Factorizing Polynomials

Factorizing Polynomials from Two Rounds of Lin's Reduced Penultimate Remainder Process with a Real Divisor

By J. W. Head*

In a recent paper[1] the author suggested that Lin's reduced penultimate remainder process could still be used to find the real or complex linear factors of polynomials of any degree, even when the process originally described by Lin[2] was divergent, provided that the initial approximation to the linear factor being sought, and the two succeeding iterates, could be regarded as differing from the root sought by a small quantity of the first order. The main principle used was that in this linear case the convergence or divergence of the successive iterates to or from the root being sought was purely geometric, so that (for a sufficiently close starting approximation) the root being sought could be deduced from two iterations; complex numbers, however, might be involved in the iteration process.

Now consider a trial divisor D_0 of any degree r, and suppose that the successive divisors arising from the iteration are $D_1, D_2 \ldots . D_n \ldots .$, and that the zeros of D_n are $\alpha_{1n}, \alpha_{2n} \ldots . \alpha_{rn}$. Then we shall show that the successive values of a particular zero α_{kn} converge to or diverge from the true value α_k geometrically, though the common ratio is different for each value of k between 1 and r. Hence α_k can be deduced from its starting value and two successive iterates. Although complex algebra is unavoidable when we wish to deduce α_k from α_{k0}, α_{k1} and α_{k2} (if these quantities are all complex) this presents no difficulty; the divisor D_0 can and should always have real coefficients, and therefore the iteration itself need only involve real numbers.

The Geometric Convergence (or Divergence) of Lin's Iteration Process

Suppose that

$$F(x)=(x-\alpha_1)(x-\alpha_2)\phi(x) \quad \ldots\ldots \quad (1)$$

is the polynomial (of degree m) whose factors are required, and that we seek successive quadratic factors approaching $(x-\alpha_1)(x-\alpha_2)$. The procedure to be described is perfectly general; the right-hand side of Eq. (1) could have had any number r of factors if we had wished, and the α's could be real or complex, but the case of a quadratic factor sufficiently indicates the general procedure.

Suppose that $(x^2+A_nx+B_n)$ is the divisor arising from the nth iteration, then the $(n+1)$th iteration process might be shortly described by the defining identity

$$F(x)=x(x^2+A_nx+B_n)Q_n(x)+\frac{F(0)}{B_{n+1}}(x^2+A_{n+1}x+B_{n+1}) \quad \ldots\ldots \quad (2)$$

where $Q_n(x)$ is a polynomial of degree $(m-3)$, the quotient when $F(x)$ is divided by $x(x^2+A_nx+B_n)$, and the rest of Eq. (2) is the remainder after this division. Now suppose that $(x^2+A_nx+B_n)$ is an approximation to $(x-\alpha_1)(x-\alpha_2)$, say $(x-\alpha_1-\xi_n)(x-\alpha_2-\eta_n)$, where ξ_n, η_n are small quantities of the first order whose squares and products may be neglected. Divide through both sides of Eq. (2) by

$$x(x-\alpha_1-\xi_n)(x-\alpha_2-\eta_n) \text{ or } x(x^2+A_nx+B_n) \quad \ldots\ldots \quad (3)$$

and substitute for $F(x)$ from Eq. (1). Now find the partial fractions associated with each side separately. For the left-hand side, we have

$$\frac{(x-\alpha_1)(x-\alpha_2)\phi(x)}{x(x-\alpha_1-\xi_n)(x-\alpha_2-\eta_n)}=Q_n(x)+\frac{\alpha_1\alpha_2\phi(0)}{x(\alpha_1+\xi_n)(\alpha_2+\eta_n)} \quad \ldots\ldots \quad (4)$$
$$+\frac{\xi_n(\alpha_1-\alpha_2+\xi_n)\phi(\alpha_1+\xi_n)}{(\alpha_1+\xi_n)(\alpha_1-\alpha_2+\xi_n-\eta_n)(x-\alpha_1-\xi_n)}$$
$$+\frac{\eta_n(\alpha_2-\alpha_1+\eta_n)\phi(\alpha_2+\eta_n)}{(\alpha_2+\eta_n)(\alpha_2-\alpha_1+\eta_n-\xi_n)(x-\alpha_2-\eta_n)}$$

