What New Axioms Could Not Be

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`numquam reformata quia numquam deformata`

Abstract

The paper exposes the philosophical and mathematical flaws in an attempt to settle the continuum problem by a new class of axioms based on probabilistic reasoning. I also examine the larger proposal behind this approach, namely the introduction of new primitive notions that would supersede the set theoretic foundation of mathematics.

1. Introduction. In *(Freiling 1986)* we are promised a "simple philosophical 'proof' of the negation of Cantor's continuum hypothesis (CH)" which asserts that the size of the set of real numbers is the first uncountable cardinal.[1] As Freiling rightly points out, a formal proof from the standard axioms of set theory ZFC (Zermelo-Fraenkel with the axiom of choice) is impossible.[2] His suggestion is to consider in addition to ZFC "intuitively clear axioms" that are justified by the symmetry in a thought experiment where 'random darts' are successively thrown at the real number line. The philosophical appeal of this approach, according to Freiling, rests on its direct consideration of the continuum itself because of the strong connections of the latter with physical reality on the one hand, and the abstract world of set theory on the other. Freiling cites this as an advantage over the usual method for justifying axioms which in his view starts from an intuition about finite or countable sets and "haphazardly" extends it to all sets.

Those ideas have caught hold not only among philosophers and cognitive scientists, but also among mathematicians outside the field of set theory. For example D. Mumford in his address to the conference *Mathematics towards the Third Millenium* held in 1999, speaks of a "beautiful stochastic argument *to disprove the continuum hypothesis*" and wonders why it "is not universally known and considered on par with the results of Gödel and Cohen."[3] By contrast, the majority of researchers working in axiomatic set theory insist that the continuum problem is open. The purpose of this article is to explain why axioms of symmetry do not count as a solution to CH and why they are not candidates for axiomhood in the first place.

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[1] *(op. cit., p. 190)*

[2] This follows from the work of *(Gödel 1938)*. The unprovability of CH from the standard axioms was established by *(Cohen 1963)*.

[3] *(Mumford 2000, p. 206, p. 208)*

Freiling's approach to the continuum problem is in fact part of a larger proposal which seeks to formulate new axioms by introducing new primitives. In the second half of the paper some recent attempts in that direction involving randomness and other probabilistic notions will be surveyed. I will compare those attempts with the picture emerging from higher set theory where new axioms use only set theoretic concepts. This will give us an idea about the prospects of Mumford's program of devising a more intuitive and powerful formalism by building probabilities and random variables into the foundations of mathematics, at least as far as axioms of symmetry are concerned.[4]

2. Axioms of Symmetry. Freiling explains the heuristics behind these principles in terms of a thought experiment where a 'random dart' is thrown on the real number line (i. e. the unit interval $[0,1]$).[5] Given a countable set $A \subset [0,1]$, the dart will land in a point outside A with probablity 1. Now throw "two random darts" on the real line one after another. In analogy to the one-dart experiment, for any countable set $A \subset [0,1]$ associated to the point hit by the first dart (such as for example the set of all rational multiples of that point), the second dart, with probability 1, will not land in A. As the two throws are independent, the situation is symmetric with regard to the order of the throws ("the real number line does not really know which dart was thrown first or second"),[6] and thus the first dart will avoid, with probability 1, any countable set assigned to the second one. This way we are led to the following 'axiom' (with $[0,1]_{\aleph_0}$ denoting the set of all countable subsets of $[0, 1]$).

$$A_{\aleph_0}: \qquad \forall f : [0,1] \to [0,1]_{\aleph_0}\ \exists x_1, x_2 \in [0,1]\ (x_2 \notin f(x_1) \wedge x_1 \notin f(x_2))$$

The intuition behind this is that x_1 and x_2 can be found *independently* throwing two random darts. Freiling emphasizes that $A_{\aleph_0}$ is weaker than the full intuition decribed above. "All it claims is that what heuristically will happen every time, can happen". Nevertheless, this is enough to obtain a two line proof of $\neg CH$, in fact *(Freiling 1986, p. 192)* shows that $A_{\aleph_0}$ is (in ZFC) provably equivalent to $\neg CH$.[7]

At this point the reader may ask why this persuasive argument doesn't settle CH. Before entering into considerations of a philosophical nature, I want to bring out what seems to be the main mathematical flaw in the reasoning leading to the postulation of $A_{\aleph_0}$ as an axiom.

[4] Cf. *(Mumford 2000, p. 198)*

[5] The reason for this standardization is that we can interpret the Lebesgue measures of subsets of $[0,1]$ as probabilities.

