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Existence results for operator equations in abstract spaces and an application

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Abstract

In this paper we apply the recursion method presented in (Heikkilä and Lakshmikantham, 1994) to derive fixed point theorems and existence results for operator equations in partially ordered sets. The obtained results, combined with related results of (Heikkilä, 2002) are then applied to operator equations in ordered Hilbert spaces and to a functional elliptic boundary value problem.

1. Introduction

Recently, it is proved in [6] that an increasing self-mapping G of a partially ordered set (poset) P has fixed points if the following hypotheses hold:

(i) Nonempty well-ordered and inversely well-ordered subsets of the range $G[P]$ of G have supremums and infimums in P.
(ii) P has a sup-center or an inf-center c, i.e., $\sup\{c, x\}$ or $\inf\{c, x\}$ exists in P for each $x \in P$.

Using the so-obtained fixed point results and their special cases new existence results are proved in [6] for equation $Lu = Nu$, where L and N are mappings from a set U to a poset P having a sup-center or an inf-center. This assumption replaces the commonly used hypothesis that equation $Lu = Nu$ has an upper and/or a lower solution (cf., e.g., [2,3,10–13]). Moreover, comparison results needed in applications are proved for the constructed

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solutions if U is also a poset and L is inverse monotone. As for applications to implicit functional differential equations in ordered Banach spaces and to elliptic boundary value problems with discontinuous data, see, e.g., [4–6]).

In this paper we complete the above cited results by proving in Section 2, for instance, that under the hypothesis (i) an increasing mapping $G : P \to P$ has a minimal fixed point if P has a sup-center, and a maximal fixed point if P has an inf-center. This result implies in particular that maximal and minimal fixed points exist for each increasing self-mapping of any closed and bounded ball of a weakly complete Banach semilattice, e.g., a reflexive Banach lattice or a UMB-lattice defined in [1].

The obtained fixed point results are then applied in Section 3 to prove existence results for maximal and minimal solutions of equation $Lu = Nu$. We also combine the above cited results and corresponding results of [6] to obtain more comprehensive information on fixed points of G and solutions of equation $Lu = Nu$.

In Section 4 we apply results of Section 3 to derive existence results for operator equations in ordered Hilbert spaces. These existence results are then applied in Section 5 to an elliptic functional boundary value problem. As for applications to integral equations in ordered Banach spaces and to equations in ordered functions spaces, see [7,8].

2. Fixed point results

Recall that a subset C of a poset $P = (P, \leqslant)$ is *well ordered* (respectively, *inversely well ordered*) if each nonempty subset of C has the least (respectively, greatest) element. In both cases C is a chain. To shorten notations, denote

$$C^{<x} = \{y \in C \mid y < x\} \quad \text{when } x \in P \text{ and } C \subseteq P. \tag{2.1}$$

A basis to our considerations is the following recursion principle (cf. [9, Lemma 1.1.1]).

Lemma 2.1. *Given a poset P, a subset $\mathcal{D}$ of $2^P = \{A \mid A \subseteq P\}$ with $\emptyset \in \mathcal{D}$, and a mapping $f : \mathcal{D} \to P$, there exists a unique well-ordered chain C in P such that*

$$x \in C \quad \textit{if and only if} \quad x = f(C^{<x}). \tag{2.2}$$

If $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of C.

Let P be a poset. We say that a subset A of P is *directed upwards* if for all $x, y \in A$ there exists $z \in A$ such that $x \leqslant z$ and $y \leqslant z$, and *directed downwards* if the reversed inequalities hold. An element z of A is called *maximal* if $x \in A$ and $z \leqslant x$ imply $x = z$, and *minimal* if $x \in A$ and $x \leqslant z$ imply $x = z$.

The recursion principle of Lemma 2.1 yields the following fixed point result.

Lemma 2.2. *Let $G : P \to P$ satisfy the following hypothesis*:

(G) *The set $P_0 = \{x \in P \mid x \leqslant Gx\}$ contains an upper bound to $G[C_0]$ whenever $C_0 \subset P_0$ and both C_0 and $G[C_0]$ are well ordered.*

Then P_0 has maximal elements, and they are maximal fixed points of G. If P_0 is directed upwards, then $\max P_0$ *exists and is the greatest fixed point of G.*

Proof. Denote

$$\mathcal{D} = \{W \subset G[P_0] \mid W \text{ is well ordered and has a strict upper bound in } P_0\}.$$

