Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE @ DIRECT

Journal of Computational and Applied Mathematics 160 (2003) 113 – 123

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

On some slowly convergent series involving the Hurwitz zeta-function

M. Hashimoto[a], S. Kanemitsu[b], Y. Tanigawa[c,*], M. Yoshimoto[c], W.-P. Zhang[d]

[a] Department of Mathematics, Wakayama National College of Technology, Nojima 77, Nada-machi, Gobo, Wakayama 644-0023, Japan
[b] Graduate School of Advanced Technology, Kinki University, Iizuka, Fukuoka 820-8555, Japan
[c] Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
[d] Department of Mathematics, Northwest University, Xi'an, Shaanxi, People's Republic of China

Received 21 October 2002; received in revised form 25 March 2003

Abstract

We shall extract the essence of the Adamchik–Srivastava generating function method (Analysis (Munich) 18 (1998) 131) by proving the most far-reaching Ramanujan–Yoshimoto formula and by showing that some of the results stated in Srivastava and Choi (Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, 2001) are simple consequences of the above-mentioned formula.

Keywords: Series; Hurwitz zeta-function

1. Introduction

In their recent book [9], Srivastava and Choi refer to the generating function method of Adamchik and Srivastava [1] implemented in Mathematica and give several applications of the method.

Our aim in this paper is:

(i) to point out in Section 3 that the generalization of their formulas can be easily deduced from the far-reaching Ramanujan–Yoshimoto formula [4–7] and

* Corresponding author.

E-mail addresses: hashimot@wakayama-nct.ac.jp (M. Hashimoto), kanemitu@fuk.kindai.ac.jp (S. Kanemitsu), tanigawa@math.nagoya-u.ac.jp (Y. Tanigawa), x02001n@math.nagoya-u.ac.jp (M. Yoshimoto), wpzhang@nwu.edu.cn (W.-P. Zhang).

doi:10.1016/S0377-0427(03)00617-4

(ii) to give the fourth proof thereof in Section 2 by the generating function method of Adamchik–Srivastava.

We shall use the following notation.

The Hurwitz zeta-function is defined by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \quad \sigma := \operatorname{Re} s > 1,$$

where α is not a non-positive integer (in what follows we assume $\operatorname{Re}\alpha > 0$).

The Riemann zeta-function is defined by

$$\zeta(s) = \zeta(s,1) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.$$

The gamma function is defined by

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-t} t^{s-1}\, \mathrm{d}t, \quad \sigma > 0.$$

The digamma function is the logarithmic derivative of the gamma function

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s)$$

and the Euler constant is given by $\gamma = -\psi(1)$.

The Bernoulli polynomial of degree n is defined by the generating function

$$\frac{z\mathrm{e}^{-xz}}{1-\mathrm{e}^{-z}} = \sum_{n=0}^{\infty} \frac{B_n(1-x)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n B_n(x)}{n!} z^n, \quad |z| < 2\pi \tag{1.1}$$

and the nth Bernoulli number is defined by

$$B_n = B_n(0), \quad n = 0,1,2,\ldots,$$

so that

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_{2k+1} = 0 \ (k = 1,2,\ldots), \quad B_4 = -\tfrac{1}{30}, \ldots$$

and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{1.2}$$

The Stirling numbers $S(n,k)$ of the second kind are defined by

$$x^n = \sum_{k=0}^{n} S(n,k) x(x-1)\cdots(x-k+1). \tag{1.3}$$

2. The Ramanujan–Yoshimoto formula

The following formula, whose prototype is found in Ramanujan's Second Notebook [2, Entry 28(b), p. 279], has a vast range of applications, and indeed, almost all parts of Chapters 3, 4, 5 of [9], some 180pp., can be deduced from it, as explained in [6,7].

Theorem 1 (Theorem A, Kanemitsu et al. [6]). *Suppose λ is a non-negative integer*, $\operatorname{Re}\alpha > 0$ *and* $|z| < |\alpha|$. *Then*

$$\sum_{m=2}^{\infty} \frac{(m-1)!}{(m+\lambda)!}(-1)^m \zeta(m,\alpha) z^{m+\lambda}$$

$$= \frac{1}{\lambda!}\zeta'(-\lambda,\alpha+z) - \frac{1}{\lambda!}\sum_{m=0}^{\lambda}\binom{\lambda}{m}\zeta'(-m,\alpha)z^{\lambda-m}$$

$$-\frac{1}{(\lambda+1)!}\sum_{m=1}^{\lambda}\binom{\lambda+1}{m}(\psi(\lambda+1)-\psi(m))B_m(\alpha)z^{\lambda+1-m}$$

$$-\frac{z^{\lambda+1}}{(\lambda+1)!}(\psi(\lambda+1)+\psi(\alpha)+\gamma). \tag{2.1}$$

Remark 1. The form given in Theorem 1 is slightly different from that in Theorem A [6].