[6] *(Freiling 1986, p. 192)*

[7] Assuming CH let $<$ be a wellorder of the continuum in order type the first uncountable cardinal ω_1. (A wellorder of a set A is a total order on A in which every nonempty subset of A has a least element.) Define $f : [0,1] \to [0,1]_{\aleph_0}$ by $f(x) = \{y \in [0,1] : y \leq x\}$. Then for all $x_1, x_2 \in [0,1]$, $x_1 \in f(x_2)$ or $x_2 \in f(x_1)$.

Conversely, assume $\neg CH$. Given $f : [0,1] \to [0,1]_{\aleph_0}$, fix an ω_1 sequence of distinct reals $\langle x_\alpha : \alpha < \omega_1 \rangle$. By $\neg CH$ we can pick a real $y \notin \bigcup_{\alpha<\omega_1} f(x_\alpha)$. Let x be the first real among the uncountably many x_α not belonging to the countable set $f(y)$. Then x and y are two reals satisfying the conclusion of $A_{\aleph_0}$.

Let us take a closer look how Freiling passes from his informal thought experiment to the formal statement $A_{\aleph_0}$. Essentially, what he uses is that a subset of $[0,1] \times [0,1]$ with probability 1 must be non-empty. More precisely: Consider the space $[0,1] \times [0,1]$ with the product algebra $\mathcal{B}^2 = \mathcal{B} \otimes \mathcal{B}$ (where $\mathcal{B}$ is the Borel algebra in $[0,1]$), let λ denote the Lebesgue measure on $\mathcal{B}$, and set $\lambda^2 = \lambda \otimes \lambda$ (the product measure). The formal statement of the conclusion of Freiling's thought experiment is

$$\forall f : [0,1] \to [0,1]_{\aleph_0} \;\; (\lambda^2(A_f) = 1 \;\wedge\; \lambda^2(A^f) = 1)$$

where

$$A_f = \{(x_1, x_2) : x_2 \notin f(x_1)\}$$

and

$$A^f = \{(x_1, x_2) : x_1 \notin f(x_2)\}.$$

From this it follows that $\lambda^2(A_f \cap A^f) = 1$, hence this intersection is nonempty.

This chain of inferences requires the measurability of A_f and A^f for which there seems to be no *a priori* reason. For example, following a classical argument of Sierpinski, under CH, let $<$ be a wellorder of the continuum in ordertype ω_1 and define $f(x) = \{y : y \leq x\}$. Then, by Fubini's theorem, neither A_f nor A^f are measurable.[8]

Thus the mathematical flaw in the transition from the thought experiment to $A_{\aleph_0}$ lies in its haphazard generalization of a plausible intuition about measurable subsets of $[0,1]$ to arbitrary subsets of $[0,1]$.

3. Two instructive comparisons. One may object that the above counterexample to the crucial measurability requirement is a pathological consequence of CH. It is precisely the kind of unnatural situation ruled out by $A_{\aleph_0}$. My reply to this is to draw a comparison with the Banach-Tarski paradox[9] and the axiom of choice (AC). The existence of paradoxical decompositions (as well as other counter-intuitive consequences of AC involving non-measurable sets) has not resulted in a rejection of the axiom of choice. Instead the former are generally conceived as expressions of the limitations of measurability.[10]

Admittedly the force of this argument is somewhat mitigated by the fact that AC is supported by an overwhelming amount of extrinsic evidence. Without the axiom of choice everyday mathematics becomes rather cumbersome. One can also quite reasonably hold that choice is self-evident in the sense that it is implied by the concept of set in its intended meaning. But if the above comparision with AC is disqualified on these grounds, this undermines the intuition behind Freiling's principles as well: the measurability assumption underlying

[8]In fact a weak version of Fubini not mentioning the product measure suffices: If $A \in \mathcal{B} \otimes \mathcal{B}$ is such that almost all (in λ) horizontal sections have measure 1 (0 resp.) then almost all (in λ) vertical sections have measure 1 (resp. 0) and vice versa.

[9]Using the axiom of choice, Banach and Tarski proved the existence of a decomposition of the unit ball in three dimensional Euclidean space into five pieces such that from these five pieces two copies of the unit ball can be reassembled using rotations.

[10]The Banach-Tarski paradox is resolved by noting that the partitioning sets are clearly not measurable.

the probabilistic motivation for $A_{\aleph_0}$ is neither intrinsically plausible nor does it enjoy an extrinsic justification independently of $A_{\aleph_0}$. In fact the moral of Sierpinski's counterexample is that 'axioms' of symmetry contain a hidden bias against CH. Their probabilistic motivation presupposes $\neg CH$.

