operatorname{dist}(z, \partial\Omega)$$

denotes the Euclidean distance from a point z to the boundary $\partial\Omega$ of Ω. The product $\lambda(z)\delta(z)$ should be viewed as the ratio of the hyperbolic and quasihyperbolic metrics at $z \in \Omega$.

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In this paper we study the analogous Möbius invariant domain constants

$$\sup \lambda\mu^{-1} = \sup_{\Omega} \lambda\mu^{-1} = \sup_{z\in\Omega} \frac{\lambda_\Omega(z)}{\mu_\Omega(z)}$$

and

$$\inf \lambda\mu^{-1} = \inf_{\Omega} \lambda\mu^{-1} = \inf_{z\in\Omega} \frac{\lambda_\Omega(z)}{\mu_\Omega(z)},$$

where now Ω is a hyperbolic domain on the Riemann sphere and $\mu = \mu_\Omega$ is the Kulkarni-Pinkall metric-density in Ω. The reader recognizes that we are considering here the ratio of two conformal metrics, which is a well defined function; local coordinates should be used when Ω is not a plane region. The metric $\mu_\Omega(z)|dz|$ was introduced in [KP94] as a canonical metric for Möbius structures on n-dimensional manifolds. In [HMM03] we employed the definition given below (see Subsection 2.3) and established various properties of this metric (both from [KP94] and some new results too) using classical function theory.

In a certain sense, the Kulkarni-Pinkall metric is a Möbius invariant analog of the quasihyperbolic metric. In this connection, it provides finer estimates for the hyperbolic metric. This point of view is borne out by the results established here.

For any hyperbolic domain, $\sup \lambda\mu^{-1} \le 1$, and $\inf \lambda\mu^{-1} \ge 1/2$ in the simply connected case; see (3.1). For the Möbius image of a convex region, equality holds in the upper bound while the lower bound can be improved to $\inf \lambda\mu^{-1} \ge \pi/4$ and this is best possible; see Theorems 3.4 and 3.11 (d). In the opposite direction, any domain with $\inf \lambda\mu^{-1} \ge \sqrt{3}/2$ must be simply connected; see Theorem 3.6. In Theorem 3.7 we reveal a close connection between $\inf \lambda\mu^{-1}$ and the domain constant $\beta(\Omega)$ introduced by Harmelin (see [Har90, HM92]).

We identify the largest constant M with $\sup \lambda\mu^{-1} \ge M$ for all hyperbolic domains. We establish this and related estimates in Theorem 3.11. In this connection, we corroborate Hilditch's Conjecture that in any hyperbolic region we always have $\sup \lambda\delta \ge 8\pi^2/\Gamma^4(1/4)$; see Theorem 4.2.

For the sake of completeness, we also provide information regarding

$$\sup \mu\delta = \sup_{\Omega} \mu\delta = \sup_{z\in\Omega} \mu_\Omega(z)\delta_\Omega(z)$$

and

$$\inf \mu\delta = \inf_{\Omega} \mu\delta = \inf_{z\in\Omega} \mu_\Omega(z)\delta_\Omega(z);$$

see Theorems 2.2 and 2.3. Here once again $\Omega \subset \mathbb{C}$ possesses at least one finite boundary point. Notice that the ratio $\lambda\mu^{-1}$ is Möbius invariant whereas the quantities $\lambda\delta$ and $\mu\delta$ are merely affine invariant.

This document is organized as follows: Section 2 contains preliminary information including basic definitions and terminology as well as elementary and/or well-known facts. We exhibit examples and examine $\inf \lambda\mu^{-1}$ and $\sup \lambda\mu^{-1}$ in

Section 3. In Section 4 we verify Hilditch's Conjecture, identify the regions in which the various metric ratios are constant, and mention some related domain constants.

2. Preliminaries

2.1. General information. Our notation is relatively standard and, for the most part, conforms with that of [HMM03]. We work in the complex plane $\mathbb{C}$; stated results are valid for the Riemann sphere $\hat{\mathbb{C}}$ in terms of local coordinates as the reader may verify. The disk centered at the point a of radius r is denoted by $D(a;r)$ and we write $\mathbb{D} = D(0;1)$ for the unit disk. We also let $\mathbb{H}$ denote the upper half-plane, and for points $a, b \in \mathbb{C}$ put

$$\mathbb{C}_{ab} := \mathbb{C} \setminus \{a, b\}$$

and write λ_{ab} and μ_{ab} for the metric-densities in $\mathbb{C}_{ab}$.

The quantity

$$\delta(z) = \delta_\Omega(z) := \operatorname{dist}(z, \partial\Omega) = \operatorname{dist}(z, \partial\Omega \cap \mathbb{C})$$

is the Euclidean distance to the boundary of Ω, and $1/\delta$ is the density for the so-called *quasihyperbolic* metric $|dz|/\delta(z)$ when $\Omega \subset \mathbb{C}$. We call $\Omega \subset \hat{\mathbb{C}}$ a *quasi-hyperbolic* domain provided $\hat{\mathbb{C}} \setminus \Omega$ contains at least two points (one of which may be the point at infinity). We make frequent use of the notation

$$D(z) = D_\Omega(z) := D(z; \delta(z)) = D(z; \delta_\Omega(z)).$$

The reader should take care not to confuse the two disks $D(z)$ and $\Delta(z)$ (the latter is defined below in Subsection 2.3) associated with a point $z \in \Omega$.

2.2. The hyperbolic metric. When $\Omega \subset \hat{\mathbb{C}}$ has at least three boundary points, usually dubbed a *hyperbolic* domain, there exists a universal covering projection $f\colon \mathbb{D} \to \Omega$ and the density $\lambda = \lambda_\Omega$ of the Poincaré hyperbolic metric $\lambda_\Omega(z)|dz|$ is determined from

$$\lambda(z) = \lambda(f(\zeta)) := \frac{2}{(1 - |\zeta|^2)|f'(\zeta)|}.$$

It is well-known, and not difficult to check, that for all points $z \in \Omega \subset \mathbb{C}$,

$$\frac{1}{2} \le \lambda(z)\delta(z) \le 2; \tag{2.1}$$

the second inequality holds for any hyperbolic domain Ω whereas the first requires that Ω be simply connected. It is also known that equality holds in (2.1) at some point z if and only if Ω is a disk centered at z for the right-hand inequality, or, Ω is the complement of a ray and z lies on the ray of symmetry for the left-hand inequality. In [FH99] one finds characterizations for the hyperbolic domains with $\sup \lambda\delta < 2$ and descriptions of the simply connected hyperbolic domains satisfying $\inf \lambda\delta > 1/2$, as well as various estimates for these domain constants.

In the sequel we require the following information about two values of the hyperbolic metric density λ_{01} in the the twice punctured plane $\mathbb{C}_{01}$:

$$\lambda_{01}(1/2) = \frac{16\pi^2}{\Gamma^4(1/4)} = 0.913893\ldots$$

and

$$\lambda_{01}(\tau) = \frac{2^{2/3}\cdot 8\pi^3}{3\,\Gamma^6(1/3)} = 0.355082\ldots,$$

where $\tau = (1+i\sqrt{3})/2$. The elliptic modular function $J\colon \mathbb{H} \to \mathbb{C}_{01}$ satisfies $J(i) = 1/2$ and $J(\tau) = \tau$; of course, $\lambda_{01}(J(\zeta)) = [\mathrm{Im}(\zeta)|J'(\zeta)|]^{-1}$. Thus, the first formula follows from Nehari's calculation [Neh75, (97),p. 329] of $|J'(i)|$ and the second formula is a consequence of Carathéodory's computation [Car60, p. 193] of $|J'(\tau)|$. The decimal expressions are easily calculated using Mathematica (which also has the built-in `ModularLambda` function permitting a direct calculation of these values).

2.3. The Möbius metric. We define the density $\mu = \mu_\Omega$ for the Kulkarni-Pinkall Möbius metric $\mu_\Omega(z)|dz|$ as follows: for $z \in \Omega \cap \mathbb{C}$,

$$\mu(z) := \inf\left\{\lambda_D(z) : z \in D \subset \Omega,\ D \text{ is a disk on } \hat{\mathbb{C}}\right\}.$$

We follow the standard convention that a disk in $\hat{\mathbb{C}}$ is either a Euclidean disk, a Euclidean half-plane, or the complement of a closed Euclidean disk together with the point at infinity. Clearly, $\mu_\Omega(z)|dz|$ is defined (and positive) for any quasihyperbolic domain $\Omega \subset \hat{\mathbb{C}}$. The 'infimum' in this definition can be replaced by 'minimum'; see [HMM03]. This metric enjoys the usual domain monotonicity property, is Möbius invariant and complete with generalized curvature between 0 and -1, and is bilipschitz equivalent to the quasihyperbolic metric. For precise statements of these results, along with various other useful facts, we refer the interested reader to [HMM03] and/or [KP94]; but, see below as well.

We mention that for each $z \in \Omega$ (a quasihyperbolic domain in $\hat{\mathbb{C}}$) there is an associated *unique extremal disk* $\Delta = \Delta(z) = \Delta_\Omega(z) \subset \Omega$ with the property that

$$\mu_\Omega(z) = \lambda_\Delta(z).$$

The extremal disk $\Delta = \Delta(z)$ is either an open Euclidean disk, an open half-plane, or the exterior of a closed disk, and is characterized by the property that $K = \partial\Delta \cap \partial\Omega$ contains two or more points and z belongs to the so-called *hyperbolically convex hull* $\widehat{K}$ (of K in Δ) defined by

$$\widehat{K} = \bigcap\{H : H \subset \Delta, \text{ the spherical closure of } H \text{ contains } K\};$$

here H is a closed (relative to hyperbolic geometry on Δ) hyperbolic half-plane in Δ. As examples: (i) if $K = \{a, b\}$, $\widehat{K}$ is the hyperbolic geodesic in Δ ending at a and b; (ii) if $K = \{a, b, c\}$, $\widehat{K}$ is the closed ideal hyperbolic triangle in Δ

with vertices a, b, c; (iii) if K contains n points, $n \geq 3$, then $\widehat{K}$ is the closed ideal hyperbolic n-gon in Δ with vertices at the points of K; (iv) when $K = \partial\Delta$, we let $\widehat{K} = \Delta$. Moreover, it turns out that such a disk Δ is the extremal disk for each point of $\widehat{K}$.