Obviously, $\emptyset \in \mathcal{D}$. Assign to each $W \in \mathcal{D}$ a fixed strict upper bound $u_W \in P_0$ of W, and define a mapping $f : \mathcal{D} \to P$ by

$$f(W) = Gu_W, \quad W \in \mathcal{D}. \tag{2.3}$$

By Lemma 2.1 and (2.3) there exists exactly one well-ordered chain C in P satisfying

$$x \in C \quad \text{if and only if} \quad x = Gu_{C^{<x}}. \tag{2.4}$$

Denote $C_0 = \{u_{C^{<x}} \mid x \in C\}$. As a consequence of the above construction one can show that if $x, y \in C_0$, $z = Gx$ and $w = Gy$, then $z \leqslant w$ implies that $x \leqslant y$. Applying this result and the fact that $C = G[C_0]$ is well ordered it is easy to verify that also C_0 is well ordered. It then follows from the hypothesis (G) that C has an upper bound b in P_0. b is a maximal element of P_0, for otherwise $f(C) = Gu_C$ would exist and would be a strict upper bound of C, which would contradict the last conclusion of Lemma 2.1.

If x is a maximal element of P_0, then $x \leqslant Gx$, whence $x = Gx$ by the definition of a maximal element, and by the hypothesis (G) with $C_0 = \{x\}$. x is also a maximal fixed point of G because each fixed point of G belongs to P_0.

Assume next that P_0 is directed upwards. Then for each $x \in P_0$ there is $y \in P_0$ such that $x \leqslant y$ and $b \leqslant y$, where $b \in P_0$ is given above. Because b is maximal, then $b = y$. Thus $x \leqslant b$, so that b is the maximum of P_0. This proves the last conclusion. □

As an application of Lemma 2.2 we obtain the following result.

Proposition 2.1. *Assume that $G : P \to P$ is increasing, that the set $P_0 = \{x \in P \mid x \leqslant Gx\}$ is nonempty, and that* $\sup G[C]$ *exists in P whenever C is a well ordered subset of P_0. Then P_0 has maximal elements which are also maximal fixed points of G. If P_0 is directed upwards, then* $\max P_0$ *exists and is the greatest fixed point of G.*

Proof. It suffices to prove that the hypothesis (G) of Lemma 2.2 holds. Assume that C is a well ordered chain in P_0. Since G is increasing, then also $G[C]$ is well ordered. Thus $y = \sup G[C]$ exists in P by a hypothesis. To show that y belongs to P_0, let $x \in C$ be given. Then $x \leqslant Gx \leqslant y$, whence $Gx \leqslant Gy$ because G is increasing. Thus Gy is an upper bound of $G[C]$. Since y is the least upper bound of $G[C]$, then $y \leqslant Gy$, i.e., y is an upper bound of $G[C]$ in P_0. This shows that G satisfies the hypothesis (G), whence Lemma 2.2 implies the assertions. □

The proof of the following result is dual to the above one.

Proposition 2.2. *Assume that $G : P \to P$ is increasing, that the set $P_1 = \{x \in P \mid Gx \leqslant x\}$ is nonempty, and that* $\inf G[D]$ *exists in P whenever D is an inversely well-ordered subset* *of P_1. Then P_1 has minimal elements which are also minimal fixed points of G. If P_1 is directed downwards, then* $\min P_1$ *exists and is the least fixed point of G.*

Next we replace the assumptions of preceding propositions by properties of P and $G[P]$. A nonempty subset B of a poset P is called *relatively well-order complete* in P if each nonempty well-ordered or inversely well-ordered subset of B has supremums and infimums in P. If $B = P$, we say that P is *well-order complete*. We say that an element c of a poset P is a *sup-center of* P if $\sup\{c, y\}$ exists and belongs to P for all $y \in P$. If $\inf\{c, y\}$ exists in P for all $y \in P$, we say that c is an *inf-center* of P. If c is both a sup-center and an inf-center of P, we call it an *order center* of P.

For instance, if $\min P$ ($\max P$) exists, it is a sup-center (an inf-center) of P. A closed disk of $\mathbb{R}^2$, ordered coordinatewise, is well-order complete, and its center is also an order center.

Now we are able to prove our main fixed point theorem.