At this point it may be instructive to recall Zermelo's famous proof that, assuming the axiom of choice, any set can be wellordered, a result know as the wellorder theorem.[11] Since a wellorder on a set immediately gives a choice function for that set, the two principles are in fact equivalent (in the standard axioms *without* choice). The key point of Zermelo's result was that the axiom of choice is evident in a way in which the wellorder theorem is not.[12] Now $A_{\aleph_0}$ and $\neg CH$ are provably equivalent (in ZFC). But again what evidence can be adduced in favor of $A_{\aleph_0}$ other than that tainted intuition underlying the above thought experiment?

Those who still believe in 'axioms' of symmetry might respond that the difficulty arises from the customary treatment of probability theory as based on set theoretic notions such as σ-algebras and measures. Indeed *(Mumford 2000)* suggests circumventing the set theoretic apparatus altogether by conceiving random variables as basic constructs. Before examining the viability of this proposal, I want to consider two strengthenings of $A_{\aleph_0}$ that cast further doubt on the reliability of Freiling's original probabilistic intuition.

4. More general axioms of symmetry. The heuristic leading to the formulation of $A_{\aleph_0}$ actually motivates a stronger principle. As Freiling points out, all that was used in the original argument was that countable sets of real numbers have Lebesgue measure 0.[13] Thus we may as well consider functions assigning measure 0 subsets of $[0,1]$ to points hit by a 'random dart', i. e.,

$$A_{null} \qquad \forall f:[0,1]\to[0,1]_{null}\ \exists x_1,x_2\in[0,1]\ (x_2\notin f(x_1)\wedge x_1\notin f(x_2))$$

(Here $[0,1]_{null}$ denotes the collection of subsets of $[0,1]$ with measure 0.) The principle A_{null} implies the failure of Martin's axiom (MA).[14] This fact enables us to contrast axioms of symmetry with recent developments in higher set theory.

MA is the first in a series of combinatorial principles called *forcing axioms* that have been intensively investigated. Even though the combinatorial content of these principles is not yet fully understood, a rather appealing structure theory for the first uncountable cardinal ω_1 has begun to emerge under the influence of the proper forcing axiom PFA and some of its modifications.[15] In a different direction, Woodin has recently found a canonical model for $\neg CH$[16]

[11] *(Zermelo 1904)*.

[12] See *(Zermelo 1908, §2a)*, in particular p. 187.

[13] *(Freiling 1986, p. 193 and 199.)*

[14] *(Freiling 1986, p. 193)*. Martin's axiom is a generalization of the Baire category theorem. The motivation behind it is to extend certain combinatorial aspects of countable sets to uncountable sets of size less than the continuum.

[15] *(Todorcevic 1989, 1996, 1997)*

[16] The model and its canonicity are explained in *(Woodin 1999)*. It is in some sense analogous to the constructible universe of *(Gödel 1938)* with respect to CH. However, because of the

in which MA (and stronger principles) are true. This leads to a definability analysis of $P(\omega_1)$ that implies among other things that the continuum has size greater than ω_1. In addition, Woodin was able to establish a fundamental asymmetry with respect to the continuum problem: there cannot be *any* analogous approach resulting in CH.

This gives us at least three more reasons to remain suspicious about 'axioms' of symmetry: First, no canonical model for them is known, second they come without an accompanying structure theory and third it remains unclear whether one could formulate other principles based on similiar intuitions which imply CH. In fact, given the evidence supporting MA it seems likely that A_{null} is *false*. Its weaker version $A_{\aleph_0}$ is of course true if $\neg CH$ holds (which is beginning to look more plausible in the light of current research). Nevertheless, the heuristics of 'random darts' should not be taken as grounds on which to adopt $A_{\aleph_0}$ as an axiom because it is equally supportive of the problematic principle A_{null}.

Further doubts about the viability of the intuition behind Freiling's principles arise from examining the standard consistency proof for A_{null}, i. e., the model of $ZFC + A_{null}$ obtained by adding $\aleph_2$ random reals to a model of CH.[17] Freiling's thought experiment entails the removal of pathologies by making many sets measurable. By contrast, in the above model exactly the opposite is happening. In the course of removing pathologies conflicting with A_{null} many non-measurable sets are generated.[18] This undermines the probabilistic motivation claimed for 'axioms' of symmetry.

In the spirit of the dubious motto – if you reject CH you are only two steps away from rejecting the axiom of choice – Freiling proposes another extension of $A_{\aleph_0}$ this time substituting subsets of 'small' cardinality rather than measure 0 sets for countable sets. Arguing "that the only thing special about a countable set of reals is that its complement is of higher cardinality and therefore infinitely more likely to be hit" he is led to the following principle which outright implies that there is no wellordering of the continuum.[19]

$$A_{<2^{\aleph_0}} \qquad \forall f : [0,1] \to [0,1]_{<2^{\aleph_0}}\ \exists x_1, x_2 \in [0,1]\ (x_2 \notin f(x_1) \land x_1 \notin f(x_2))$$

(Here $[0,1]_{<2^{\aleph_0}}$ denotes the sets of all subsets of $[0,1]$ of cardinality less than the continuum.) This type of reasoning is defective in several respects. The most obvious one is an unwarranted equivocation of the combinatorial notion of smallness ('small' cardinality) with the probabilistic one (measure 0). For one thing, assuming A_{null} there may be small sets that are not measurable. With respect to those sets "statements like 'its complement is infinitely more likely to be hit' do not have any meaning in the usual sense," as Freiling himself is

limitations imposed by forcing, in the $\neg CH$ case it is the theory of the model, rather than the model itself which is canonical.