For detailed information and proofs regarding extremal disks we refer to [HMM03, Thms. 3.4, 3.5, 3.7, 4.1, 4.2]. In particular, Theorems 3.4 and 4.2 provide explicit descriptions for the extremal disks (and formulae for the Kulkarni-Pinkall metric thereof) in the regions obtained by puncturing the Riemann sphere at two and three points, respectively.

We point out that if $z \in D \subset \Omega$ with D a disk in $\hat{\mathbb{C}}$, and z on the hyperbolic geodesic in D ending at points $a, b \in \partial D \cap \partial\Omega$, then

$$\mu_\Omega(z) = \lambda_D(z) = \mu_{ab}(z) = \frac{|a-b|}{|z-a||z-b|}.$$

2.4. Estimates for $\mu\delta$. As already mentioned, for plane domains the Kulkarni-Pinkall metric is bilipschitz equivalent to the quasihyperbolic metric. In particular we have the following explicit information.

Theorem 2.2. *Let $\Omega \subset \hat{\mathbb{C}}$ be a quasihyperbolic domain and fix $z \in \Omega \cap \mathbb{C}$. Then*

(a) $\mu(z) \leq \dfrac{2}{\delta(z)}$, *and equality holds if and only if $\Delta(z) = D(z)$;*

(b) $\mu(z) \leq \dfrac{\sqrt{2}}{\delta(z)}$ *if $\infty \in \Delta(z)$, and the constant $\sqrt{2}$ is best possible;*

(c) $\mu(z) \geq \dfrac{1}{\delta(z)}$ *if $\Delta(z) \subset \mathbb{C}$, and equality holds if and only if $\Delta(z)$ is a Euclidean half-plane H with $\delta_\Omega(z) = \delta_H(z)$.*

Proof. The inequalities in parts (a) and (c) were essentially verified in [HMM03, Thm. 3.3]. However, there we assumed that $\Omega \subset \mathbb{C}$. For the sake of completeness, we sketch the proofs. The inequality in (a) follows readily from the observation that $\mu_\Omega(z) \leq \lambda_D(z)$ when $D = D(z)$. If $\Delta(z) = D$, then equality clearly holds; if $\Delta(z) \neq D$, then $\mu(z) < \lambda_D(z) = 2/\delta(z)$.

The inequality in (c) follows exactly as in [HMM03, Thm. 3.3]. Moreover, the argument also reveals that equality forces $\Delta(z)$ to be a half-plane H with $\delta_\Omega(z) = \delta_H(z)$ as asserted. Clearly, if $\Delta(z)$ is such a half-plane H, then equality does hold.

Let us look at (b). We may assume $\Delta = \{z \in \hat{\mathbb{C}} : |z| > 1\} \subset \Omega$ is the extremal disk associated with some point $z \in (1,\infty) \cap \Omega$. Let $e^{\pm i\theta}$ be the points where the hyperbolic geodesic in Δ through z meets $\partial\Delta = \partial\mathbb{D}$; here we take $0 < \theta < \pi/2$ and find that said geodesic is a subarc of the circle centered at $c = \sec\theta$ and of radius $r = \tan\theta$ (so $z = \sec\theta + \tan\theta$). There must be a point of $\partial\delta \cap \partial\Omega$ on $\partial\mathbb{D} \cap \bar{D}(c; r)$ and a short calculation reveals that

$$\mu(z)\delta(z) \leq \frac{2|z - e^{i\theta}|}{|z|^2 - 1} = \sqrt{2}\frac{\cos\theta}{\sqrt{1 + \sin\theta}} \leq \sqrt{2}$$

as desired. To see that $\sqrt{2}$ is best possible, consider $\Omega_\theta = \hat{\mathbb{C}} \setminus \{-1, e^{i\theta}, e^{-i\theta}\}$ and notice that $\lim_{\theta\to 0}(\mu\delta_\theta)(\sec\theta + \tan\theta) = \sqrt{2}$ (where $\mu\delta_\theta = \mu_{\Omega_\theta}\delta_{\Omega_\theta}$). ■

Notice that (b) and (c) above provide improved estimates for λ in terms of δ (compared with (2.1)).

Next we present uniform estimates for the $\mu\delta$ ratio. To help understand part (c) below, we point out that for any $z \in \mathbb{C} \cap \Omega$,

$$\Delta(z) = D(z) \qquad \Longleftrightarrow \qquad \sup\{\ell(A) : A \subset \Omega \cap \partial D(z),\ A \text{ an arc}\} \leq \pi\delta(z).$$

The condition in (c) below is a uniform version of this; it is also a direct analog of a corresponding necessary and sufficient condition for $\sup\lambda\delta < 2$ to hold (see [FH99, Thm. 4.2]).

Theorem 2.3. *Let $\Omega \subset \hat{\mathbb{C}}$ be a quasihyperbolic domain. Then*

(a) $\inf\mu\delta = 1$ *if* $\Omega \subset \mathbb{C}$*;*
(b) $\inf_{\mathbb{C}\cap\Omega}\mu\delta = 0$ *if* $\infty \in \Omega$*;*
(c) $\sup_{\mathbb{C}\cap\Omega}\mu\delta < 2$ *if and only if there exists a constant* $\kappa > 0$ *such that for each* $z \in \mathbb{C} \cap \Omega$ *there is a semi-circle* $C \subset \Omega \cap \partial D(z)$ *with* $\operatorname{dist}(C, \partial\Omega) \geq \kappa\delta(z)$*.*

Proof. First we verify (b). Suppose ∞ belongs to Ω. Assume that $1 \in \partial\Omega$ and that $\Delta = \{z \in \hat{\mathbb{C}} : |z| > 1\} \subset \Omega$. Then for all $z \in \Delta \cap \mathbb{C}$, say with $z > 1$, we have $\delta_\Omega(z) = |z| - 1$ and so as $z \to \infty$,

$$\mu_\Omega(z)\delta_\Omega(z) \leq \lambda_\Delta(z)\delta_\Omega(z) = \frac{2}{|z|^2 - 1}(|z| - 1) = \frac{2}{|z| + 1} \to 0.$$

For (a), suppose $\Omega \subset \mathbb{C}$. Thanks to Theorem 2.2 (c), we know that $\inf\mu\delta \geq 1$. By fixing a disk $D = D(a) \subset \Omega$, choosing a point $b \in \partial D \cap \partial\Omega$, and letting z move along the line segment $[a, b)$ to b, we readily find that $\inf\mu\delta \leq 1$.

Finally, let us corroborate (c). The sufficiency of the stated condition follows from the calculations in the proof of [HMM03, Thm. 4.1(a)]. We leave the details to the interested reader. Suppose now that the stated condition fails to hold. Then there are points $z_n \in \mathbb{C} \cap \Omega$ with the property that each semi-circle $C \subset \partial D(z_n)$ has $\operatorname{dist}(C, \partial\Omega) \leq (1/n)\delta(z_n)$.

Select affine transformations φ_n sending z_n, $D(z_n)$, Ω to 0, $\mathbb{D}$, Ω_n respectively, and also such that $1 \in \partial\Omega_n$ for all n. Then each semi-circle $C \subset \mathbb{D}$ has

$\mathrm{dist}(C, \partial\Omega_n) \le 1/n$. Also, since $\delta_{\Omega_n}(0) = 1$, $\mu_\Omega(z_n)\delta_\Omega(z_n) = \mu_{\Omega_n}(0)$. We show that the latter has upper limit 2 as $n \to \infty$. Put

$$\alpha_n = \sup\{\theta : 0 \le \theta \le \pi,\ \mathrm{dist}(e^{i\theta}, \partial\Omega_n) \le 1/n\},$$
$$\beta_n = \inf\{\theta : \pi \le \theta \le 2\pi,\ \mathrm{dist}(e^{i\theta}, \partial\Omega_n) \le 1/n\}$$

and choose $a_n, b_n \in \partial\Omega_n$ with $|a_n - e^{i\alpha_n}| \le 1/n$, $|b_n - e^{i\beta_n}| \le 1/n$. Then for all $\theta \in (\alpha_n, \beta_n)$, $\mathrm{dist}(e^{i\theta}, \partial\Omega_n) > 1/n$; therefore $\beta_n - \alpha_n \le \pi$.

Extracting subsequences where necessary permits us to assume that as $n \to \infty$: $a_n, e^{i\alpha_n} \to a = e^{i\alpha}$ and $b_n, e^{i\beta_n} \to b = e^{i\beta}$. Then

$$\Omega_n \subset \hat{\mathbb{C}} \setminus \{1, a_n, b_n\} \to \hat{\mathbb{C}} \setminus \{1, a, b\} = \Omega'.$$

Now either $\alpha = \beta$ (in which case $a = -1 = b$) or $0 < \beta - \alpha \le \pi$ (and $a \ne b$). In both cases we see that Ω' is a quasihyperbolic domain and moreover, $\mathbb{D}$ is the extremal disk for Ω' associated with the origin (because $K = \partial\mathbb{D} \cap \partial\Omega' = \{1, a, b\}$ enjoys $0 \in \widehat{K}$). Thus, according to [HMM03, Thm. 3.9],

$$\mu_{\Omega_n}(0) \ge \mu_{\hat{\mathbb{C}}\setminus\{1,a_n,b_n\}}(0) \to \mu_{\Omega'}(0) = 2. \qquad \blacksquare$$

3. The ratio $\lambda\mu^{-1}$

In [HMM03, Thms. 3.6, 6.3] we demonstrated that for all points $z \in \Omega \subset \hat{\mathbb{C}}$,

$$\frac{1}{2} \le \frac{\lambda(z)}{\mu(z)} \le 1; \tag{3.1}$$

the second inequality holds for any hyperbolic domain Ω whereas the first requires that Ω be simply connected. It is also known that equality holds at some point z for the right-hand inequality in (3.1) if and only if Ω is a disk in $\hat{\mathbb{C}}$. Below we explain the equality situation for the left-hand inequality. Notice that (3.1) in conjunction with the inequalities from Theorem 2.2 imply the classical inequalities (2.1). In particular, we have

$$\frac{1}{2\delta} \le \frac{\mu}{2} \le \lambda \le \mu \le \frac{2}{\delta};$$

of course, the left-hand inequalities must be appropriately interpreted.