For the proof we need two lemmas.

Lemma 1 (Adamchik and Srivastava [1, p. 136]). *Suppose the sequence $\{f(k)\}$ satisfies $f(k) = O(1/k)$. Then the infinite sum $\Omega(\alpha) = \sum_{k=1}^{\infty} f(k)\zeta(k+1,\alpha)$ can be expressed as the Laplace transform of $F(t)/(1-\mathrm{e}^{-t})$:*

$$\Omega(\alpha) = \int_0^{\infty} \frac{F(t)}{1-\mathrm{e}^{-t}} \mathrm{e}^{-\alpha t}\, \mathrm{d}t,$$

where $F(t)$ is the generating function of $f(k)$ defined by

$$F(t) = \sum_{k=1}^{\infty} \frac{f(k)}{k!} t^k.$$

The following lemma evaluates the Laplace transform of the function required for the proof.

Lemma 2. *Suppose* $\operatorname{Re}\alpha > 0$ *and that* $|z| \leqslant |\alpha|$, $z \neq -\alpha$. *Then for* $n \in \mathbb{N}$, *we have*

(i)

$$\int_0^{\infty} \left(\frac{t\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}} - \sum_{k=0}^{n} (-1)^k \frac{B_k(z)}{k!} t^k \right) \mathrm{e}^{-\alpha t} t^{-n-1}\, \mathrm{d}t$$

$$=\frac{(-1)^{n-1}}{(n-1)!}\zeta'(1-n,\alpha+z)$$

$$+\frac{(-1)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}B_k(z)\alpha^{n-k}\{\psi(n)-\psi(n+1-k)+\log\alpha\}$$

(ii) *and*

$$\int_0^\infty\left(\frac{t\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}-1\right)\mathrm{e}^{-\alpha t}t^{-1}\,\mathrm{d}t=-\psi(\alpha+z)+\log\alpha.$$

Proof. Recall the integral representation for $\zeta(s,\alpha)$, $\sigma>1$:

$$\zeta(s,\alpha)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t\mathrm{e}^{-\alpha t}}{1-\mathrm{e}^{-t}}t^{s-2}\,\mathrm{d}t$$

and that $(t\mathrm{e}^{-\alpha t})/(1-\mathrm{e}^{-t})$ is a generating function of Bernoulli polynomials.

Extracting the first $(n+1)$ terms gives

$$\zeta(s,\alpha+z)=\sum_{k=0}^{n}\frac{\Gamma(s+k-1)}{\Gamma(s)k!}(-1)^kB_k(z)\alpha^{-s-k+1}$$

$$+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{t\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}-\sum_{k=0}^{n}\frac{B_k(z)}{k!}(-t)^k\right)\mathrm{e}^{-\alpha t}t^{s-2}\,\mathrm{d}t, \tag{2.2}$$

where the integral is analytically continued to $\sigma>-n$.

In view of

$$\frac{\Gamma(s+k-1)}{\Gamma(s)}=\frac{\Gamma(1-s)}{\Gamma(2-s-k)}(-1)^{k-1},$$

we may write the first term on the right-hand side as

$$-\sum_{k=0}^{n}\frac{\Gamma(1-s)}{\Gamma(2-s-k)k!}B_k(z)\alpha^{-s-k+1}.$$

Now we differentiate both sides with respect to s and then let $s\to 1-n$. Differentiated form of (2.2) is

$$\zeta'(s,\alpha+z)$$

$$=\sum_{k=0}^{n}\frac{\Gamma(1-s)}{\Gamma(2-s-k)k!}B_k(z)\alpha^{-s-k+1}\{\psi(1-s)-\psi(2-s-k)+\log\alpha\}$$

$$+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{t\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}-\sum_{k=0}^{n}\frac{B_k(z)}{k!}(-t)^k\right)\mathrm{e}^{-\alpha t}t^{s-2}(\log t-\psi(s))\,\mathrm{d}t. \tag{2.3}$$

As $s \to 1-n$, Formula (2.3) becomes

$$\zeta'(1-n,\alpha+z)$$
$$=\sum_{k=0}^{n}\frac{(n-1)!}{(n-k)!k!}B_k(z)\alpha^{n-k}\{\psi(n)-\psi(n+1-k)+\log\alpha\}$$
$$+(-1)^{n-1}(n-1)!\int_0^\infty\left(\frac{t\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}-\sum_{k=0}^{n}\frac{B_k(t)}{k!}(-t)^k\right)\mathrm{e}^{-\alpha t}t^{-n-1}\,\mathrm{d}t,$$

whence (i) follows.