[17] Cf. *(Freiling 1986, p. 193)*

[18] One manifestation of this is Woodin's proof (unpublished) of an extension of the Fubini-Tonelli theorem in that model. (A similar result is proved in *(Friedman 1980)*.) For example the set of random reals (over the ground model) has outer measure 1, yet fails to be measurable.

[19] *(Freiling 1986, p. 192)*

willing to concede.[20]

5. The larger proposal. The fact that many fundamental statements of set theory as well as propositions in other areas of mathematics are undecidable on the basis of the standard axioms motivates the search for additional axioms. The axiomatic extensions which have turned out most useful so far (large cardinal axioms, axioms of determinacy and forcing axioms) are all expressible in the language of set theory with *set* and *membership* as primitive notions. A different way of introducing axioms is to expand the language of set theory by adding symbols for new primitives and to formulate plausible principles about them. From those principles in the expanded language solutions to problems stated in the original language of set theory may be obtainable. *(Kreisel 1969, p. 100)* argues for one such proposal (namely the one of adding a truth predicate), and remarks that "[a] more interesting, but also more problematic, expansion in the literature *(Kruse 1967)* concerns the primitive predicate of being a *random sequence*." A recent articulation of this point of view is contained in the aformentioned programmatic article by Mumford. He suggests that treating the concept of *random variable* as a basic construct will render the customary set theoretic foundation of mathematics obsolete and result in a more intuitive and powerful formalism.

> The reductionist approach defines random variables in terms of measures, which are defined in terms of the theory of the reals, which are defined in terms of set theory, which is defined on top of predicate calculus. I'd like to propose instead that it should be possible to put random variables into the very foundations of both logic and mathematics and arrive at a more complete and more transparent formulation of the stochastic point of view. *(ibid, p. 206)*

Mumford concedes that he lacks a complete formulation of his program, and offers what he calls "a sketch which draws on two sources [he] find[s] very provocative. The first is the development by E. T. Jaynes of the foundations of Bayesian probability and statistics *(Jaynes 1996 - 2000)*; the second is a beautiful stochastic argument due to Christopher Freiling to *disprove the continuum hypothesis (Freiling 1986)*."

An appraisal of the viability of Jaynes' viewpoint falls outside the scope of this article. At any rate his primary concern is not the resolution of undecidable statements of infinitary mathematics (which he regards as meaningless altogether), but rather to devise a general calculus of plausible reasoning applicable to all inferences arising from incomplete information.[21] Suffice it to say that this is a highly contested area among philosophers of science, especially

[20] *(loc. cit.)* In this context recall also Gödel's remarks on the implausible consequences of CH described in terms of covering properties of small sets in *(Gödel 1947, p. 185f)* as well as the commentary of *(Martin 1976, p. 87)* and the subsequent work of *(Todorcevic 1997)*.

[21] See *(Jaynes 1996 - 2000, preface)*. In that sense it is a quantitative formulation of the approach taken by *(Polya 1945, 1954a,b)* for treating general problems of scientific methodology.

with respect to the Bayesian approach favored by Jaynes. For example, it has repeatedly been pointed out that models of quantitative confirmation employing Bayesian theories of belief change are plagued with persistent failures to match our intuitive judgements about evidence.[22] Similarly, at present far too little is known about the way we think, in order to arrive at definite conclusions whether stochastic models and statistical reasoning are more relevant to understanding the computations in our own minds than exact models and logical reasoning.[23]

In what follows I will concentrate on the other source cited by Mumford, namely the probabilistically motivated 'axioms' of symmetry. My aim is to examine Mumford's claim that they establish grounds for making random variables one of the basic elements of mathematics, and to question whether

> it follows that the C.H. is false and we will get rid of one of the meaningless conundrums of set theory. The continuum hypothesis is surely similiar to the scholastic issue of how many angels can stand on the head of a pin: an issue which disappears if you change your point of view. *(Mumford 2000, p. 208)*

In view of the diffculties created by non-measurable sets and Sierpinski's counterexample, Mumford points out that a stochastic reformulation of set theory requires dropping either the axiom of choice or the power set axiom. His reasons for dropping the latter and keeping the former are that "the existence of random objects is a sort of axiom of random choice" together with his belief that "mathematics really needs, for each set X, [...] not the huge set 2^X but the set of sequences $X^{\mathbb{N}}$ in X." Moreover, any definable subset A of the continuum must be measurable so that a plausibility may be assigned to the event of a random variable taking a value in A.[24] In support of his proposal, Mumford mentions some recent work in higher set theory.[25] Actually, as we shall see, these results provide hints for appraising the prospects of Mumford's program. But before doing so, I shall sketch two modern attempts of incorporating probabilistic notions into the foundations of mathematics.