It is not hard to guess that equality should hold for the left-hand inequality in (3.1) if and only if Ω is a Möbius image of a slit plane. Observe that such a region is precisely $\hat{\mathbb{C}} \setminus A$ where A is a non-degenerate closed subarc of some circle C on $\hat{\mathbb{C}}$. For the reader's convenience, we sketch a proof.

Lemma 3.2. *For a simply connected hyperbolic domain $\Omega \subset \hat{\mathbb{C}}$, $\lambda(z) \ge \mu(z)/2$ for all $z \in \Omega$, and equality holds at a point $z \in \Omega$ if and only if $\Omega = \hat{\mathbb{C}} \setminus A$, where A is a non-degenerate closed subarc of some circle $C \subset \hat{\mathbb{C}}$, and z lies on $C \cap \Omega$.*

Proof. The domain $\mathbb{C}\setminus(-\infty,0]$ has $\lambda(z)\delta(z)=1/2$ for all $z>0$. The sufficiency of the above condition now follows from Möbius invariance in conjunction with $\lambda\delta\geq\lambda\mu^{-1}$. Let us verify the necessity. The stated inequality was established in [HMM03, Thm. 6.3] (see Remark 3.8(i) also), which in turn was based on Theorem 5.1 in that paper. We require its proof to validate our assertion concerning equality.

Assume Ω is a plane domain, $a\in\Omega$, and $f\colon(\mathbb{D},0)\to(\Omega,a)$ is a conformal map of the form $f(\zeta)=\zeta+a_2\zeta^2+\cdots$ with $0\leq a_2\leq 2$. Thus $a=f(0)=0$ and $\lambda(a)=2/|f'(0)|=2$. If $a_2=2$, then $f=k$ is the Koebe function and everything works; assume $0\leq a_2<2$. Given $b\in\mathbb{C}\setminus\Omega$,

$$g(\zeta)=\frac{bf(\zeta)}{b-f(\zeta)}=\zeta+(a_2+b^{-1})\zeta^2+\cdots\,,$$

belongs to the class $\mathcal{S}$ (of normalized univalent functions), so $|a_2+b^{-1}|\leq 2$. Now $b^{-1}\in\bar{D}(-a_2;2)$ tells us $b\notin D=D(c;r)$, where $c=a_2/(4-a_2{}^2)$ and $r=2/(4-a_2{}^2)$. Since this is valid for all points $b\in\mathbb{C}\setminus\Omega$, we have $a=0\in D\subset\Omega$, which yields

$$\mu(a)\leq\lambda_D(0)=4,\qquad\text{so}\qquad\lambda(a)\geq\frac{1}{2}\mu(a).$$

Suppose there is equality at a; i.e. that $\mu(a)=4$. Because of our normalization, we can check that $\Omega=\mathbb{C}\setminus\big((-\infty,-1/(2+a_2)]\cup[1/(2-a_2),+\infty)\big)$. Indeed, $\mu(a)=4$ implies that $\Delta(a)=D=D(c;r)$. In particular there must be some $b\in\partial D\cap\partial\Omega$. For this point b, $|a_2+b^{-1}|=2$, which in turn means that the associated function g is a 'rotation' of the Koebe function:

$$g(\zeta)=k_\theta(\zeta):=e^{i\theta}k(e^{-i\theta}\zeta)$$

where $e^{i\theta}(a_2+b^{-1})=2$. Therefore $f(\zeta)=bk_\theta(\zeta)/(b+k_\theta(\zeta))$, and so

$$\Omega=f(\mathbb{D})=\psi\left(\mathbb{C}\setminus\{-re^{i\theta}\,:\,r\geq 1/4\}\right)$$

where $\psi(z)=bz/(b+z)$. Since $\psi(-b)=\infty\notin\Omega$, $-b\notin k_\theta(\mathbb{D})$, so $b=re^{i\theta}$ for some $r\geq 1/4$. Straightforward calculations now reveal that $e^{i\theta}=\pm1$ and $\partial D\cap\partial\Omega=\{\psi(-e^{i\theta}/4),\psi(\infty)\}=\{-1/(a_2\pm 2)\}$. ■

3.1. Examples for $\lambda\mu^{-1}$. Here we mention a few simple examples where one can actually calculate $\inf\lambda\mu^{-1}$ and $\sup\lambda\mu^{-1}$. For the *infinite strip*

$$\Sigma_0:=\left\{x+iy\,:\,|y|<\frac{\pi}{2}\right\}$$

we have

$$\lambda(x+iy)=\sec(y)$$

and

$$\mu(x+iy)=\frac{\pi}{(\pi/2)^2-y^2};$$

from this we see that for $x+iy\in\Sigma_0$, $\lambda\mu^{-1}(x+iy)\geq\pi/4$ with equality if and only if $y=0$, and $\lim_{|y|\to\pi/2}\lambda\mu^{-1}(x+iy)=1$. Thus $\inf\lambda\mu^{-1}=\pi/4$ and $\sup\lambda\mu^{-1}=1$.

Next, we consider the *infinite sectors*

$$\Sigma_\alpha := \left\{ re^{i\theta} : r > 0, |\theta| < \frac{\alpha\pi}{2} \right\}$$

where $0 < \alpha \le 2$. Careful calculations reveal that for all $0 < \alpha \le 2$ we have $\sup \lambda\mu^{-1} = 1$, while $\inf \lambda\mu^{-1} = \alpha^{-1}\tan(\alpha\pi/4)$ for $0 < \alpha \le 1$ and $\inf \lambda\mu^{-1} = \alpha^{-1}$ for $1 \le \alpha \le 2$. We make repeated use of this information in the convex case, so let us indicate the calculations involved. In fact we have

$$\lambda(re^{i\theta}) = \frac{\sec(\theta/\alpha)}{\alpha r}$$

and

$$\mu(re^{i\theta}) = \frac{\sin(\alpha\pi/2)}{r[\cos(\theta) - \cos(\alpha\pi/2)]}.$$

A standard conformal change of variables gives the first formula, which holds for all $0 < \alpha \le 2$. The second formula is only valid for $0 < \alpha \le 1$ and can be established using the fact that $re^{i\theta}$ lies on the geodesic through the points $re^{\pm i\alpha}$. Thus for $0 < \alpha \le 1$ we obtain

$$\frac{\lambda(re^{i\theta})}{\mu(re^{i\theta})} = \frac{\cos(\theta) - \cos(\alpha\pi/2)}{\alpha\cos(\theta/\alpha)\sin(\alpha\pi/2)}.$$

The minimum is attained when $\theta = 0$. (For calculations, the two identities $2\sin[(\varphi+\psi)/2]\sin[(\varphi-\psi)/2] = \cos\psi - \cos\varphi$ and $(1-\cos\varphi)/\sin\varphi = \tan(\varphi/2)$ come in handy!)

Another simple, although important, example is the *punctured disk* $\mathbb{D} \setminus \{0\}$ for which

$$\frac{\lambda(z)}{\mu(z)} = \frac{1-|z|}{|\log|z||}$$

yielding

$$\inf \lambda\mu^{-1} = 0 \qquad \text{and} \qquad \sup \lambda\mu^{-1} = 1.$$

Similarly, $\inf \lambda\mu^{-1} = 0$ for any hyperbolic region possessing an isolated boundary point.

Finally, we take a close look at the *annulus*

$$A(R) := \left\{ z : \frac{1}{R} < |z| < R \right\}.$$

A routine computation produces

$$\lambda_{A(R)}(z) = \frac{\pi/2}{|z|\log R} \sec\left(\frac{\pi}{2}\frac{\log|z|}{\log R}\right)$$

and

$$\mu_{A(R)}(z) = \frac{R - 1/R}{(R-|z|)(|z|-1/R)}.$$

Letting $|z| \to R$ (or $|z| \to 1/R$) we find that $\sup \lambda\mu^{-1} = 1$, while evaluating on $|z| = 1$ gives

$$\inf \lambda\mu^{-1} \leq \frac{\pi/2}{\log R}\frac{R-1}{R+1}.$$

In fact, equality holds in the above as we now demonstrate. This assertion is equivalent to

$$\left(R + \frac{1}{R}\right) - \left(r + \frac{1}{r}\right) \geq \frac{(R-1)^2}{R}\cos\left(\frac{\pi}{2}\frac{\log r}{\log R}\right) \qquad \text{for } \frac{1}{R} \leq r \leq R.$$

To verify that this holds for $1 \leq r \leq R$ (which suffices), write $R = \exp(\rho)$ and change variables to $u = \log r/\log R$, so the above inequality becomes

$$\cosh(\rho) - \cosh(\rho u) \geq (\cosh(\rho) - 1)\cos\left(\frac{\pi u}{2}\right),$$

or equivalently,

$$\begin{aligned} 0 &\leq (\cosh(\rho) - 1)\left(1 - \cos\left(\frac{\pi u}{2}\right)\right) - (\cosh(\rho u) - 1) \\ &= 4\sinh^2\left(\frac{\rho}{2}\right)\sin^2\left(\frac{\pi u}{4}\right) - 2\sinh^2\left(\frac{\rho u}{2}\right) \\ &= 2\left(\sqrt{2}\sinh\left(\frac{\rho}{2}\right)\sin\left(\frac{\pi u}{4}\right) + \sinh\left(\frac{\rho u}{2}\right)\right) \\ &\quad \cdot \left(\sqrt{2}\sinh\left(\frac{\rho}{2}\right)\sin\left(\frac{\pi u}{2}\right) - \sinh\left(\frac{\rho u}{2}\right)\right). \end{aligned}$$

Thus our claim follows from the fact that, for all $0 \leq u \leq 1$,

$$\psi(u) = \sqrt{2}\sinh\left(\frac{\rho}{2}\right)\sin\left(\frac{\pi u}{4}\right) - \sinh\left(\frac{\rho u}{2}\right) \geq 0,$$

which in turn is a consequence of $\psi(0) = 0 = \psi(1)$ together with $\psi''(u) \leq 0$. The significance of these calculations is that we now have

$$\inf \lambda\mu^{-1} = \frac{\pi/2}{\log R}\frac{R-1}{R+1} \to \frac{\pi}{4} \qquad \text{as } R \to 1.$$

3.2. Estimating $\inf \lambda\mu^{-1}$. We remind the reader that a hyperbolic region is convex if and only if $\inf \lambda\delta = 1$. However given any $\varepsilon > 0$ there exists non-simply connected regions (thin annuli) with $\inf \lambda\delta \geq 1 - \varepsilon$; see [HM92, Thm. 4], [Hil84, Thm. 2.2, Ex. 2.4]. Here we present analogs of these facts for the $\lambda\mu^{-1}$ ratio. We demonstrate that $\inf \lambda\mu^{-1} \geq \pi/4$ for any Möbius image of a convex region. Then we show that any region with $\inf \lambda\mu^{-1} \geq \sqrt{3}/2$ must be simply connected. We complete this subsection by establishing a connection between $\inf \lambda\mu^{-1}$ and the domain constant $\beta(\Omega)$ introduced and studied by Harmelin and Minda [Har90, HM92].