while the right-hand side reduces to

$$Q_n(x)+\frac{F(0)(x-\alpha_1-\xi_{n+1})(x-\alpha_2-\eta_{n+1})}{(\alpha_1+\xi_{n+1})(\alpha_2+\eta_{n+1})x(x-\alpha_1-\xi_n)(x-\alpha_2-\eta_n)}$$
$$=Q_n(x)+\frac{F(0)}{(\alpha_1+\xi_{n+1})(\alpha_2+\eta_{n+1})}\psi(x) \quad \ldots\ldots \quad (5)$$

where

$$\psi(x)=\frac{(\alpha_1+\xi_{n+1})(\alpha_2+\eta_{n+1})}{(\alpha_1+\xi_n)(\alpha_2+\eta_n)x}+\frac{(\xi_n-\xi_{n+1})(\alpha_1-\alpha_2+\xi_n-\eta_{n+1})}{(\alpha_1+\xi_n)(\alpha_1-\alpha_2+\xi_n-\eta_n)(x-\alpha_1-\xi_n)}$$
$$+\frac{(\eta_n-\eta_{n+1})(\alpha_2-\alpha_1+\eta_n-\xi_{n+1})}{(\alpha_2-\alpha_1+\eta_n-\xi_n)(\alpha_2+\eta_n)(x-\alpha_2-\eta_n)} \quad \ldots\ldots \quad (6)$$

The coefficients of $1/x$ in Eqs. (4) and (5) are the same since Eq. (1) with $x=0$ equates $F(0)$ and $\alpha_1\alpha_2\phi(0)$. Equating coefficients of $1/(x-\alpha_1-\xi_n)$ and dropping all terms of order above the first in ξ_n, ξ_{n+1}, η_n, and η_{n+1}, we find

$$\frac{\xi_{n+1}}{\xi_n}=1-\frac{\phi(\alpha_1)}{\phi(0)} \quad \ldots\ldots \quad (7)$$

so that, to the first order, ξ_{n+1}/ξ_n is constant, that is, the difference between the value of the zero, $\alpha_1+\xi_n$, obtained from the divisor $x^2+A_nx+B_n$ and the true value α_1 of this zero, changes in geometric progression at each iteration. Similarly, by equating coefficients of $1/(x-\alpha_2-\eta_n)$, we have

$$\frac{\eta_{n+1}}{\eta_n}=1-\frac{\phi(\alpha_2)}{\phi(0)} \quad \ldots\ldots \quad (8)$$

so that, again, the second zero converges to or diverges from its limit geometrically, but the common ratio involved in Eq. (8) is different from that involved in Eq. (7). In the general case, where the divisor is of degree r, a similar argument would show that each zero approached its limit geometrically, but each with a different common ratio.† It follows that each zero β can be determined from a starting approximation β_0 and two iterated values $\beta_1\beta_2$ by means of Aitken's formula

$$\beta=\frac{\beta_1{}^2-\beta_0\beta_2}{2\beta_1-\beta_0-\beta_2} \quad \ldots\ldots \quad (9)$$

so that only two rounds of iteration are required. If the starting approximation is good, Eq. (9) should yield a closer approximation to the required root, which can if necessary be used as a further starting approximation.

For successful application of the iteration process, we require that the

Summary

If a trial divisor of any degree is used to seek a factor of a polynomial by iteration according to Lin's reduced penultimate remainder process, each zero of that trial divisor converges to or diverges from the corresponding true zero of the polynomial geometrically. The true zero may therefore be closely estimated by means of the starting approximation and two iterates. The iterative process need not be convergent; it is only necessary that the starting approximation and the first two iterates differ by a small quantity of the first order from their true values.

The common ratio in the geometric progression varies from one zero of the trial divisor to another; its modulus is small if the zero of the trial divisor is small compared to all zeros of the polynomial unrelated to those of the trial divisor, but large if a zero of the trial divisor has modulus large compared to one of the unrelated zeros of the polynomial. Lin's iteration process is therefore most economically used for finding any cluster of roots having moduli small compared to those of the remaining roots. Any factor found should be divided out, as its continued presence hinders the search for roots of larger modulus.

REFERENCES TO LITERATURE

(1) J. W. Head. 'Widening the Applicability of Lin's Iteration Process for Determining Quadratic Factors of Polynomials'. (*Quart. J. Mech. App. Math.* Feb. 1957, Vol. 10, Part 1, pp. 122-128.)

(2) Shih-Nge Lin. 'A Method of Successive Approximations of Evaluating the Real and Complex Roots of Cubic and Higher Order Equations', *J. Math. Phys.*, Vol. 20 (1941), 231.