To prove (ii), we subtract $1/(s-1)$ from both sides of (2.2) with $n=0$, and then let $s\to1$, upon noting the fact

$$\lim_{s\to1}\left\{\zeta(s,\alpha+z)-\frac{1}{s-1}\right\}=-\psi(\alpha+z).$$

This completes the proof. □

Remark 2. Srivastava and Choi [9] state an integral expression (25), p. 93, for $\zeta(s,\alpha)$ similar to (2.2) above, with $z=0$, but the range of s in their formula should read $\mathrm{Re}\,s>-n$ rather than $\mathrm{Re}\,s>-(2n-1)$. The reason why such an error occurs is most probably that they use two kinds of definition of Bernoulli numbers, i.e., although they define Bernoulli numbers by ($z=0$ in (1.1))

$$\frac{t}{\mathrm{e}^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n \quad (|t|<2\pi)$$

(2), p. 59, they use $(-1)^{k-1}B_k$ instead of B_{2k} in the Stirling formula (54), p. 8, (thus they take the first $(2n-1)$ terms), and as a result, Formula (56), p. 22, a consequence of (54) p. 8, contradicts (37), p. 6, where B_{2k}'s are used. We also note that in (55), p. 8, the numerator of the coefficients of z^{-6} should be 5,246,819.

Proof of Theorem 1. In Lemma 1 we take

$$f(k)=\frac{k!}{(k+\lambda+1)!}(-1)^{k+1}z^{k+\lambda+1}. \tag{2.4}$$

Then the generating function $F(t)$, being the infinite series

$$(-1)^{\lambda}t^{-\lambda-1}\sum_{k=\lambda+2}^{\infty}\frac{(-zt)^k}{k!},$$

has a finite expression

$$F(t)=(-1)^{\lambda}\,\mathrm{e}^{-zt}t^{-\lambda-1}-\sum_{k=0}^{\lambda+1}\frac{(-1)^{\lambda+k}}{k!}z^kt^{k-\lambda-1}. \tag{2.5}$$

Hence

$$\frac{F(t)}{1-\mathrm{e}^{-t}}=\frac{t\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}(-1)^{\lambda}t^{-\lambda-2}+\sum_{k=0}^{\lambda+1}\frac{(-1)^kz^{\lambda+1-k}}{(\lambda+1-k)!}\frac{t}{1-\mathrm{e}^{-t}}t^{-k-1}. \tag{2.6}$$

We transform the generating function parts on the right-hand side of (2.6) in the same way as we did in the proof of Lemma 2, i.e. we extract the first $(\lambda+2)$ terms from $te^{-zt}/(1-e^{-t})$ and the first $(n+1)$ terms from $t/(1-e^{-t})$. After simplification, we have

$$\frac{F(t)}{1-e^{-t}} = (-1)^{\lambda}\left\{\frac{te^{-zt}}{1-e^{-t}} - \sum_{k=0}^{\lambda+1}\frac{B_k(z)}{k!}(-t)^k\right\}t^{-\lambda-2} + \sum_{n=0}^{\lambda+1}\frac{(-1)^n z^{\lambda+1-n}}{(\lambda+1-n)!}\left\{\frac{t}{1-e^{-t}} - \sum_{k=0}^{n}\frac{B_k}{k!}(-t)^k\right\}t^{-n-1} + \sum_{k=0}^{\lambda+1}\frac{B_k(z)}{k!}(-1)^{\lambda+k}t^{k-\lambda-2} + \sum_{n=0}^{\lambda+1}\sum_{k=0}^{n}\frac{(-1)^{n-k}B_k}{(\lambda+1-n)!k!}z^{\lambda+1-n}t^{k-n-1}. \tag{2.7}$$