First we consider (the interior of) triangles; this lemma and its proof will be used in the subsequent proof.

Lemma 3.3. *Let Ω be the interior of a Euclidean triangle. Then* $\inf \lambda\mu^{-1} > \pi/4$.

Proof. Let Δ be the interior of the incircle Γ for the triangle $\partial\Omega$. Let $I = \widehat{K}$ be the ideal hyperbolic triangle in Δ, where $K = \Gamma \cap \partial\Omega$ (the three points opposite the vertices of $\partial\Omega$). Notice that $\Omega \setminus I$ consists of three circular sectors.

First, assume $z \in \Omega$ belongs to one of the circular sectors of $\Omega \setminus I$. This circular sector lies in an infinite sector Σ which is affine equivalent to some Σ_α, where $\alpha\pi$ is the angle at the vertex and $0 < \alpha < 1$. Since $\mu_\Omega(z) = \mu_\Sigma(z)$, the second example in Subsection 3.1 yields

$$\frac{\lambda_\Omega(z)}{\mu_\Omega(z)} \geq \frac{\lambda_\Sigma(z)}{\mu_\Sigma(z)} \geq \frac{1}{\alpha}\tan\left(\frac{\pi\alpha}{4}\right) > \frac{\pi}{4}.$$

Now, assume $z \in I$. In this case, $\mu_\Omega(z) = \lambda_\Delta(z)$. Also, for $z \in \partial I \cap \Omega$, we have $\lambda(z)/\mu(z) > \pi/4$. Next we show that for each $b \in K = \partial I \cap \partial\Omega$,

$$\lim_{\substack{z\to b\\ z\in I}} \frac{\lambda_\Omega(z)}{\lambda_\Delta(z)} = 1.$$

To see this, choose a Möbius transformation φ with $\varphi(b) = \infty$ and $\varphi(\Delta) = \mathbb{H}$. Then $I' = \varphi(I)$ is contained in a vertical half-strip above the real axis, and $\Omega' = \varphi(\Omega) \subset \{w : \operatorname{Im} w > -a\}$ for some constant $a > 0$. Thus for $w = \varphi(z)$ we have

$$\frac{\lambda_\Omega(z)}{\mu_\Omega(z)} = \frac{\lambda_\Omega(z)}{\lambda_\Delta(z)} = \frac{\lambda_{\Omega'}(w)}{\lambda_{\Delta'}(w)} \geq \frac{\operatorname{Im}(w)}{\operatorname{Im}(w) + a},$$

and the latter quantity clearly tends to 1 as w tends to infinity through points of I'.

Finally, define

$$v(z) = \frac{\lambda_\Omega(z)}{\lambda_\Delta(z)} \qquad \text{for } z \in I$$

and put $m = \inf\{v(z) : z \in I\}$. Choose $z_n \in I$ with $v(z_n) \to m$. Assume $z_n \to c \in \overline{I}$. If $c \in \partial I$, then $m > \pi/4$. Suppose c is an interior point of I. Then $\log v$ attains its minimum value at c, which, in conjunction with $\lambda(z)|dz|$ having curvature -1 (so $\Delta \log \lambda = \lambda^2$), yields

$$0 \leq \Delta \log v(c) = \Delta \log \lambda_\Omega(c) - \Delta \log \lambda_\Delta(c) = \lambda_\Omega(c)^2 - \lambda_\Delta(c)^2.$$

Hence, $\lambda_\Delta(c) \leq \lambda_\Omega(c)$, which is impossible. Thus $c \in \partial I$ and $m > \pi/4$. ∎

Notice that the above proof actually furnishes a better lower bound if we assume some lower bound on the interior angles.

Theorem 3.4. *Suppose Ω is a Möbius image of a Euclidean convex region. Then $\inf \lambda\mu^{-1} \geq \pi/4$. However, given $\varepsilon > 0$ there is a hyperbolic domain Ω with $\inf \lambda\mu^{-1} \geq \pi/4 - \varepsilon$ and Ω is not the Möbius image of a convex region.*

Proof. The last assertion follows by looking at thin annuli; see the end of Subsection 3.1. Suppose now that Ω is a Euclidean convex hyperbolic region in $\mathbb{C}$. Let $z \in \Omega$ and put $\Delta = \Delta(z)$, $K = \partial\Delta \cap \partial\Omega$.

First, we consider the special case when $\partial\Omega$ consists of a finite line segment and two infinite rays, each from one end point of the line segment. Here K contains either two points or three points. Suppose K contains two points. Then we can choose an infinite sector Σ such that $\Omega \subset \Sigma$ and Δ is still the extremal disk in Σ associated with z. The second example in Subsection 3.1 yields $\lambda_\Omega(z)/\mu_\Omega(z) \geq \lambda_\Sigma(z)/\mu_\Sigma(z) > \pi/4$. On the other hand, when K contains three points, then, since $z \in \widehat{K}$, we can argue as at the end of the proof of Lemma 3.3, to see that $\lambda_\Omega(z)/\mu_\Omega(z) \geq \pi/4$; this is valid even if two of the three points are diametrically opposite.

Now, we deal with the general case. Since any convex region is the kernel of a sequence of bounded convex polygons, we may assume that Ω is a bounded convex polygon (see [Hej74], [HMM03, Thm. 3.9]). Suppose K contains exactly two points. Then we can choose either an infinite strip or an infinite sector Σ which contains Ω and is such that Δ is still the extremal disk in Σ associated with z. As above, the first and second examples in Subsection 3.1 produce the desired conclusion.

Our remaining case is when K contains finitely many, but at least three, points. Since $z \in \widehat{K}$, z lies inside or on the boundary of one ideal hyperbolic triangle with its three vertices in K. We construct an enveloping region G by intersecting the three supporting half-planes for $\partial\Delta$ at these three vertices. Then ∂G is either a Euclidean triangle or consists of a finite line segment and two infinite rays, each from one end point of the line segment. Lemma 3.3 furnishes the desired result in the former case, while the latter situation is handled by the special case considered at the beginning of our proof. ■

Remarks 3.5. (i) It is straightforward to check that $\Omega \subset \hat{\mathbb{C}}$ is a Möbius image of a Euclidean convex region if and only if there is some point $c \in \hat{\mathbb{C}} \setminus \Omega$ with the property that for all points $a, b \in \Omega$, the subarc of the circle through a, b, c which joins a, b and does not contain c lies in Ω. (ii) We will see below (Remarks 3.8(iii)) that given $\varepsilon > 0$, there is even a hyperbolic Ω with $\inf \lambda\mu^{-1} > 1-\varepsilon$, yet Ω is not the Möbius image of a convex region.

Now we show that $\inf \lambda\mu^{-1}$ cannot be too small unless Ω is multiply connected.

Theorem 3.6. *Suppose a hyperbolic region Ω satisfies $\inf \lambda\mu^{-1} \geq \sqrt{3}/2$. Then Ω is simply connected.*

Proof. Assume Ω is multiply connected; we show $\inf \lambda\mu^{-1} < \sqrt{3}/2$.

Thanks to Möbius invariance, we can assume $\infty \in \Omega$, and thus, we can partition $\hat{\mathbb{C}} \setminus \Omega$ into two disjoint compact sets C_1 and C_2. Select two points $c_i \in C_i$ with $|c_1 - c_2| = \operatorname{dist}(C_1, C_2)$. We can further assume that $c_i = (-1)^i$. Then $G = D(-1;2) \cap D(1;2) \subset \Omega$, so we see that $\Delta(0) = \mathbb{D}$ and $\mu(0) = 2$. A routine calculation gives $\lambda_G(0) = \sqrt{3}$, and thus $\lambda(0)/\mu(0) < \sqrt{3}/2$. ■

It is natural to ask for the least possible constant m such that $\inf \lambda\mu^{-1} \geq m$ implies simple connectivity of Ω. Clearly, $\sqrt{3}/2$ is not sharp. On the other hand, our examples illustrate that $m \geq \pi/4$, which we believe is the best possible value.

Conjecture. *For a hyperbolic region* Ω*,* $\inf \lambda\mu^{-1} \geq \pi/4$ *implies that* Ω *is simply connected.*

Now we reveal a close connection between the ratio λ/μ and the Schwarzian derivative $\mathcal{S}_f$ of any holomorphic covering $f\colon \mathbb{D} \to \Omega$. Following [HM92] and [Har90], we define

$$\beta(\Omega) := 2 \sup_{z\in\Omega} \frac{|\mathcal{S}_\lambda(z)|}{\lambda(z)^2} = \frac{1}{2} \sup_{\zeta\in\mathbb{D}} (1-|\zeta|^2)^2 |\mathcal{S}_f(\zeta)|;$$

here $\Omega \subset \hat{\mathbb{C}}$ is a hyperbolic region and $f\colon \mathbb{D} \to \Omega$ any holomorphic covering projection. In general, the *Schwarzian norm* of a locally univalent holomorphic $g\colon \mathbb{D} \to \mathbb{C}$ is the quantity $\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^2|\mathcal{S}_g(\zeta)|$. Recently, Barnard *et al.* have determined the maximal Schwarzian norm, say N, of a hyperbolically convex map; see [BCPWta, Thm. 1.1]. (In fact, $N = 2.383635\ldots$.)