Theorem 2.1. *Assume that P is a poset, that $G : P \to P$ is increasing, and that the range $G[P]$ of G is relatively well-order complete in P.*

(a) *If P has a sup-center, then G has minimal fixed points.*
(b) *If P has an inf-center, then G has maximal fixed points.*

Proof. (a) Assume that c is a sup-center of P, and define

$$Fx := \sup\{c, Gx\}, \quad x \in P.$$

F is increasing because G is, and the set $P_2 = \{x \in P \mid x \leqslant Fx\}$ is nonempty because $c \leqslant Fc$. Let C be a well-ordered chain in P. Since G is increasing, then $G[C]$ is a well-ordered chain in P. Thus $y = \sup G[C]$ exists in P by a hypothesis. Because c is a sup-center of P, then $b := \sup\{c, y\}$ exists and belongs to P. Since b is an upper bound of $Fx = \sup\{c, Gx\}$ for each $x \in C$, then b is an upper bound of $F[C]$. If z is an upper bound of $F[C]$, then for each $x \in C$ we have $Fx = \sup\{c, Gx\} \leqslant z$, so that $c \leqslant z$ and $y \leqslant z$, which implies that $b \leqslant z$. Thus $b = \sup F[C]$.

The above proof shows that F satisfies the hypotheses given for G in Proposition 2.1, whence F has a maximal fixed point z. In particular, $Gz \leqslant Fz = z$, so that the set $P_1 = \{x \in P \mid Gx \leqslant x\}$ is nonempty. If D is an inversely well-ordered chain in P_1, then $\inf G[D]$ exists by a hypothesis. Thus G has by Proposition 2.2 minimal fixed points.

(b) The proof that in the case (b) G has maximal fixed points is dual to the above one. □

Combining the results of Theorem 2.1 and [6, Proposition 2.1] we obtain the following fixed point result.

Proposition 2.3. *Let P be a poset having an order center c, and $G : P \to P$ an increasing mapping whose range $G[P]$ is relatively well-order complete in P. Then G has*

(a) *minimal and maximal fixed points*;

(b) *least and greatest fixed points in* $[a_-, b_+]$, *where* a_- *is any minimal solution of* $x = \inf\{c, Gx\}$ *and* b_+ *is any maximal solution of* $x = \sup\{c, Gx\}$;
(c) *least and greatest fixed points* x_* *and* x^* *in* $[a, b]$, *where* a *is the greatest solution of* $x = \inf\{c, Gx\}$ *and* b *is the least solution of* $x = \sup\{c, Gx\}$.

Moreover, x_* *and* x^* *are increasing with respect to* G.

Proof. (a) is a consequence of Theorem 2.1, whereas (c) and the last conclusion of theorem follow from [6, Proposition 2.1]. To prove (b), let a_- be a minimal solution of $x = \inf\{c, Gx\}$ and b_+ a maximal solution of $x = \sup\{c, Gx\}$. These solutions exist by the proof of Theorem 2.1, and $a_- \leqslant c \leqslant b_+$. If $x \in [a_-, b_+]$, then

$$a_- = \inf\{c, Ga_-\} \leqslant Ga_- \leqslant Gx \leqslant Gb_+ \leqslant \sup\{c, Gb_+\} = b_+,$$

so that $G[a_-, b_+] \subseteq [a_-, b_+]$. Moreover, $G[a_-, b_+]$ is relatively well-order complete in $[a_-, b_+]$. It then follows from [6, Lemma 2.1], with P replaced by $[a_-, b_+]$ that G has least and greatest fixed points in $[a_-, b_+]$. □

In the case when P is well-order complete we get the following result.

Corollary 2.1. *If* P *is a well-order complete poset possessing an order center* c, *then the results of Proposition* 2.3 *hold for each increasing mapping* $G : P \to P$.

Example 2.1. Denote $P = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid |x_1|^p + \cdots + |x_m|^p \leqslant R^p\}$, where $p, R \in (0, \infty)$, and assume that P is ordered coordinatewise. Show that each increasing mapping $G : P \to P$ has minimal and maximal fixed points.

Solution. To show that the origin $c = (0, \ldots, 0)$ of $\mathbb{R}^m$ is a sup-center of P, let $x = (x_1, \ldots, x_m) \in P$ be given. Since $\sup\{c, x\} = (\max\{0, x_1\}, \ldots, \max\{0, x_m\})$ and $|\max\{0, x_i\}| \leqslant |x_i|$ for each $i = 1, \ldots, m$, it follows that $\sup\{c, x\} \in P$. Similarly, it can be shown that c is an inf-center of P. Moreover, P is a closed and bounded subset of $\mathbb{R}^m$, whence it is well-order complete. Thus Corollary 2.1 implies the asserted results. □

Under the hypotheses of Theorem 2.1 the set of fixed points of G is increasing with respect to G in the following sense.