(3) A. C. Aitken. 'On the Factorization of Polynomials by Iterative Methods', *Proc. Roy. Soc. Edinburgh*, A, Vol. 63 (1951), 174.

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† This result is included in a more general one which Aitken[3] obtained by matrix methods, but the present approach is simpler and more elementary than Aitken's.

right-hand sides of Eqs. (7) and (8) should preferably have moduli appreciably less than 1. Eq. (9) can, however, still be used successfully if these moduli exceed 1, but are not so great that two iterations yield approximations to a zero which differ by more than a first-order quantity from the true value. Now the expression $\phi(\alpha_1)/\phi(0)$ on the right-hand side of Eq. (7) is made up of a number of factors

$$(\alpha_s-\alpha_1)/\alpha_s \qquad (10)$$

where $\alpha_s(s=3, 4 \ldots)$ are the various zeros, real or complex, of $\phi(x)$. If $|\alpha_1|$ is small compared to $|\alpha_s|$, the factor (10) will be near unity, but if $|\alpha_1|$ is large compared to $|\alpha_s|$, the factor (10) will have large modulus. If therefore all the roots being sought have moduli small relative to those of the remaining roots (those of $\phi(x)=0$), $\phi(x)/\phi(0)$ will for each root sought be a complex number not very different from unity. If, however, any of the roots being sought have moduli large compared to those of $\phi(x)=0$, $\phi(x)/\phi(0)$ will have some factors of the form (10) of large modulus, and therefore $\phi(x)/\phi(0)$ will itself have large modulus, and violent divergence will occur. This suggests that Lin's iteration process is most economically applied to any collection of roots whose moduli are small compared to those of the remaining roots. Initially we may not know how many roots of relatively small modulus are present. If we first seek a small real root, our starting divisor can be derived from the last two terms of the polynomial in the absence of other information. If divergence too violent to enable us to use Eq. (9) successfully occurs, we shall know that there are one or more pairs of complex roots of smaller modulus, and therefore try a quadratic divisor (which can be derived from the last three terms of the polynomial in the absence of better information). Excessive divergence now would suggest a greater cluster of roots having moduli all of the same order of magnitude, so that a divisor of higher degree r is required; the starting divisor can be the last $(r+1)$ terms of the equation. Before a value of r over 2 is considered, the above processes should be applied to the 'reciprocal' polynomial $x^m F(1/x)$. Once a successful divisor is found (and this only means one for which violent divergence does not occur) two rounds of iteration with it will give us three iterates of a number of roots to each of which we can apply Eq. (9), and deduce a better starting divisor of the same degree. Any factor found should be divided out, because its continued presence complicates the finding of the remaining factors. If the original equation has high degree, it will be necessary to find very accurately the factors divided out early.

If the zeros of the original polynomial are well separated in modulus, the above procedure should extract them readily. If they are well separated but several have moduli of the same order of magnitude, a change of variable to increase or decrease all the roots remaining to be found by a constant amount may be desirable. If the original polynomial has equal or clustered roots, special techniques may be required, as briefly mentioned in Ref. (1). Unless, however, all the zeros of the original polynomial are clustered in this way, the application of the processes here suggested should at least considerably reduce the degree of the polynomial to which such special techniques must be applied.

It is worth noting that in Eqs. (7) and (8), the common ratio of the geometric progression can only be unity if $\phi(\alpha_1)$ or $\phi(\alpha_2)$ is zero, that is to say, if a zero sought by means of the trial divisor is also a zero of $\phi(x)$, so that the original polynomial has a repeated zero. In this case Eq. (9) will have differences between unusually equal quantities in numerator and denominator. Early warning is thus available if zeros are so clustered that the special techniques are required.

Acknowledgment

The author wishes to thank the Chief Engineer of the British Broadcasting Corporation for permission to publish this paper.

TRADE ANNOUNCEMENTS

B.W.R.A. Research Director

The British Welding Research Association has announced that Richard Weck has been appointed Director of Research as from June 17, 1957.

New London Offices of George Ellison Ltd. and Tufnol Ltd.

The associated companies of George Ellison Ltd. and Tufnol Ltd. have announced that their respective London offices in Victoria Street have been transferred to Ellison House, Connaught Place, London, W2. From this address will be conducted the Southern Area sales and technical service for Ellison switchgear and hydraulic valves, and for Tufnol, the non-metallic material for engineering components.