Theorem 3.7. *For any hyperbolic region* $\Omega \subset \hat{\mathbb{C}}$*,* $\beta = \beta(\Omega)$ *satisfies*

$$\frac{2}{N}\beta \leq \frac{1}{(\inf \lambda\mu^{-1})^2} \leq 1 + \beta.$$

The right-hand inequality is best possible when $0 \leq \beta \leq 3$*.*

Proof. Without loss of generality, we may assume $\Omega \subset \mathbb{C}$. Put $m = \inf \lambda\mu^{-1}$. We first prove the estimate $m \geq 1/r$ where $r = \sqrt{1+\beta}$. Fix a point $a \in \Omega$ and let $f\colon (\mathbb{D}, 0) \to (\Omega, a)$ be a holomorphic covering projection. By affine invariance, we may assume $f(\zeta) = \zeta + a_2\zeta^2 + \cdots$ with $a_2 \geq 0$; thus, $a = f(0) = 0$ and $\lambda(0) = 2/|f'(0)| = 2$. We show there is a disk or half-plane D with $0 \in D \subset \Omega$ and $\lambda_D(0) = 2r$. Then, since $\mu(0) \leq \lambda_D(0)$, we obtain $\lambda(0)/\mu(0) \geq 1/r$ as desired.

The definition of $\beta = \beta(\Omega)$ gives us $(1-|\zeta|^2)^2|\mathcal{S}_f(\zeta)| \leq 2\beta$. Select any $b \notin \Omega$ and put

$$g(\zeta) = \frac{bf(\zeta)}{b - f(\zeta)} = \zeta + \left(a_2 + b^{-1}\right)\zeta^2 + \cdots.$$

Then g also satisfies $(1-|\zeta|^2)^2|\mathcal{S}_g(\zeta)| \leq 2\beta$. According to Proposition 3.9 below, we have $|a_2 + b^{-1}| \leq \sqrt{1+\beta} = r$, so $b^{-1} \in \bar{D}(-a_2; r)$. Thus $b \in \hat{\mathbb{C}} \setminus D$ where either $D = D(c; s)$ with $c = a_2/(r^2 - {a_2}^2)$, $s = r/(r^2 - {a_2}^2)$ when $a_2 < r$; or $D = \{z : \operatorname{Re}(z) > -1/2r\}$ when $a_2 = r$. In both cases, $0 \in D \subset \Omega$ and $\lambda_D(0) = 2r$.

Next, to verify the bound $\beta \leq (N/2)m^{-2}$, it suffices to confirm that for every $z \in \Omega$,

$$\frac{|\mathcal{S}_\lambda(z)|}{\lambda(z)^2} \leq \frac{N}{4}m^{-2}.$$

Again, fix $a \in \Omega$ and let $f\colon (\mathbb{D},0) \to (\Omega,a)$ be a holomorphic covering projection. By Möbius invariance, we may assume that $a = f(0) = 0$ and that $\Delta(0) = \mathbb{D}$; thus $\mu(0) = 2$. Now $|\mathcal{S}_\lambda(z)|/\lambda(z)^2 = (1/4)(1-|\zeta|^2)^2|\mathcal{S}_f(\zeta)|$ for $z = f(\zeta)$, so we must demonstrate that $|S_f(0)| \le N/m^2$. Since $m \le \lambda(0)/\mu(0) = 1/|f'(0)|$, it suffices to show $|\mathcal{S}_f(0)| \le N|f'(0)|^2$.

It is known that disks and half-planes are hyperbolically convex subsets of containing domains (see [Jør56]). In particular, $\mathbb{D}$ is a hyperbolically convex subset of Ω. Let $g = f^{-1}$ denote the branch of the inverse that is defined in $\mathbb{D}$ with $g(0) = 0$. Then g is a hyperbolically convex function; i.e. g is univalent in $\mathbb{D}$ and $g(\mathbb{D})$ is a hyperbolically convex region in $\mathbb{D}$. According to [BCPWta, Thm. 1.1], $(1-|\zeta|^2)^2|\mathcal{S}_g(\zeta)| \le N$ for all $\zeta \in \mathbb{D}$. Thus, $|\mathcal{S}_g(0)| \le N$. Now, $g \circ f = \mathrm{id}$ in $g(\mathbb{D}) \subset \mathbb{D}$, which implies $0 = S_{g\circ f} = (S_g \circ f)(f')^2 + S_f$. Therefore, $|S_f(0)| = |S_g(0)f'(0)^2| \le N|f'(0)|^2$ as desired.

It remains to confirm the sharpness of the lower bound when $0 \le \beta \le 3$. Consider $\Omega = h_\beta(\mathbb{D})$, where

$$h_\beta(\zeta) = \frac{1}{2\sqrt{1+\beta}}\left[\left(\frac{1+\zeta}{1-\zeta}\right)^{\sqrt{1+\beta}} - 1\right] = \zeta + \sqrt{1+\beta}\,\zeta^2 + \cdots .$$

Note that h_β has Schwarzian derivative $\mathcal{S}_{h_\beta}(\zeta) = -2\beta/(1-\zeta^2)^2$. This means $\beta(\Omega) = \beta$, so $\inf \lambda\mu^{-1} \ge 1/\sqrt{1+\beta}$. Moreover, Ω contains the half-plane described by $\mathrm{Re}(z) > -1/(2\sqrt{1+\beta})$ and $-1/(2\sqrt{1+\beta}) \in \partial\Omega$. Hence we have $\mu(0) = 2\sqrt{1+\beta}$ and therefore $\lambda(0)/\mu(0) = 1/\sqrt{1+\beta}$. ∎

Remarks 3.8. (i) When Ω is simply connected, we have $\beta(\Omega) \le 3$, and from above we obtain $\inf \lambda\mu^{-1} \ge 1/2$; this was established by a different means in [HMM03, Thm. 6.3]. However, note that there are many non-simply connected regions Ω with $\beta(\Omega) \le 3$. (ii) Similarly, if Ω is a Nehari domain, then $\beta(\Omega) \le 1$ and $\inf \lambda\mu^{-1} \ge 1/\sqrt{2}$. (iii) For $0 \le \beta \le 3$, we have shown that the domain $\Omega = h_\beta(\mathbb{D})$ satisfies $\inf \lambda\mu^{-1} = 1/\sqrt{1+\beta}$, which increases to 1 as β decreases to 0. It follows that for any $\varepsilon > 0$, there is a hyperbolic domain Ω with $\inf \lambda\mu^{-1} > 1-\varepsilon$ and Ω is not the Möbius image of a convex region.

Proposition 3.9. *If $f(\zeta) = \zeta + a_2\zeta^2 + \cdots$ is holomorphic and locally univalent in $\mathbb{D}$ and satisfies $(1-|\zeta|^2)^2|\mathcal{S}_f(\zeta)| \le 2\beta$ for all $\zeta \in \mathbb{D}$, then $|a_2| \le \sqrt{1+\beta}$; also, equality holds for h_β and its 'rotations'.*

Proof. Since the quantity $(1-|\zeta|^2)^2|\mathcal{S}_f(\zeta)|$ is invariant under 'rotations' of f, it is enough to show that $\mathrm{Re}(a_2) \le \sqrt{1+\beta}$. The collection of all such functions forms a compact normal family (because such functions are uniformly locally univalent in the hyperbolic sense), so there exists an extremal function f for the problem of maximizing $\mathrm{Re}(a_2)$. By applying the Koebe transformation, we see that the extremal function f must satisfy the Marty relation [Dur83, pp. 59–60]

$$3a_3 - 2{a_2}^2 - 1 = 0,$$

or equivalently

$$a_2{}^2 - 1 = 3(a_2{}^2 - a_3) = -\frac{1}{2}\mathcal{S}_f(0).$$

From $|\mathcal{S}_f(0)| \le 2\beta$, we have $|a_2|^2 \le 1+\beta$, which implies $\mathrm{Re}(a_2) \le \sqrt{1+\beta}$ for the extremal function f. ■

3.3. Estimating $\sup \lambda\mu^{-1}$**.** Here we demonstrate that there is a positive lower bound for $\sup \lambda\mu^{-1}$ valid for any hyperbolic region $\Omega \subset \hat{\mathbb{C}}$. In fact, for such Ω

$$\sup \lambda\mu^{-1} \ge M := \frac{2^{2/3}\cdot 4\pi^3}{\sqrt{3}\,\Gamma^6(1/3)} = 0.307510\ldots,$$

and this is best possible. When we know additional information concerning Ω, we can produce a larger lower bound. We begin by determining the value of $\sup \lambda\mu^{-1}$ for the domain $\mathbb{C}_{01} := \mathbb{C}\setminus\{0,1\}$; recall that λ_{01} and μ_{01} are the metric-densities in $\mathbb{C}_{01}$.

Example 3.10. For $\mathbb{C}_{01}$, $\sup \lambda\mu^{-1} = \dfrac{\lambda_{01}(\tau)}{\mu_{01}(\tau)} = M$, where $\tau = (1+i\sqrt{3})/2$.

Proof. We utilize a number of monotonicity properties of λ_{01} which can be found in [Hem79] (see also [Min87]). According to [HMM03, Thm. 4.2], $\mu_{01}(\tau) = 2/\sqrt{3}$, so $\lambda_{01}(\tau)/\mu_{01}(\tau) = (\sqrt{3}/2)\lambda_{01}(\tau) = M$; see the last paragraph of Subsection 2.2. We first prove that

$$\sup_{\mathbb{C}_{01}} \lambda_{01}\mu_{01}^{-1} = \sup_{I} \lambda_{01}\mu_{01}^{-1},$$

where $I = \left\{z : 0 \le \mathrm{Re}\, z \le 1,\ \left|z-\frac{1}{2}\right| \ge \frac{1}{2}\right\}\setminus\{0,1\}$ (a closed ideal hyperbolic triangle). By symmetry, the maximum value of $\lambda_{01}\mu_{01}^{-1}$ on $I^* = \{\bar{z} : z \in I\}$ is the same as on I.