6. Randomness and independence. An immediate difficulty when one tries to define *random numbers* is that their elementary properties seem to contradict each other. One way to avoid this is by switching from classical to intuitionistic logic where *lawless sequences* provide another interpretation of 'arbitrary'.[26] A related approach using classical logic was suggested by Myhill's axiomatization of random numbers[27] which treats randomness as an intensional notion, i.e., a notion concerning the mode in which infinite sequences are given to us.

[22] Some of the problems with Bayesian accounts of confirmation are described in *(Christensen 1999*. See also *(Glymour 1980)*.

[23] Cf. *(Mumford 2000, p. 198)*. Recent empirical studies *(Kahneman and Tversky 1996)* suggest that the process of scientific discovery does not rest on probabilistic inductive reasoning.

[24] Cf. *(ibid)*

[25] Namely *(Shelah and Woodin 1990)*

[26] *(Troelstra and van Dalen 1988)*

[27] See *(Kruse 1967)*

Finally, two attempts to axiomatize randomness and independence within classical logic are made in *(van Lambalgen 1992)* taking into account extensional as well as intensional aspects of infinite sequences. His idea is to view random sequences as obtained by infinitely many independent choices as opposed to defining randomness in terms of statistical properties such as the stability of relative frequencies over many repetitions. Thus the new primitive is *independence*. The notation $R(x, \vec{y})$ expresses that x is independent of $\vec{y}$, or that $\vec{y}$ has no information about x. In other words, the language at our disposal does not allow us to define (using the parameters $\vec{y}$) a 'small' set containing x. Adding axioms about R to the ZF axioms, Lambalgen arrives at a system called $ZF\mathcal{R}$.[28] A drawback of $ZF\mathcal{R}$ is that it denies that the continuum can be wellordered and thereby refutes the axiom of choice. The two-line proof *(op. cit., theorem 1.3)* makes essential use of the axiom of extensionality, and van Lambalgen takes this as an indication that randomness should be treated as an intensional concept. He speculates that the system $ZF\mathcal{R}$ minus the axiom of extensionality is consistent with choice, and proposes intensional set theory along the lines of (*Beeson 1985)* as a suitable framework for further investigations of consistency. The system $ZF\mathcal{R}$ itself is consistent relative to ZF since R has a straightforward interpretation in a model of Solovay obtained by adding uncountably many random reals to a model of CH. (cf. *(Solovay 1970)* and *(Kunen 1984)*) In fact this interpretation strongly suggests that $ZF\mathcal{R}$ is merely an abstraction of combinatorics for the continuum in these models. This contrasts with van Lambalgen's claim that $ZF\mathcal{R}$ describes "a non-artificial notion of set for which we do not have perfect information about all sets", and that it tells us " what goes on in universes of set theory (like the world of AD) in which all sets are Lebesgue measurable, or have the property of Baire."[29] In reality, models of full AD are far more complicated than random real extensions,

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[28] Van Lambalgen's axioms are as follows *(van Lambalgen 1992, p. 1279ff)*

R0 Axioms and inference rules of classical predicate logic

R1 $\exists x\, R(x)$

R2 $R(x, \vec{y}\vec{z}) \to R(x, \vec{z})$

R3 (a) $R(x, \vec{y}) \to R(x, \pi\vec{y})$ for any permutation π
(b) $R(x, y\vec{z}) \to R(x, yy\vec{z})$

R4 $R(x, y) \to x \neq y$

R5 Suppose $\phi(x, \vec{y})$ is in $L_{\in,=,R}$, where x ranges over reals and all parameters are listed among the $\vec{y}$. Then

$$\exists x(R(x, \vec{y}) \wedge \phi(x, \vec{y})) \to \exists x(R(x, z\vec{y}) \wedge \phi(x, \vec{y})).$$

(Here z must be different from x.)