Proposition 2.4. *Let* P *be a poset having an order center, let* $G, \tilde{G} : P \to P$ *be increasing mappings whose ranges* $G[P]$ *and* $\tilde{G}[P]$ *are relatively well-order complete in* P, *and assume that*

$$\tilde{G}x \leqslant Gx \quad \textit{for all } x \in P. \tag{2.5}$$

Then the following results hold:

(a) *For each fixed point* $\tilde{y}$ *of* $\tilde{G}$ *there is a fixed point* y *of* G *such that* $\tilde{y} \leqslant y$.
(b) *For each fixed point* z *of* G *there is a fixed point* $\tilde{z}$ *of* $\tilde{G}$ *such that* $\tilde{z} \leqslant z$.

Proof. To prove (a), let $\tilde{y}$ be a fixed point of $\tilde{G}$. It follows from (2.5) that $\tilde{y}$ belongs to the set $P_0 = \{x \in P \mid x \leqslant Gx\}$. Either $\tilde{y}$ is a fixed point of G, or $\tilde{y} < G\tilde{y}$, in which case we can choose $u_\emptyset = \tilde{y}$ in the proof of Lemma 2.2 and obtain a fixed point y of G such that $\tilde{y} < y$. This proves (a), the proof of the case (b) being similar. □

Remarks 2.1. Example 2.1 with $0 < p < 1$ shows that also nonconvex subsets of $\mathbb{R}^m$ may posses order centers.

The results of Proposition 2.3 are applied in [7] to discontinuous functional integral equations of Urysohn type in ordered Banach spaces.

3. Existence results for equation $Lu = Nu$

As an application of Theorem 2.1 we shall first prove existence results for minimal and maximal solutions of the operator equation $Lu = Nu$. Combining the so obtained results with those of [6, Proposition 2.2] we get a more comprehensive existence theorem for equation $Lu = Nu$.

Theorem 3.1. *Given posets U and P and mappings $L, N : U \to P$ assume that the following hypotheses hold*:

(H0) *L is a bijection, and both L^{-1} and N are increasing.*
(H1) *$N[U]$ is relatively well-order complete in P.*

(a) *If P has a sup-center, then equation $Lu = Nu$ has minimal solutions.*
(b) *If P has an inf-center, then equation $Lu = Nu$ has maximal solutions.*

Proof. (a) Hypothesis (H0) implies that $G := N \circ L^{-1}$ is an increasing self-mapping of P. Since

$$G[P] = N\big[L^{-1}[P]\big] = N\big[L^{-1}\big[L[U]\big]\big] = N[U],$$

then the hypothesis (H1) implies that $G[P]$ is relatively well-order complete in P. Thus $G = N \circ L^{-1}$ satisfies the hypotheses of Theorem 2.1. Consequently, if P has a sup-center, G has minimal fixed points. Let x be such a minimal fixed point. Define $u = L^{-1}x$; then $x = Lu$ and $x = Nu$, so that $Lu = Nu$.

To prove that u is a minimal solution of $Lu = Nu$, let $v \in P$ satisfy $Lv = Nv$, and assume that $v \leqslant u$. Denote $y = Lv$; then $y \in P$, and

$$y = Lv = Nv = N(L^{-1}Lv) = G(Lv) = Gy.$$

Thus y is a fixed point of $G := N \circ L^{-1}$ in P. Since $v \leqslant u$, and N is increasing, then $y = Lv = Nv \leqslant Nu = Lu = x$. But x is a minimal fixed point of G, whence $y = x$, and thus $v = L^{-1}y = L^{-1}x = u$. This proves that $u = L^{-1}x$ is a minimal solution of $Lu = Nu$.

The above proof shows that if P has a sup-center, then the equation $Lu = Nu$ has minimal solutions.

(b) If P has an inf-center, then $G = N \circ L^{-1}$ has a maximal fixed point x, and $u = L^{-1}x$ is a maximal solution of $Lu = Nu$. □

As a consequence of Theorem 3.1 and [6, Proposition 2.2], and Theorem 2.2 we obtain the following existence results for equation $Lu = Nu$.

Proposition 3.1. *Let U and P be posets, assume that P has an order center c, and let $L, N : U \to P$ satisfy the hypotheses* (H0) *and* (H1). *Then the equation $Lu = Nu$ has*

(a) *minimal and maximal solutions*;
(b) *least and greatest solutions in $[v_-, w_+]$, where v_- is any minimal solution of $Lu = \inf\{c, Nu\}$ and w_+ is any maximal solution of $Lu = \sup\{c, Nu\}$*;
(c) *least and greatest solutions u_* and u^* in $[v, w]$, where v is the greatest solution of $Lu = \inf\{c, Nu\}$ and w is the least solution of $Lu = \sup\{c, Nu\}$*.