For $\mathrm{Re}\, z \le 0$, $z \ne 0$, $\mu_{01}(z) = 1/|z|$ so $\lambda_{01}(z)/\mu_{01}(z) = |z|\lambda_{01}(z)$. If we fix $\theta \in [\pi/2, 3\pi/2]$, then $|z|\lambda_{01}(z) = r\lambda_{01}(re^{i\theta})$ is increasing for $0 < r \le 1$ and decreasing for $r \ge 1$. Also, $\lambda_{01}(e^{i\theta})$ is decreasing on $(0,\pi]$. Since $\lambda_{01}/\mu_{01}(z)$ is symmetric about $\mathbb{R}$, we have

$$\sup\left\{\frac{\lambda_{01}(z)}{\mu_{01}(z)} : \mathrm{Re}\, z \le 0, z \ne 0\right\} = \frac{\lambda_{01}(i)}{\mu_{01}(i)} = \frac{\lambda_{01}(-i)}{\mu_{01}(-i)} =: M_0.$$

Similarly,

$$\sup\left\{\frac{\lambda_{01}(z)}{\mu_{01}(z)} : \mathrm{Re}\, z \ge 1,\ z \ne 1\right\} = \frac{\lambda_{01}(1+i)}{\mu_{01}(1+i)} = \frac{\lambda_{01}(1-i)}{\mu_{01}(1-i)} = M_0.$$

As $S(z) = 1/z$ maps $\mathbb{C}_{01}$ onto itself and maps $\{z : \mathrm{Re}\, z \ge 1\}\setminus\{1\}$ onto $\overline{D(1/2;1/2)}\setminus\{0,1\}$, we likewise obtain

$$\sup\left\{\frac{\lambda_{01}(z)}{\mu_{01}(z)} : |z-1/2| \le 1/2,\ z \ne 0,1\right\} = \frac{\lambda_{01}((1\pm i)/2)}{\mu_{01}((1\pm i)/2)} = M_0.$$

In summary, on $\{z : \operatorname{Re} z \leq 0 \text{ or } |z-1/2| \leq 1/2 \text{ or } \operatorname{Re} z \geq 1\} \setminus \{0,1\}$, the maximum value of $\lambda_{01}\mu_{01}^{-1}$ equals M_0 and is attained at the six points: $\pm i$, $1 \pm i$, $(1 \pm i)/2$.

Write $z = x + iy = re^{i\theta}$. Note that in I, $\lambda_{01}(z)/\mu_{01}(z) = y\,\lambda_{01}(z)$ and by symmetry

$$\sup_I \lambda_{01}\mu_{01}^{-1} = \sup_{I_1} y\lambda_{01}(z),$$

where $I_1 = \{z : 0 \leq \operatorname{Re} z \leq 1/2, |z-1/2| \geq 1/2\} \setminus \{0\}$. We now verify that

$$\sup_{I_1} y\lambda_{01}(z) = \sup_{\partial I_1} y\lambda_{01}(z)$$

by confirming that $y\,\lambda_{01}(x+iy)$ does not have any relative extrema inside of I_1. For suppose there exists such a relative extremal point z_0 in the interior of I_1. Then at z_0,

$$\frac{\partial\lambda_{01}}{\partial x} = 0 \qquad \text{and} \qquad y\frac{\partial\lambda_{01}}{\partial y} + \lambda_{01} = 0,$$

so,

$$r\frac{\partial\lambda_{01}}{\partial r} + \lambda_{01} = x\frac{\partial\lambda_{01}}{\partial x} + y\frac{\partial\lambda_{01}}{\partial y} + \lambda_{01} = 0.$$

If $|z_0| < 1$, this contradicts the fact that $|z|\lambda_{01}(z)$ is strictly increasing in $0 < |z| < 1$; if $|z_0| > 1$, it contradicts the fact that $|z|\lambda_{01}(z)$ is strictly decreasing in $|z| > 1$; and when $|z_0| = 1$, we get a contradiction to

$$(x-1)\frac{\partial\lambda_{01}}{\partial x} + y\frac{\partial\lambda_{01}}{\partial y} + \lambda_{01} < 0$$

in $|z-1| > 1$.

We derive this last inequality as follows: for $u + iv = w = 1 - z = 1 - x - iy$, $\lambda_{01}(w) = \lambda_{01}(z)$, $|w|\lambda_{01}(w)$ is strictly decreasing for $|w| > 1$, and

$$\begin{aligned} 0 > |w|\frac{\partial\lambda_{01}}{\partial|w|}(w) + \lambda_{01}(w) &= u\frac{\partial\lambda_{01}}{\partial u}(w) + v\frac{\partial\lambda_{01}}{\partial v}(w) + \lambda_{01}(w) \\ = -u\frac{\partial\lambda_{01}}{\partial x}(z) - v\frac{\partial\lambda_{01}}{\partial y}(z) + \lambda_{01}(z) &= (x-1)\frac{\partial\lambda_{01}}{\partial x}(z) + y\frac{\partial\lambda_{01}}{\partial y}(z) + \lambda_{01}(z). \end{aligned}$$

We know that the maximum value of $y\lambda_{01}(z)$ is M_0 on $z = iy$, $0 < y < \infty$, and on $z = (1 + e^{i\theta})/2$, $0 < \theta < \pi$. We claim that $y\lambda_{01}(x+iy) \to 0$ as $y \to \infty$ uniformly on $0 \leq x \leq 1/2$. Indeed, for $|z| > 1$, $\lambda_{01}(1/z) \leq |z|/\log|z|$, so $\lambda_{01}(z) = \lambda_{01}(1/z)/|z|^2 \leq 1/|z|\log|z|$. Thus, $y\lambda_{01}(x+iy) \leq y/|z|\log|z|$, which clearly tends uniformly to 0 as $y \to \infty$.

Finally, we prove that

$$\max\left\{y\lambda_{01}(\frac{1}{2} + iy) : 0 \leq y < \infty\right\} = \frac{\sqrt{3}}{2}\lambda_{01}(\tau).$$

Actually, we show that $y\lambda_{01}(1/2 + iy)$ is increasing on $[0, \sqrt{3}/2]$ and decreasing on $[\sqrt{3}/2, \infty)$. Set $z = re^{i\theta} = x + iy$. Since $|z|\lambda_{01}(z)$ is increasing for $0 < |z| < 1$,

$r\partial\lambda_{01}/\partial r + \lambda_{01} > 0$. Note that $(\partial\lambda_{01}/\partial x)(1/2+iy) = 0$ and $|1/2+iy| < 1$ if and only if $|y| < \sqrt{3}/2$. Using

$$r\frac{\partial\lambda_{01}}{\partial r} = x\frac{\partial\lambda_{01}}{\partial x} + y\frac{\partial\lambda_{01}}{\partial y},$$

we get

$$y\frac{\partial\lambda_{01}}{\partial y} = r\frac{\partial\lambda_{01}}{\partial r} > -\lambda_{01}$$

on $z = 1/2 + iy$, $|y| < \sqrt{3}/2$. Thus for $0 \le y < \sqrt{3}/2$,

$$\frac{\partial}{\partial y}\left[y\lambda_{01}\right]\left(\frac{1}{2}+iy\right) = y\frac{\partial\lambda_{01}}{\partial y}\left(\frac{1}{2}+iy\right) + \lambda_{01}\left(\frac{1}{2}+iy\right) > 0.$$

Similarly, using the fact that $|z|\lambda_{01}(z)$ is decreasing for $|z| > 1$, we obtain

$$\frac{\partial}{\partial y}\left[y\lambda_{01}\right]\left(\frac{1}{2}+iy\right) < 0 \qquad \text{when } y > \frac{\sqrt{3}}{2}.$$ ■

Notice that $\mathbb{C}_{01}$ is Möbius equivalent to $\hat{\mathbb{C}} \setminus \{\omega, e^{2\pi i/3}\omega, e^{4\pi i/3}\omega\}$ for any ω with $|\omega| = 1$; for any such region, $\sup \lambda\mu^{-1}$ is attained at the origin.

Below are the aforementioned lower bounds on $\sup \lambda\mu^{-1}$. Recall that $\Omega \subset \mathbb{C}$ is a *Bloch* domain if $R(\Omega) = \sup_{z\in\Omega} \delta_\Omega(z)$ is finite. Minda [Min85, Thm. 2] demonstrated that $2/R(\Omega) \ge \Lambda(\Omega) \ge 1/R(\Omega)$, where $\Lambda(\Omega) = \inf_{z\in\Omega} \lambda_\Omega(z)$; thus Ω is Bloch if and only if $\Lambda(\Omega) > 0$. Also, B denotes the Bloch constant for the class $\mathcal{S}$; it is known that $B \ge 0.57088$ [Zha89].

Theorem 3.11. *Let $\Omega \subset \hat{\mathbb{C}}$ be a hyperbolic domain. Then*

(a) $\sup \lambda\mu^{-1} \ge M$, *and equality holds if* $\Omega = \mathbb{C}_{01}$;

(b) $\sup \lambda\mu^{-1} \ge \frac{1}{2}$ *if* $\bar{\Omega} \ne \hat{\mathbb{C}}$ *or if* Ω *is Bloch;*

(c) $\sup \lambda\mu^{-1} \ge B$ *if* Ω *is simply connected and* $\Lambda(\Omega)$ *is attained in* Ω;

(d) $\sup \lambda\mu^{-1} = 1$ *if* Ω *is the Möbius image of a convex region.*

Proof. We start by confirming (d). We may assume Ω is convex. Exactly as in the proofs of (Theorem 2.3(a), [Hil84, Thm. 2.1], [HM92, Thm. 4]), we fix a disk $D = D(a) \subset \Omega$, choose a point $b \in \partial D \cap \partial\Omega$, and let z move along the line segment $[a, b)$ to b; we readily find that $\limsup_{z\to b} \lambda(z)\mu(z)^{-1} \ge 1$.