Hymatic Appointment

Mr G. B. Longbottom has been apponted Deputy Technical Manager of The Hymatic Engineering Co. Ltd., Redditch, Worcs. In his new position Mr Longbottom will act as deputy to Hymatic's Technical Manager, Mr H. R. Haerle, but continues his responsibilities as Senior Development Engineer.

British Thermo-Electric De-icing System to be made in the United States

Under an agreement signed recently, PacAero Inc. of Santa Monica, California, has acquired the right to manufacture and sell the Napier Spraymat electrical system of de-icing in the United States, its territories and possessions.

Napier is to supply all the necessary drawings, reports, technical information and assistance needed to enable PacAero to establish both manufacturing and servicing facilities, and will take all reasonable steps to promote the sale of Spraymat by PacAero.

For its part, PacAero will make available to Napier details of all refinements and improvements which may result from development processes applied to the system by the American company.

Ministry of Supply Appointment

The Minister of Siupply, with the agreement of the Secretary of State for Air, has appointed Air Commodore H. B. Wrigley to be Director of Guided Weapons (Projects).

Appointment of Private Secretary

The Minister of Transport and Civil Aviation, the Rt Hon Harold Watkinson, M.P., has appointed Mr J. M. Moore to be one of his Joint Principal Private Secretaries, in place of Mr O. F. Gingell, who has been promoted.

Formation of New Anglo-American Company

The formation of a new company to manufacture transistors and other semi-conductors in England has been announced. To be known as Semiconductors Limited, the new company has been formed by the Plessey Company Limited and Philco Corporation of U.S.A. It is anticipated that production of transistors manufactured under Philco patents, covering a wide range of applications in the industrial and consumer fields, will begin early in 1958.

New Name for Range of Curing Agents

Epikure is the new name to be given to the range of six curing agents for liquid Epikote resins which are part of the Shell Chemical Company's customer service in resin sales. The range of Shell curing agents will then be Epikure MPD, DDM, 2, K·61B, T, BF_3-400.

New Plannair Appointment

The appointment of Mr Greslé Farthing as Sales Engineer has been announced by Plannair Limited. For the past three years he has been employed by Air Control Limited as a sales engineer.

Plessey Licensing Agreement

A licensing agreement which provides for the future manufacture of automatic machine tool control components has been concluded between the Plessey Company Limited and Farrand Controls Inc., of New York. Under the agreement, Plessey will develop in this country during the next two years, applications for the existing Farrand system of electronic machine tool control. The basis of the Farrand system is a position control element known as the Linear Inductosyn which comprises a scale and slider allowing positional accuracy of better than ·0001 in. with a repeatability of approximately 25 micro inches.

CORRECTIONS

Thermal Stresses in Disks of Constant Thickness

It is regretted that certain errors appeared in the captions to this article in the May issue, as indicated below:

Fig. 2 '(r=2)' should read '(n=2)'

Fig. 11.

Upper diagram

- - - - - - - - - represents $\frac{1}{2}\frac{T_{mr}}{T_2}\left\{1-(r_1/r)^2\right\}$, not $\frac{1}{2}\frac{T_{mr}}{T_2}\left\{1-(r_1/r_2)\right\}$

-·-·-·-·-·-·- represents $\frac{1}{2}\cdot\frac{T_{mr2}}{T_2}\left\{1-\frac{r_1^2}{r_2^2}\right\}\left\{\frac{(1+\sigma)/(1-\sigma)-(r_1^2/r^2)}{(1+\sigma)/(1-\sigma)+(r_1^2/r_2^2)}\right\}$

The sum of the above quantities is given by the line bounding the hatched area on the side opposite to the line T/T_2. The hatched area represents the variation of $f_t/\alpha ET_2$ with r/r_2.

Lower diagram

- - - - - - - - - represents $\frac{1}{2}\cdot\frac{T_{mr}}{T_2}\left\{1-(r_1/r)^2\right\}$, as above.

-·-·-·-·-·-·-·- represents $\frac{1}{2}\cdot\frac{T_{mr2}}{T_2}\left\{1-\frac{r_1^2}{r_2^2}\right\}\left\{\frac{(1+\sigma)/(1-\sigma)+(r_1^2/r^2)}{(1+\sigma)/(1-\sigma)+(r_1^2/r_2^2}\right\}$

The difference of the above quantities is shown by the hatched area, which represents the variation of $f_r/\alpha ET_2$ with r/r_2.