First we simplify the last term on the right-hand side of (2.7). Changing the order of summation and squeezing out the binomial coefficients, we see that the last term becomes

$$\sum_{k=0}^{\lambda+1}\frac{(-1)^k t^{-k-1}}{(\lambda+1-k)!}\sum_{n=k}^{\lambda+1}\binom{\lambda+1-k}{n-k}B_{n-k}z^{\lambda+1-k-(n-k)},$$

which is

$$\sum_{k=0}^{\lambda+1}\frac{(-1)^k t^{-k-1}}{(\lambda+1-k)!}B_{\lambda+1-k}(z),$$

or

$$-\sum_{k=0}^{\lambda+1}\frac{(-1)^{\lambda+k}t^{k-\lambda-2}}{k!}B_k(z),$$

and this cancels the third term on the right-hand side of (2.7). Thus we end up with the first two terms on the right-hand side of (2.7):

$$\frac{F(t)}{1-e^{-t}} = (-1)^{\lambda}\left\{\frac{te^{-zt}}{1-e^{-t}} - \sum_{k=0}^{\lambda+1}\frac{B_k(z)}{k!}(-t)^k\right\}t^{-\lambda-2} + \sum_{n=1}^{\lambda+1}\frac{(-1)^n z^{\lambda+1-n}}{(\lambda+1-n)!}\left\{\frac{t}{1-e^{-t}} - \sum_{k=0}^{n}\frac{B_k}{k!}(-t)^k\right\}t^{-n-1} + \frac{z^{\lambda+1}}{(\lambda+1)!}\left\{\frac{t}{1-e^{-t}} - 1\right\}t^{-1}. \tag{2.8}$$

Now the Laplace transform of the right-hand side of (2.8) is computed in Lemma 2, (i), (ii), so that

$$\Omega(\alpha) = \frac{1}{\lambda!}\zeta'(-\lambda, \alpha + z) - \sum_{n=1}^{\lambda+1} \frac{z^{\lambda+1-n}}{(\lambda+1-n)!(n-1)!}\zeta'(1-n, \alpha) - \frac{z^{\lambda+1}}{(\lambda+1)!}\psi(\alpha) - S_0 + S_1 + S_2 - S_3 + S_4, \tag{2.9}$$

say, where

$$\begin{aligned} S_0 &= \frac{1}{(\lambda+1)!}\psi(\lambda+1)\sum_{k=0}^{\lambda+1}\binom{\lambda+1}{k}B_k(z)\alpha^{\lambda+1-k}, \\ S_1 &= \frac{1}{(\lambda+1)!}\sum_{k=0}^{\lambda+1}\binom{\lambda+1}{k}\alpha^k\psi(k+1)B_{\lambda+1-k}(z), \\ S_2 &= \sum_{n=1}^{\lambda+1}\frac{z^{\lambda+1-n}}{(\lambda+1-n)!n!}\psi(n)\sum_{k=0}^{n}\binom{n}{k}B_k\alpha^{n-k}, \\ S_3 &= \sum_{n=1}^{\lambda+1}\frac{z^{\lambda+1-n}}{(\lambda+1-n)!n!}\sum_{k=0}^{n}\binom{n}{k}B_k\alpha^{n-k}\psi(n+1-k) \end{aligned} \tag{2.10}$$

and

$$S_4 = \left(-\frac{1}{(\lambda+1)!}\sum_{k=0}^{\lambda+1}\binom{\lambda+1}{k}B_k(z)\alpha^{\lambda+1-k} + \sum_{n=1}^{\lambda+1}\sum_{k=0}^{n}\frac{z^{\lambda+1-n}}{(\lambda+1-n)!n!}\binom{n}{k}B_k\alpha^{n-k} + \frac{z^{\lambda+1}}{(\lambda+1)!}\right)\log\alpha.$$

We transform S_3 slightly to get

$$S_3 = \frac{1}{(\lambda+1)!}\sum_{n=0}^{\lambda+1}\binom{\lambda+1}{n}z^{\lambda+1-n}\sum_{k=0}^{n}\binom{n}{k}B_k\alpha^{n-k}\psi(n-k+1) - \frac{\psi(1)}{(\lambda+1)!}z^{\lambda+1}.$$

Changing the order of summation and using the relation

$$\binom{\lambda+1}{n}\binom{n}{k} = \binom{\lambda+1}{k}\binom{\lambda+1-k}{n-k},$$

we deduce that

$$S_3 = \frac{1}{(\lambda+1)!}\sum_{k=0}^{\lambda+1}\binom{\lambda+1}{k}\alpha^k\psi(k+1)\sum_{n=k}^{\lambda+1}\binom{\lambda+1-k}{n-k}B_{n-k}z^{\lambda+1-k-(n-k)} - \frac{\psi(1)}{(\lambda+1)!}z^{\lambda+1}.$$