Next, we verify (c). Suppose Ω is simply connected and $\Lambda(\Omega)$ is attained in Ω. Thus there is a point $a \in \Omega$ such that $\lambda(z) \ge \lambda(a)$ for all $z \in \Omega$. Let $f\colon \mathbb{D} \to \Omega$ be conformal with $f(0) = a$; so $\lambda(a) = 2/|f'(0)|$. The mapping defined by $g(z) := (f(z) - a)/f'(0)$ belongs to the class $\mathcal{S}$. As $g(\mathbb{D})$ contains some disk $D(b; B)$ we have $f(\mathbb{D}) \supset D(c; B|f'(0)|) = D$ where $c = a + f'(0)b$. This implies $\mu_\Omega(c) \le \lambda_D(c) = 2/B|f'(0)| = \lambda_\Omega(a)/B$ and $\lambda(c)/\mu(c) \ge \lambda(a)/\mu(c) \ge B$.

To justify (b), first observe that $\bar{\Omega} \neq \hat{\mathbb{C}}$ permits us to assume that Ω is a bounded (hence Bloch) domain; this follows by making a Möbius change of variable $w = 1/(z-c)$ where $c \in \mathbb{C} \setminus \bar{\Omega}$. For such a region Ω, there is a point $a \in \Omega$ with $\delta(a) = R(\Omega)$. Since $\mu(a) \leq 2/\delta(a) = 2/R(\Omega)$,

$$\lambda(a) \geq \Lambda(\Omega) \geq \frac{1}{R(\Omega)} \geq \frac{1}{2}\mu(a).$$

For the case of a Bloch domain Ω, we use Theorem 2.2(a) in conjunction with the aforementioned fact that $\Lambda(\Omega)R(\Omega) \geq 1$.

Finally, we establish (a). Suppose there is an extremal disk D in Ω such that $\partial D \cap \partial\Omega$ contains at least three points. By Möbius invariance, we may assume $D = \mathbb{H}$ and $0, 1, \infty \in \partial\Omega$. Then $I = \{z : 0 \leq \operatorname{Re} z \leq 1, |z - 1/2| \geq 1/2, z \neq 0, 1\}$ is a subset of the closed hyperbolic convex hull of $\partial\mathbb{H} \cap \partial\Omega$ in $\mathbb{H}$, so for all $z \in I$, $\mu_\Omega(z) = \lambda_{\mathbb{H}}(z) = \mu_{01}(z)$. Thus

$$\frac{\lambda_\Omega(z)}{\mu_\Omega(z)} = \frac{\lambda_\Omega(z)}{\lambda_{\mathbb{H}}(z)} \geq \frac{\lambda_{01}(z)}{\mu_{01}(z)}.$$

By Example 3.10 we know $\sup \lambda_{01}\mu_{01}^{-1}$ is attained at $\tau \in I$, so $\sup_\Omega \lambda\mu^{-1} \geq M$.

It remains to consider hyperbolic regions Ω with the property that each extremal disk in Ω has exactly two boundary points on $\partial\Omega$. Select any extremal disk Δ in Ω. By Möbius invariance, we may assume $\Delta = \mathbb{H}$ and $0, \infty \in \partial\Omega$. Since Ω is hyperbolic, there is a point of $\hat{\mathbb{C}} \setminus \Omega$ in the lower half-plane. Recall that all points of $\mathbb{R} \setminus \{0\}$ must lie in Ω since $\partial\mathbb{H} \cap \partial\Omega = \{0, \infty\}$.

Suppose there were some $\theta \in (0, \pi)$ such that $e^{-i\theta}\mathbb{H} = \{e^{-i\theta}z : z \in \mathbb{H}\} \subset \Omega$. Then $\theta_0 = \sup\{\theta \in (0, \pi) : e^{-i\theta}\mathbb{H} \subset \Omega\}$ would be such that there is a point of $\partial\Omega$ in the lower half-plane that lies on the boundary of $e^{-i\theta_0}\mathbb{H}$. But then $e^{-i\theta_0}\mathbb{H}$ would be an extremal disk that has three common boundary points with Ω, a contradiction.

Thus, for every $\theta \in (0, \pi)$, $e^{-i\theta}\mathbb{H}$ meets $\partial\Omega$. This implies that there exists $r_n e^{-i\theta_n} \in \partial\Omega$ with $\theta_n \to 0$. Set $\Omega_n = r_n^{-1}\Omega = \{z/r_n : z \in \Omega\}$ and $b_n = e^{-i\theta_n}$. Then $0, b_n, \infty \in \partial\Omega_n$ and $\sup_\Omega \lambda\mu^{-1} = \sup_{\Omega_n} \lambda\mu^{-1}$ for all n.

Let $f_n : \mathbb{D} \to \Omega_n$ be a covering with $f_n(0) = i$ and $f_n'(0) > 0$; then we have $\lambda_{\Omega_n}(i) = 2/f_n'(0)$. Note that since f_n omits $0, b_n, \infty$ in $\mathbb{D}$, an extended version of Montel's normality criterion (cf. [Car60, p. 202]) implies that $\{f_n : n \geq 1\}$ is a normal family in $\mathbb{D}$.

Now for all n, $\mathbb{H} \subset \Omega_n \subset \mathbb{C}_{0b_n}$, so $1 = \lambda_{\mathbb{H}}(i) \geq \lambda_{\Omega_n}(i) \geq \lambda_{0b_n}(i)$, which provides the estimates $2 \leq f_n'(0) \leq 2/\lambda_{0b_n}(i)$. Recalling that $\lambda_{01}(-1)$ is the minimum value of λ_{01} on the unit circle, we obtain

$$2 \leq f_n'(0) \leq \frac{2}{\lambda_{01}(-1)}.$$

Because $\{f_n : n \geq 1\}$ is a normal family, we may assume that $f_n \to f$ locally uniformly in $\mathbb{D}$. The preceeding inequalities guarantee that f is not a constant.

This means that f is a covering of $\mathbb{D}$ onto $\Omega' = f(\mathbb{D})$ and that $\Omega_n \to \Omega'$ in the sense of kernel convergence (say, with respect to the point i). According to [Hej74] and [HMM03, Thm. 3.9] respectively, we know that λ_{Ω_n} and μ_{Ω_n} converge pointwise in Ω' to $\lambda_{\Omega'}$ and $\mu_{\Omega'}$ respectively. Notice that $\mathbb{H} \subset \Omega'$ and $0, 1, \infty \in \partial\Omega'$. This means that $\mathbb{H}$ is an extremal disk in Ω' and $\Omega' \subset \mathbb{C}_{01}$; therefore for all $z \in I = \{z : 0 \leq \operatorname{Re} z \leq 1,\ |z - 1/2| \geq 1/2,\ z \neq 0, 1\}$ we have

$$\frac{\lambda_{\Omega'}(z)}{\mu_{\Omega'}(z)} = \frac{\lambda_{\Omega'}(z)}{\lambda_{\mathbb{H}}(z)} \geq \frac{\lambda_{01}(z)}{\lambda_{\mathbb{H}}(z)} = \frac{\lambda_{01}(z)}{\mu_{01}(z)}.$$

Once again we appeal to Example 3.10 to conclude that $\sup_\Omega \lambda\mu^{-1} \geq M$. ■

Remarks 3.12. (i) The technique used to prove part (d) applies to many other domains. For example, if Ω has a boundary point at which there exist two disks tangent to each other with one inside Ω and the other inside of $\mathbb{C} \setminus \Omega$, then (by mapping the outside disk onto a half-plane) we see that $\sup \lambda\mu^{-1} = 1$. (ii) It is not difficult to see that the domain $\mathbb{C} \setminus \mathbb{Z}$ has $\sup \lambda\mu^{-1} = 1$. There are many similar examples.

The reader has no doubt observed that so far all of our examples enjoy the property that $\sup \lambda\mu^{-1} = 1$.

Example 3.13. The domain $\mathbb{C} \setminus \mathbb{Z}^2$ has $\sup \lambda\mu^{-1} < 1$.

Proof. Using the formula for the hyperbolic metric in a punctured disk, in conjunction with $\lambda\mu^{-1} \leq \lambda\delta$, we easily find that $\lambda(z) \leq \mu(z)/\log 3$ for all $z \in \mathbb{C} \setminus \mathbb{Z}^2$ with $\delta(z) \leq 1/3$. Thus either $\sup \lambda\mu^{-1} \leq 1/\log 3 < 1$, or the supremum is a maximum, attained at some point $a \in \mathbb{C} \setminus \mathbb{Z}^2$ with $\delta(a) \geq 1/3$, in which case $\lambda(a) < \mu(a)$ because $\mathbb{C} \setminus \mathbb{Z}^2$ is not a disk on $\hat{\mathbb{C}}$. ■

Conjecture. *For* $\mathbb{C} \setminus \mathbb{Z}^2$ *we have* $\sup \lambda\mu^{-1} = \lambda\mu^{-1}((1 + i)/2)$.

4. Miscellaneous

4.1. Hilditch's Conjecture. Hilditch [Hil84] demonstrated that

$$\inf_\Omega \sup \lambda\delta > 0,$$

where the infimum is taken over all hyperbolic domains $\Omega \subset \mathbb{C}$. He conjectured that twice punctured planes (i.e. similarity images of $\mathbb{C}_{01}$) are the extremal regions and he also believed that $(1/2)\lambda_{01}(1/2)$ gives the maximum value of $\lambda_{01}(z)\delta_{01}(z)$, where $\delta_{01}(z) = \delta_{\mathbb{C}_{01}}(z)$. In this section we substantiate both of these convictions. In fact we show that

$$H := \frac{8\pi^2}{\Gamma^4(1/4)} = 0.456947\ldots$$

provides a universal lower bound for $\sup \lambda\delta$ valid for any hyperbolic domain in $\mathbb{C}$. See also [FH99, Prop. 4.1] where $\inf_\Omega \sup \lambda\delta \geq \lambda_{01}(-1) = 0.22847\ldots$ is proved and other lower bounds (for $\sup \lambda\delta$) are presented for various special classes of hyperbolic regions.