Moreover, u_* and u^* are increasing with respect to N.

Proof. (a) is a consequence of Theorem 3.1.

(b) Applying the reasoning used in the proof of Theorem 2.1 and the hypothesis (H1) it is easy to show that the ranges of the operators $u \mapsto \inf\{c, Nu\}$ and $u \mapsto \sup\{c, Nu\}$ are relatively well-order complete in P. Because both these operators are also increasing by (H0), they satisfy the hypotheses given for N in Theorem 3.1. This result implies that equation $Lu = \inf\{c, Nu\}$ has a minimal solution v_- and equation $Lu = \sup\{c, Nu\}$ has a maximal solution w_+. Since $Lv_- \leqslant c \leqslant Lw_+$, and L^{-1} is increasing, then $v_- \leqslant w_+$. Assume that $x \in [Lv_-, Lw_+]$. Because L^{-1} and N are increasing, then $u = L^{-1}x \in [v_-, w_+]$ and

$$Lv_- = \inf\{c, Nv_-\} \leqslant Nv_- \leqslant Nu \leqslant Nw_+ \leqslant \sup\{c, Nw_+\} = Lw_+. \tag{3.1}$$

Thus $Nu = NL^{-1}x \in [Lv_-, Lw_+]$ so that $NL^{-1}[Lv_-, Lw_+] \subseteq [Lv_-, Lw_+]$. Moreover, $NL^{-1}[Lv_-, Lw_+]$ is relatively well-order complete in $[Lv_-, Lw_+]$, whence $G = N \circ L^{-1}$ has least and greatest fixed points x_- and x_+ in $[Lv_-, Lw_+]$ (cf. the proof of Proposition 2.3). Denoting $u_\pm = L^{-1}x_\pm$ we have $Lu_\pm = x_\pm = NL^{-1}x_\pm = Nu_\pm$. Since $Lv_- \leqslant x_\pm \leqslant Lw_+$, then $v_- \leqslant L^{-1}x_\pm = u_\pm \leqslant w_+$ because L^{-1} is increasing. Thus $u_\pm$ are solutions of $Lu = Nu$ in $[v_-, w_+]$. If $u \in [v_-, w_+]$ and $Lu = Nu$, then $x = Lu$ is a fixed point of $G = N \circ L^{-1}$ and (3.1) holds, whence $x = NL^{-1}x \in [Lv_-, Lw_+]$. Because x_- and x_+ are least and greatest fixed points of G in $[Lv_-, Lw_+]$, then $x_- \leqslant x \leqslant x_+$. Since L^{-1} is increasing, then $L^{-1}x_- \leqslant L^{-1}x \leqslant L^{-1}x_+$, or equivalently, $u_- \leqslant u \leqslant u_+$. This proves that u_- and u_+ are least and greatest solutions of $Lu = Nu$ in $[v_-, w_+]$, i.e., (b) holds.

(c) It follows from [6, Proposition 2.2], that the greatest solution v of equation $Lu = \inf\{c, Nu\}$ and the least solution w of equation $Lu = \sup\{c, Nu\}$ exist. Replacing v_- by v and w_+ by w in the proof of (b) we see that $Lu = Nu$ has least and greatest solutions u_* and u^* in $[v, w]$, i.e., (c) holds.

As for the proof of the last assertion of theorem see the proof of the corresponding assertion in [6, Theorem 2.2]. □

In the case when P is well-order complete we get the following result.

Corollary 3.1. *If P is a well-order complete poset possessing an order center, then the results of Theorem* 3.2 *hold whenever L, N are mappings from a poset U to P satisfying the hypothesis* (H0).

Example 3.1. Denote $P = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid |x_1|^p + \cdots + |x_m|^p \leqslant R^p\}$, where $p, R \in (0, \infty)$, and assume that P is ordered coordinatewise. Show that the equation $Lu = Nu$ has minimal and maximal solutions whenever L, N are mappings from a poset U to P satisfying the hypothesis (H0).

Solution. By the solution of Example 2.1, P is well-order complete and 0 is its order center. Thus Corollary 3.1 implies the asserted results. □

Remark 3.1. The results of Propositions 2.3 and 3.1 are applied in [8] to operator equations in ordered function spaces.