R6 $R(y, \vec{z}) \wedge R(x, y\vec{z}) \to R(y, x\vec{z})$ (Steinitz exchange principle)

The system consisting of axioms R0 - R6 is denoted by $\mathcal{R}$. If the $\vec{y}$ range only over reals, this is indicated by $\mathcal{R}^0$. $ZF\mathcal{R}$ is obtained by adding $\mathcal{R}$ and similiarly $ZF\mathcal{R}^0$. In both cases R may occur in the schemata of ZF, and the first argument of R is always a real.

[29] *(van Lambalgen 1992, p. 1278, p. 1282f). AD* stands for the *axiom of determinacy* which stipulates that in a certain class of infinite two-person games of perfect information, one of the two players has a winning strategy. (Cf. *(Moschovakis 1980)*).

and their essential features apparently have nothing to do with randomness as explicated in $ZF\mathcal{R}$.[30]

Another fact which, according to van Lambalgen, renders the extensional theory of randomness difficult is that "practical certainty is not absolute certainty", and this motivates formalizing the properties of "practically certain" itself and incorporating this notion into ZF.[31] For the purpose of illustration suppose we have a stochastic mechanism (such as a 'fair coin') randomly producing infinite binary sequences. Add to the language of set theory a generalized quantifier Q, where the intended interpretation of $Qx\,\phi(x)$ is "if x is randomly generated, it is practically certain that $\phi(x)$". This leads to a system dubbed $ZF\mathcal{Q}$ with the same drawback as $ZF\mathcal{R}$, namely that the continuum is no longer wellorderable.[32] Lambalgen shows that the theories $ZF\mathcal{R}$ and $ZF\mathcal{Q}$ are bi-interpretable, and he describes in detail how Freiling's axioms of symmetry can be 'embedded' into a suitable fragment of $ZF\mathcal{Q}$.[33]

To summarize: the attempts examined above to axiomatize probabilistic notions directly instead of defining them set theoretically depart from the concept of set as codified by the standard axioms ZFC.[34] Unless one is willing to adopt intuitionistic logic, either the axiom of choice or the axiom of extensionality has to be abandoned. Both alternatives are unattractive. My reasons for holding this opinion are not so much that classical mathematics (in particular real analysis) cannot be developed in intuitionistic or constructive set theory. After all several case studies beginning with Weyl's *Das Kontinuum* in 1918, have demonstrated that a fair amount of 'scientifically applicable' mathematics can be formalized in weak subsystems of set theory that are reducible to Peano Arithmetic.[35] But the results obtained in the course of this program

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[30] Some of these features are explored in *(Woodin, Mathias and Hauser forthcoming)*

[31] *(van Lambalgen 1992, p. 1285)*

[32] Variables bound by Q range over reals. The following axioms for Q are given in *(van Lambalgen 1992, sec. 2)*

Q0 Axioms and inference rules of classical logic

Q1 $\neg Qx\; x \neq x$

Q2 $Qx\; x \neq y$

Q3 $Qx\,\phi(\ldots, x, \ldots) \rightarrow Qy\,\phi(\ldots, y, \ldots)$

Q4 $Qx\phi \wedge Qx(\phi \rightarrow \psi) \rightarrow Qx\psi$

Q5 $Qx\phi \wedge Qx\psi \rightarrow Qx(\phi \wedge \psi)$

Q6 $QxQy\phi \leftrightarrow QyQx\phi$

$\mathcal{Q}$ is the system consisting of Q0 - Q6. $ZF\mathcal{Q}$ is obtained by adding $\mathcal{Q}$ to ZF where Q is allowed to occur in the schemata.

[33] The fragment is denoted by $ZF\mathcal{E}_2$ and basicly allows two iterations of Q. $ZF\mathcal{E}_2$ proves $A_{\aleph_0}$, and conversely $ZF\mathcal{E}_2 + AC$ is interpretable in $ZFC + A_{\aleph_0}$. *(van Lambalgen 1992, §6.2)*

[34] The reason for restricting the discussion to the axiomatizations in *(van Lambalgen 1992)* was their connection with Freiling's axioms of symmetry. Another approach to randomness is based on the notion of algorithmic *complexity* along the lines of A. Kolmogorov and A. Solomonoff. Cf. *(Li and Vitanyi 1997)*. This allows a definition of randomness without any recourse to physical reality whereas *(Freiling 1986, p. 199)* regards the crucial symmetry argument in his thought experiment as an "almost physical intuition".