We begin by determining the maximum value of $\lambda_{01}(z)\delta_{01}(z)$.

Example 4.1. For $\mathbb{C}_{01}$, $\sup \lambda\delta = \dfrac{1}{2}\lambda_{01}(1/2) = H$.

Proof. That $H = (1/2)\lambda_{01}(1/2)$ follows from the last paragraph of Subsection 2.2. By symmetry,

$$\sup_{\mathbb{C}_{01}} \lambda\delta = \sup\left\{\lambda_{01}(z)\delta_{01}(z) : \operatorname{Re} z \le \frac{1}{2},\ \operatorname{Im} z \ge 0,\ z \ne 0\right\}.$$

Note that $\delta_{01}(z) = |z|$ when $\operatorname{Re} z \le 1/2$. Since $w = 1/z$ is a conformal self-map of $\mathbb{C}_{01}$,

$$\lambda_{01}(z) = \lambda_{01}(w)\left|\frac{dw}{dz}\right| = \frac{\lambda_{01}(w)}{|z|^2},$$

which implies $\lambda_{01}(1/z)|1/z| = \lambda_{01}(z)|z|$. Thus, $\sup_{\mathbb{C}_{01}} \lambda\delta = \sup_{z\in I} \lambda_{01}(z)|z|$, where

$$I = \left\{z = re^{i\theta} : 0 < r\cos\theta \le \frac{1}{2} \text{ for } 0 \le \theta \le \frac{\pi}{3},\ 0 < r \le 1 \text{ for } \frac{\pi}{3} \le \theta \le \pi\right\}.$$

For fixed θ, $r\lambda_{01}(re^{i\theta})$ is an increasing function of r on $(0, 1]$ (cf. [Hem79] or [Min87]), so

$$\sup_{\mathbb{C}_{01}} \lambda\delta = \sup\left\{r\lambda_{01}(re^{i\theta}) : r\cos\theta = \frac{1}{2} \text{ for } 0 \le \theta \le \frac{\pi}{3}, r = 1 \text{ for } \frac{\pi}{3} \le \theta \le \pi\right\}.$$

Next, $\lambda_{01}(e^{i\theta})$ is an decreasing function of θ on $(0, \pi]$, so we conclude that

$$\sup_{\mathbb{C}_{01}} \lambda\delta = \sup\left\{r\lambda_{01}(re^{i\theta}) : r\cos\theta = \frac{1}{2} \text{ for } 0 \le \theta \le \frac{\pi}{3}\right\}.$$

Now we use polarization to prove that $|z|\lambda_{01}(z) \le (1/2)\lambda_{01}(1/2)$ for $z = 1/2 + iy$ with $0 \le y \le \sqrt{3}/2$. Fix $z = 1/2 + iy$, let $w = 1 + 2iy$, let L be the horizontal line through z, and put $\Omega = \mathbb{C}_{0w} = \mathbb{C} \setminus \{0, w\}$. The reflection of Ω with respect to L is $\Omega^* = \mathbb{C} \setminus \{2iy, 1\}$. Then (see [Sol97]) $\lambda_{P\Omega}(\zeta) \le \lambda_\Omega(\zeta)$ for all $\zeta \in \Omega^+$, where Ω^+ consists of all points in Ω that are on or above L and

$$P\Omega = (\Omega \cup \Omega^*)^+ \cup (\Omega \cap \Omega^*)^-.$$

It is easy to see that $P\Omega = \mathbb{C}_{01}$. Since $z \in \Omega^+$,

$$\lambda_{01}(z) = \lambda_{P\Omega}(z) \le \lambda_\Omega(z) = \lambda_{0w}(z).$$

Therefore,

$$|z|\lambda_{01}(z) \le |z|\lambda_{0w}(z) = \frac{1}{2}\lambda_{01}(1/2). \qquad \blacksquare$$

Theorem 4.2. *For any hyperbolic region Ω in $\mathbb{C}$, $\sup \lambda\delta \ge H$. In fact, unless Ω is a twice punctured plane, there exists $z \in \Omega$ such that $\lambda_\Omega(z)\delta_\Omega(z) > H$.*

Proof. Thanks to Koebe's One-Quarter Theorem, we may assume that Ω is not simply connected. Next, suppose ∞ is an isolated boundary point of Ω and $\mathbb{C}\setminus\Omega$ is a continuum. Then $\Omega' = \Omega\cup\{\infty\}\subset\hat{\mathbb{C}}$ is simply connected, so by (3.1), $\lambda_{\Omega'} \geq (1/2)\mu_{\Omega'}$. Fix $a\in\Omega$ and choose $b, c\in\mathbb{C}\setminus\Omega$ with $\delta(a) = |a-b|$ and $b\neq c$. Now $\Omega'\subset\hat{\mathbb{C}}\setminus\{b,c\}$ and thus for any $z\in[a,b)$,

$$\lambda_\Omega(z)\delta_\Omega(z) \geq \lambda_{\Omega'}(z)\delta_\Omega(z) \geq \frac{1}{2}\mu_{\Omega'}(z)\delta_\Omega(z) \geq \frac{1}{2}\frac{|b-c|}{|z-b||z-c|}\delta_\Omega(z) = \frac{|b-c|}{2|z-c|}.$$

Therefore, letting z tend to b along $[a,b)$, we see that $\sup\lambda\delta \geq 1/2$.

We now consider the remaining cases: ∞ a non-isolated boundary point, or $\mathbb{C}\setminus\Omega$ not a continuum. In both situations we can partition $\hat{\mathbb{C}}\setminus\Omega$ into disjoint closed subsets C_1, C_2 such that $0 < \operatorname{dist}(C_1, C_2) < \infty$ and $\infty\in C_2$. This can be seen as follows.

Suppose ∞ is not an isolated point of $\hat{\mathbb{C}}\setminus\Omega$. Since Ω is not simply connected, $\hat{\mathbb{C}}\setminus\Omega$ is not connected. Hence, there exist disjoint compact (relative to $\hat{\mathbb{C}}$) subsets C_1 and C_2 with $\hat{\mathbb{C}}\setminus\Omega = C_1\cup C_2$. We may assume $\infty\in C_2$. Because ∞ is not an isolated point of $\hat{\mathbb{C}}\setminus\Omega$, $C_2\setminus\{\infty\}$ is not empty. So $0 < \operatorname{dist}(C_1, C_2) < \infty$.

If ∞ is an isolated point of $\hat{\mathbb{C}}\setminus\Omega$ and $\mathbb{C}\setminus\Omega$ is not a continuum, then $\mathbb{C}\setminus\Omega$ is bounded in $\mathbb{C}$ and not connected. In this case we can write $\mathbb{C}\setminus\Omega = C_1\cup C_2'$, where C_1, C_2' are disjoint non-empty compact sets, and now C_1 and $C_2 = C_2'\cup\{\infty\}$ form the desired decomposition.

Choose $c_i\in C_i$ with $|c_1 - c_2| = \operatorname{dist}(C_1, C_2)$. By using an affine transformation if necessary, we may assume $c_1 = 0$ and $c_2 = 1$. Then $D(1/2; 1/2)\subset\Omega\subset\mathbb{C}_{01}$, which implies $\lambda_\Omega(1/2) \geq \lambda_{01}(1/2)$ with strict inequality unless $\Omega = \mathbb{C}_{01}$. Also, $\delta_\Omega(1/2) = \delta_{01}(1/2)$. Therefore,

$$\lambda_\Omega(1/2)\delta_\Omega(1/2) \geq \lambda_{01}(1/2)\delta_{01}(1/2) = \frac{1}{2}\lambda_{01}(1/2)$$

with strict inequality unless $\Omega = \mathbb{C}_{01}$. ∎

4.2. Constant ratios. Here we demonstrate that the metric ratios $\lambda\delta$, $\lambda\mu^{-1}$, $\mu\delta$ cannot be constant except in special domains. Recall that the (Gaussian) curvature of a (smooth) conformal metric $\rho(z)|dz|$ can be defined as $-\rho^{-2}\Delta\log\rho$.

Theorem 4.3. *If $\lambda\delta$, $\lambda\mu^{-1}$, or $\mu\delta$ is identically a constant k in Ω, then $k = 1$ and, respectively, Ω is a half-plane, a disk on $\hat{\mathbb{C}}$, or $\mathbb{C}\setminus C$ for some Euclidean convex set $C\subset\mathbb{C}$. Each converse holds too.*

Proof. First, suppose that $\lambda\delta = k$ for some hyperbolic region $\Omega\subset\mathbb{C}$. Then $1/\delta = \lambda/k$ which has constant curvature $-k^2$. The asserted conclusion now follows from [MO86, Thm. 3.23].

Next, assume $\lambda\mu^{-1} = k$ for some hyperbolic region $\Omega \subset \hat{\mathbb{C}}$. As above we deduce that the Kulkarni-Pinkall metric $\mu(z)|dz|$ has constant curvature $-k^2$. In this setting, the asserted conclusion follows from [HIMpp, Thm. D].

Finally, consider a quasihyperbolic domain $\Omega \subset \mathbb{C}$ with $\mu\delta = k$. Thanks to Theorem 2.3(a) we have $k = \inf \lambda\delta = 1$, and then by Theorem 2.2(c) we know that every extremal disk in Ω is a Euclidean half-plane. From this it follows that each point $z \in \Omega$ has a unique closest boundary point. Thus, according to [MO86, Thm. 3.11], $\mathbb{C} \setminus \Omega$ is convex. ■

4.3. Other domain constants. There are also connections between $\inf \lambda\mu^{-1}$, linear invariance, and radii of univalence. For example, the domain constant $\eta = \eta(\Omega) = 2 \sup |(\partial/\partial z) \log \lambda|/\lambda$ (see [HM92] and [Har90]) satisfies

$$\frac{1}{\sqrt{\eta^2 + 2}} \leq \inf \lambda\mu^{-1} \leq \frac{4}{\sqrt{\eta^2 + 8} + \eta}.$$

The authors will return to these matters in the future.

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