4. Application to an equation in ordered Hilbert spaces

In this section we apply the results of Proposition 3.1 to the equation

$$Au = F(u), \tag{4.1}$$

where $A : W \subseteq V \to H$ and $F : V \to H$. The spaces V, H are assumed to posses the following properties:

(H) $H = (H, (\cdot|\cdot)_H, \leqslant)$ is a lattice-ordered real Hilbert space, $\|u^+\|_H \leqslant \|u\|_H$ for each $u \in H$, where $u^+ = \sup\{0, u\}$, and

$$(u|v)_H \geqslant 0 \quad \forall u, v \in H_+ = \{u \in H \mid u \geqslant 0\}. \tag{4.2}$$

(V) V is a reflexive Banach space which is continuously and densely embedded in H and lattice ordered by the ordering $\leqslant$ of H.

For instance, these properties are valid for the following a.e. pointwise-ordered spaces where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary:

- $H = L^2(\Omega)$ and V is $L^p(\Omega)$, $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, $2 \leqslant p < \infty$;
- H is $W^{1,2}(\Omega)$ (respectively, $W_0^{1,2}(\Omega)$) and V is $W^{1,p}(\Omega)$ (respectively, $W_0^{1,p}(\Omega)$), $2 \leqslant p < \infty$.

Since V is continuously embedded in H, there exists a positive constant K such that

$$\|v\|_H \leqslant K\|v\|_V \quad \forall v \in V. \tag{4.3}$$

To define the operator A, let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form which has the following properties:

(a1) (continuity) $|a(u,v)| \leqslant C\|u\|_V\|v\|_V$ for all $u, v \in V$, where $C \geqslant 0$;
(a2) (coercivity) $a(u,u) \geqslant \kappa\|u\|_V^2$ for all $u \in V$ and for some $\kappa > 0$;
(a3) (orthogonality) $a(u^-, u^+) = 0$ for all $u \in V$, where $u^\pm = \sup\{0, \pm u\}$.

Applying the properties (a1)–(a3), (H) and (V) and Lax–Milgram theorem it is shown in [5] that the following result holds.

Lemma 4.1. *Relations*

(W) $W := \{u \in V$ *there is* $h \in H$ *such that* $a(u,v) = (h|v)_H$ *for all* $v \in V\}$;
(A) $(Au|v)_H := a(u,v)$, $u \in W$, $v \in V$,

define a subset W *of* V *and an operator* $A : W \to H$ *which is a linear bijection. Moreover, the inverse* A^{-1} *of* A *is increasing and satisfies*

$$\|A^{-1}\| \leqslant \frac{K}{\kappa}, \quad \text{where } \kappa \text{ and } K \text{ are the constants in (a2) and (4.3).} \tag{4.4}$$

The mapping $F : V \to H$ is assumed to satisfy the following hypotheses:

(F0) F is increasing.
(F1) $\|F(u)\|_H \leqslant M + \mu\|u\|_V^r$ for all $u \in V$, where $M > 0$, $\mu \geqslant 0$, $r \in [0,1]$, and if $r = 1$, then $K\mu < \kappa$, where κ and K are constants in (a2) and (4.3).

By means of Proposition 3.1 and Lemma 4.1 we prove the following existence result for Eq. (4.1).

Theorem 4.1. *Assume that* $W \subseteq V$ *and* $A : W \to H$ *are defined by* (W) *and* (A), *and that a mapping* $F : V \to H$ *satisfies the hypotheses* (F0) *and* (F1). *Then Eq.* (4.1) *has*

(a) *minimal and maximal solutions in* W;
(b) *least and greatest solutions in* $[v_-, w_+]$, *where* v_- *is any minimal solution of* $Au = \inf\{0, F(u)\}$ *and* w_+ *is any maximal solution of* $Au = \sup\{0, F(u)\}$;
(c) *least and greatest solutions in* $[v, w]$, *where* v *is the greatest solution of* $Au = \inf\{0, F(u)\}$ *and* w *is the least solution of* $Au = \sup\{0, F(u)\}$.