[35] See for example *(Feferman 1993)*. However, the complete dispensibility of set theory in scientific applications of mathematics has not yet been established: The mathematical

contribute very little to the epistemological analysis of the abstract ideas guiding mathematical progress. The essence of mathematics resides in the broadest possible abstraction, idealization and generalization. In this regard intuitionism and constructivism with their restrictions on acceptable mathematical reasoning fall short of providing a foundation for mathematics. Therefore we should remain suspicious of proposals to introduce new primitive notions if we are told that they are best studied within those schools of thought. Van Lambalgen's axioms for randomness and independence belong to that category. He suggests constructive set theory as a suitable framework for capturing the intensional aspects of randomness,[36] and he repeatedly compares random sequences with lawless sequences in the intuitionistic sense in order to justify their properties. In fact he shows that there is a precise correspondence between the two.[37] I do not deny the usefulness of studies of randomness and independence be it in classical logic or in intuitionistic and constructivistic systems. But I do not see why this should result in a more intuitive and powerful formalism than the one provided by set theory.[38]

Similar remarks apply with respect to the first one of the four "self-evident philosophical principles" cited by Freiling in support of his 'axioms' of symmetry, namely their connection with physical reality.

> Choosing reals at random is a physical reality, or at least an intuition mathematics should embrace to the extent possible. *(Freiling 1986, p. 199)*

For one thing how are we supposed to construe the physical existence of real numbers given that actual measurements produce only rational numbers as outcomes?[39] On more general grounds, one may even question whether randomness should be regarded as a *fundamental* entity in the description of physical reality.[40] In any case, mathematical research primarily pursues problems for the sake of their intrinsic interest independently of irrelevant considerations about physical reality.[41] Mathematics, as Weyl remarked,[42] is characterized by the theoretical desire toward totality which is incomprehensible from a strictly phenomenal point of view.

7. Beyond the standard axioms. The ZFC axioms are an adequate formal framework of mathematics in the sense that (i) every mathematical statement

treatment of quantum field theory, for example, requires at least some impredicative notions. In any case I regard conjectures about the (in)dispensibility of set theory as highly speculative for the simple reason that the future shape of physics or other scientific fields depending on mathematics is unpredictable.

[36] *(van Lambalgen 1992, §3)*

[37] *(van Lambalgen 1992, §5)*

[38] Recall that $ZF\mathcal{R}$ can readily be interpreted in terms of Solovay-randomness, a fact pointed out by van Lambalgen himself. *(van Lambalgen 1992, 1.6)*

[39] Already at this point, those who are claiming that integers are less problematic than sets because they can be 'embedded' into physical reality, are faced with an analogous difficulty.

[40] See for example *(Dürr 2001)*.

[41] The continuum problem which in the words of *(Hilbert 1925, p. 180)* "is distinguished by its unique character and inner beauty", serves as an illustrative example.

[42] *(Weyl 1949, p. 66)*

is expressible in the language of set theory with *set* and *membership* as the only primitives and (ii) every theorem of classical mathematics is formally derivable from ZFC. Of course alternative foundations using different primitives may be possible and equally successful. However, at least for now, there is no viable axiomatization of mathematics in terms of probabilistic notions. The purpose of this section is to outline another aspect of the foundations of mathematics where set theory has made significant advances. An alternative foundation such as the one envisioned by Mumford would have to reproduce those advances in one way or another if it were to be taken seriously as an alternative.

Soon after Cohen's proof of the underivability of CH from the standard axioms, it became clear that independence is a concern of mathematical practice. Not only many fundamental questions of set theory, but also important problems in other areas of mathematics cannot be resolved within ZFC.[43] In anticipation of that situation and motivated by philosophical reflections, *(Gödel 1947)* proposed a search for new axioms which are implied by the concept of set and strong enough to settle questions beyond the reach of ZFC.

> First of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of". These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers... *(Gödel 1947, p. 181)*

Some large cardinal axioms had already been introduced at the beginning of the twentieth century – that is almost three decades before the standard axioms were given their present form. But their systematic investigation did not begin until the 1960s with the advent of forcing and the importation of techniques from other areas of mathematical logic. Since then the subject has grown enormously, and the resulting theory is commonly viewed as the correct superstructure for ZFC. The principal reason for this is the remarkable fact that – despite the ostensible disparity of the various motivations underlying their formulation – these axioms are arranged into a *linear* hierarchy ordered by increasing logical strength.[44] Moreover, the large cardinal hierarchy functions as a yardstick for measuring logical strength. Propositions transcending ZFC (to the extent they have been analyzed so far) are equiconsistent with some large cardinal axiom - *including statements not containing set theoretic vocabulary.*

Along this hierarchy canonical 'halting points' have been identified. One of them concerns the definability theory of the continuum which is commonly known as second order number theory. Here the underlying idea is to obtain information about the continuum by studying the sets definable in this structure, namely the *projective sets*. For example, we may ask whether there is a

[43] Examples include the Kaplanski Conjecture *(Dales and Woodin 1987)*, the Whitehead problem *(Shelah 1974)*, and the S- and L-Space Problems *(Todorcevic 1989)*.

[44] For an overview of the theory of large cardinals see *(Kanamori 1994)*. The logical strength of a mathematical proposition is measured in terms of relative consistency. A proposition A_1 is stronger than A_2 if the consistency of $ZFC + A_1$ implies the consistency of $ZFC + A_2$. Note that there is no *a priori* reason why A_1 and A_2 should be comparable this way. The fact that members of a wide ranging class of propositions are comparable this way is quite remarkable.

projective counterexample to CH: an uncountable projective set which is not equinumerous with $\mathbb{R}$. For their systematic investigation the projective sets are classified according to the logical complexity of their definitions. Since these sets are precisely the ones arising in real analysis it will be important to know that many of them are well behaved. For the first two levels of the projective sets, ZFC yields the classical regularity properties such as Lebesgue measurability. At higher levels, the presence of regularity properties is independent from ZFC, so stronger axioms are needed. If one stipulates that all infinite two-person games with projective pay-off sets are determined (meaning that in any such game one of the two players has a winning strategy), the structure theory for the first two levels can be extended to a complete structure theory for all projective sets giving the kind of answers that are most useful to analysts. For instance, under projective determinacy (PD), all projective sets are Lebesgue measurable.[45]

On the other hand, acceptance of PD as an *axiom* is hampered by its lack of plausibility in contrast with the intrinsic evidence supporting large cardinal axioms. The apparent difficulty is that determinacy as a 'local' phenomenon occuring in the power set of the real numbers apparently has no connection with 'global' considerations about higher regions of the set theoretic universe motivating large cardinals. By the end of the 1980s this difficulty could be overcome, and in a series of dramatic advances a precise level-by-level correspondence between determinacy and large cardinals was established.[46] Nowadays, we see that PD is ubiquitous. There are a vast number of combinatorial statements implying PD which are seemingly unrelated to it. Moreover, in virtually all cases of interest where a principle has higher consistency strength than PD, it actually proves PD. Thus when extending ZFC by any of these stronger principles we automatically pass through second order number theory as given by PD.

The work done on projective sets in the last fifteen years has also resulted in a remarkable interaction of set theory with other areas of mathematics. On the one hand the notions and dichotomies in descriptive set theory turned out to carry meanings in disciplines ranging from harmonic analysis, Banach space theory and topological dynamics to control theory and mathematical economics. On the other hand sophisticated techniques from other branches of mathematics have led to the solution of purely set theoretic problems and furnished new insights into the relationship of descriptive set theory with other branches of mathematical logic.[47] Admittedly, the effect of these developments on strong hypotheses is somewhat mitigated by the fact that many of the applications in-

[45]Among other things this means there are no paradoxical decompositions of the unit sphere into projective pieces. *(Moschovakis 1980)* is the standard reference to second order number theory under projective determinacy

[46]Ironically, the results in *(Shelah and Woodin 1990)* cited by Mumford in support of his own proposal, acted as a catalyst by identifying the relevant large cardinal concept. See *(Kanamori 1994)* for a lively account of these developments. Proofs and further references are given in *(Martin and Steel 1989)* and *(Woodin, Mathias and Hauser forthcoming)*.

[47]Examples of such branches are recursion theory (in connection with the global structure of Turing degrees) and model theory (topological Vaught Conjecture). For an overview and further references see *(Kechris 2001)*.

volve only the low levels of the projective hierarchy, for which ZFC suffices.[48] However, with increasing sophistication the number of genuine uses of stronger hypotheses is likely to grow. In any case, irrespective of considerations about logical strength, the full theory of the projective sets performs a crucial epistemic function by virtue of their strong closure properties as the natural framework for modeling purposes. This fact (which is already in evidence in the set theoretic foundation of analysis) is philosophically significant.

As far as the continuum problem is concerned, by *(Levy and Solovay 1967)* there is no hope of duplicating the success of large cardinals achieved in second order number theory. Nevertheless, I do not see any compelling basis for the view that CH is an ill-posed question as alleged in *(Mumford 2000, p. 208)*[49] In my opinion, the mathematical evolution of set theory in the post-Cohen era gives no reason to regard the hope as irrational that the continuum problem will some day be settled. In the end a comprehensive theory as a whole rather than an intrinsically plausible axiom may be accepted as a *solution* to CH (or alternatively as an *explanation* why there is no fact of the matter about the size of the continuum). But this much is already certain: 'axioms' of symmetry fall short in both respects.

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[48] In some sense this echoes the speculative remarks on the real needs of mathematics in *(Mumford 2000, p. 208)* quoted above.

[49] Strangely, in that passage Mumford acknowledges that CH is false while at the same time asserting that it is meaningless. For a more systematic evaluation of arguments for and against the position that CH has determinate truth value see *(Fefermann 2000)*, *(Hauser 2002)* and *(Steel 2000)*.

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