Proof. It follows from the hypothesis (F1) that the function

$$q(x) = M + \frac{\mu K^r}{\kappa^r} x^r, \quad x > 0, \tag{4.5}$$

is strictly increasing and has exactly one fixed point $x = R$. Define

$$P := \{h \in H \mid \|h\|_H \leqslant R\}, \qquad U := \{u \in W \mid \|Au\|_H \leqslant R\}. \tag{4.6}$$

The growth condition of (F1) and (4.4) imply that for each $u \in U$,

$$\|F(u)\|_H \leqslant M + \mu\|A^{-1}Au\|_V^r \leqslant q(\|Au\|_H) \leqslant q(R) = R, \tag{4.7}$$

so that $F[U] \subseteq P$. Thus the restrictions $L = A|_U$ and $N = F|_U$ satisfy the hypothesis (H) of Theorem 3.1. Since $\|\sup\{0,u\}\|_H \leqslant \|u\|_H$ for all $u \in H$ by (H0) and $\inf\{0,u\} = -\sup\{0,-u\}$, then 0 is an order center of P. Moreover, H is a reflexive by (H) and P is bounded, closed and convex, and hence also weakly closed. These properties ensure that all monotone sequences of P converge weakly in P, whence [3, Lemma A.3.1] implies that P is well-order complete. Thus the results of Proposition 3.1 with $c = 0$ hold by Corollary 3.1. In particular, Eq. (4.1) has minimal and maximal solutions in U. If $u \in W$ and $Au = F(u)$, it follows from (4.7) that

$$\|Au\|_H = \big\|F(u)\big\|_H \leqslant q\big(\|Au\|_H\big),$$

where q is defined by (4.5). This implies that $\|Au\|_H \leqslant R$, i.e., $u \in U$, since $q(x)/x$ is strictly decreasing and $q(R)/R = 1$. Thus all the solutions of (4.1) are contained in U, whence (4.1) and equation $Lu = Nu$ have same solutions. Similarly one can show that all the solutions of $Au = \inf\{0, F(u)\}$ and $Au = \sup\{0, F(u)\}$ belong to U, so that they have the same solutions as equations $Lu = \inf\{0, Nu\}$ and $Lu = \sup\{0, Nu\}$. These results and the results of Proposition 3.1 imply the assertions. □

Remark 4.1. According to the hypotheses (F0) and (F1) the nonlinear operator F may be discontinuous and noncompact. Moreover, the growth condition (F1) does not provide means to construct a priori upper and/or lower solutions for Eq. (4.1). Thus the standard theories such as the theory of monotone operators due to Brezis and Browder, Schauder's fixed point theorem or the method of upper and lower solutions are, in general, not applicable under the hypotheses given above to solve (4.1).

5. Application to an elliptic boundary value problem

In this section we shall apply Theorem 4.1 to the following functional elliptic boundary value problem (BVP):

$$\Lambda u(x) = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.1}$$

where $f : \Omega \times W_0^{1,2}(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ being a bounded domain with Lipschitz boundary $\partial\Omega$, and where the operator Λ is defined by

$$\Lambda u(x) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \alpha(x)u(x), \tag{5.2}$$

with coefficients $a_{ij}, \alpha \in L^\infty(\Omega)$ satisfying for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \gamma|\xi|^2 \quad \text{and} \quad \alpha(x) \geqslant \beta \quad \text{for a.e. } x \in \Omega\ (\gamma, \beta > 0). \tag{5.3}$$

Definition 5.1. Defining for all $u, v \in W_0^{1,2}(\Omega)$,

$$a(u,v) := \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j} + \alpha(x)u(x)v(x)\right)dx, \tag{5.4}$$

we say that $u \in W_0^{1,2}(\Omega)$ is a weak solution of (5.1) iff

$$a(u,v) = \int_\Omega f(x,u)v(x)\,dx \quad \text{for each } v \in W_0^{1,2}(\Omega). \tag{5.5}$$

Assuming that $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$ are equipped with a.e. pointwise ordering we get the following existence result as a consequence of Theorem 4.1.

Theorem 5.1. *Assume that $f : \Omega \times W_0^{1,2}(\Omega) \to \mathbb{R}$ satisfies the following hypotheses*:

(f0) *$f(\cdot,u)$ is measurable in Ω for all $u \in W_0^{1,2}(\Omega)$.*
(f1) *$f(x,\cdot)$ is increasing for a.e. $x \in \Omega$.*
(f2) *$\|f(\cdot,u)\|_{L^2(\Omega)} \leqslant M + \mu\|u\|^r_{W_0^{1,2}(\Omega)}$ for all $u \in W_0^{1,2}(\Omega)$, where $M > 0$, $\mu \geqslant 0$ and $r \in [0,1]$, and if $r = 1$, then $\mu < \min\{\gamma,\beta\}$, where γ and β are the constants in* (5.3).

Then the BVP (5.1), (5.2) *has*

(a) *minimal and maximal weak solutions*;
(b) *least and greatest weak solutions in $[v_-, w_+]$, where v_- is any minimal weak solution of*

$$\Lambda u(x) = \min\{0, f(x,u)\} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.6}$$

and w_+ is any maximal weak solution of

$$\Lambda u(x) = \max\{0, f(x,u)\} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega; \tag{5.7}$$

(c) *least and greatest weak solutions in $[v,w]$, where v is the greatest weak solution of* (5.6) *and w is the least weak solution of* (5.7).

Proof. Choose $V = W_0^{1,2}(\Omega)$ and $H = L^2(\Omega)$, equipped with the inner products

$$(h|v)_H := \int_\Omega h(x)v(x)\,dx, \quad h, v \in H, \tag{5.8}$$

$$(u|v)_V := \int_\Omega \left(\sum_{i,j=1}^N \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j} + u(x)v(x) \right) dx, \quad u, v \in V. \tag{5.9}$$

These spaces have properties (H) and (V) given in Section 4. Moreover, it follows from (5.8) and (5.9) that (4.3) is valid when $K = 1$. The function a defined in (5.4) has properties (a1)–(a3). For instance, since all the products

$$\frac{\partial u^-(x)}{\partial x_i}\frac{\partial u^+(x)}{\partial x_j} \quad \text{and} \quad u^-(x)u^+(x)$$

vanish for all $u, v \in W_0^{1,2}(\Omega)$, then property (a3) holds. Moreover, it follows from (5.3) and (5.4) that

$$a(u,u) \geqslant \int_\Omega \left(\gamma \sum_{i=1}^N \left(\frac{\partial u(x)}{\partial x_i} \right)^2 + \beta u(x)^2 \right) dx \geqslant \min\{\gamma, \beta\} \|u\|_V^2$$

for each $u \in V$, whence we can choose $\kappa = \min\{\gamma, \beta\}$ in (a2). The hypotheses (f0)–(f2) imply that the hypotheses (F0) and (F1) are satisfied for the mapping F defined by

$$F(u) := f(\cdot, u), \quad u \in W_0^{1,2}(\Omega). \tag{5.10}$$

Consequently, all the hypotheses of Theorem 4.1 hold for A and F, given by (A) and (5.10). Definition 5.1 and the above notations imply that $u \in V = W_0^{1,2}(\Omega)$ is a weak solution of (5.1) if and only if $Au = F(u)$. Thus the assertions follow from Theorem 4.1. □

References

[1] G. Birkhoff, Lattice Theory, in: Amer. Math. Soc. Publ., vol. 25, American Mathematical Society, Providence, RI, 1940.
[2] S. Carl, S. Heikkilä, Operator and differential equations in ordered spaces, J. Math. Anal. Appl. 234 (1999) 31–54.
[3] S. Carl, S. Heikkilä, Nonlinear Differential Equations in Ordered Spaces, Chapman & Hall/CRC Press, London, 2000.
[4] S. Carl, S. Heikkilä, Elliptic problems with lack of compactness via a new fixed point theorem, J. Differential Equations 186 (2002) 122–140.
[5] S. Carl, S. Heikkilä, Existence results for equations in ordered Hilbert spaces and elliptic BVP's, Nonlinear Funct. Anal. Appl. 74 (2002) 531–546.
[6] S. Heikkilä, Existence and comparison results for operator and differential equations in abstract spaces, J. Math. Anal. Appl. 274 (2002) 586–607.
[7] S. Heikkilä, Discontinuous functional integral equations in abstract spaces, in: R.P. Agarwal, D. O'Regan (Eds.), Advances in Integral Equations, Dynam. Systems Appl., 2004, pp. 595–616.
[8] S. Heikkilä, Operator equations in ordered function spaces, in: R.P. Agarwal, D. O'Regan (Eds.), Nonlinear Analysis and Applications, Kluwer Academic, Dordrecht, in press.
[9] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Dekker, New York, 1994.
[10] E. Liz, Monotone iterative techniques in ordered Banach spaces, Nonlinear Anal. 30 (1997) 5179–5190.
[11] E. Liz, Abstract monotone iterative techniques and applications to impulsive differential equations, Dynam. Contin. Discrete Impuls. Systems 30 (1997) 443–452.
[12] E. Liz, J.J. Nieto, An abstract monotone iterative method and applications, Dynam. Systems Appl. 7 (1998) 365–376.
[13] J.J. Nieto, An abstract monotone iterative technique, Nonlinear Anal. 20 (1997) 1923–1933.