By (1.2), the inner sum is $B_{\lambda+1-k}(z)$, so that by change of variable, we see that

$$S_3 = S_1 - \frac{\psi(1)}{(\lambda+1)!} z^{\lambda+1}. \tag{2.11}$$

Similarly, using (1.2), we can compute S_0, S_2 and S_4 to be

$$\begin{aligned} S_0 &= \frac{\psi(\lambda+1)}{(\lambda+1)!} \sum_{k=1}^{\lambda+1} \binom{\lambda+1}{k} B_k(\alpha) z^{\lambda+1-k} + \frac{z^{\lambda+1}}{(\lambda+1)!}, \\ S_2 &= \frac{1}{(\lambda+1)!} \sum_{n=1}^{\lambda+1} \binom{\lambda+1}{n} \psi(n) z^{\lambda+1-n} B_n(\alpha) \end{aligned} \tag{2.12}$$

and

$$S_4 = 0.$$

From (2.9)–(2.12), it follows that

$$\begin{aligned} \Omega(\alpha) = {} & \frac{1}{\lambda!} \zeta'(-\lambda, \alpha+z) - \frac{1}{\lambda!} \sum_{k=0}^{\lambda} \binom{\lambda}{k} \zeta'(-k,\alpha) z^{\lambda-k} \\ & - \frac{1}{(\lambda+1)!} \psi(\lambda+1) \sum_{k=1}^{\lambda+1} \binom{\lambda+1}{k} B_k(\alpha) z^{\lambda+1-k} \\ & + \frac{1}{(\lambda+1)!} \sum_{k=1}^{\lambda+1} \binom{\lambda+1}{k} \psi(k) B_k(\alpha) z^{\lambda+1-k} \\ & - \left(\frac{\psi(\lambda+1)}{(\lambda+1)!} z^{\lambda+1} - \frac{\psi(1)}{(\lambda+1)!} z^{\lambda+1} + \frac{\psi(\alpha)}{(\lambda+1)!} z^{\lambda+1} \right). \end{aligned} \tag{2.13}$$

From (2.13) we can readily read off the formula in Theorem 1, thus completing the proof. □

3. Some consequences

As is explained in [4], Theorem 1 transforms into Theorem B in [6] giving the formula for $\sum_{m=2}^{\infty} [(\zeta(m,\alpha))/(m+\lambda)] z^m$.

The following example illustrates the power of Theorem 1.

Example (Srivastava and Choi [9, (712), p. 248], Kanamitra et al. [6]).

$$\begin{aligned} & \sum_{m=2}^{\infty} \frac{(-1)^m \zeta(m,\alpha)}{m(m+1)(m+2)} z^{m+2} \\ & \quad = \log \Gamma_3(z+\alpha) + \left(z+\alpha-\tfrac{3}{2}\right) \log \Gamma_2(z+\alpha) + \tfrac{1}{2}(z+\alpha-1)^2 \log \Gamma(z+\alpha) \end{aligned}$$

$$-\log\Gamma_3(\alpha)-\left(\alpha-\tfrac{3}{2}\right)\log\Gamma_2(\alpha)-\tfrac{1}{2}(\alpha-1)^2\log\Gamma(\alpha)$$
$$-\left(\log\Gamma_2(\alpha)+(\alpha-1)\log\Gamma(\alpha)+\frac{\alpha^2}{4}-\frac{\alpha}{4}+\zeta'(-1)+\frac{1}{24}\right)z$$
$$-\frac{1}{2}\left(\log\Gamma(\alpha)+\frac{3\alpha}{2}-\frac{1}{2}\log 2\pi-\frac{3}{4}\right)z^2-\frac{1}{6}\left(\psi(\alpha)+\frac{3}{2}\right)z^3.$$

The special case $\lambda=0$ of (2.1) reads

$$\sum_{k=2}^{\infty}\zeta(k,\alpha)\frac{t^k}{k}=\log\Gamma(\alpha-t)-\log\Gamma(k)+t\psi(\alpha),\qquad |t|<|\alpha| \tag{3.1}$$

which is a well-known formula stated in [3, p. 358] and [10, p. 276].

Corollary 1 (Generalizations of Proposition 3.3, Srivastava and Choi [9, p. 153]). *Suppose* $\operatorname{Re}\alpha>0$ *and* $|\omega|<|\alpha|$. *Then for any integer* $q\geqslant 2$, *we have*

$$\sum_{n=1}^{\infty}\frac{\omega^{nq}}{n}\zeta(nq,\alpha)=\sum_{r=1}^{q}\log\Gamma(\alpha-\mathrm{e}^{2\pi ir/q}\omega)-q\log\Gamma(\alpha).$$

We also have

$$\sum_{n=1}^{\infty}\frac{\zeta(nq+1,\alpha)}{nq+1}=\frac{1}{q}\sum_{r=1}^{q}\mathrm{e}^{-2\pi ir/q}\log\Gamma(\alpha-\mathrm{e}^{2\pi ir/q})+\psi(\alpha)$$

and

$$\sum_{n=0}^{\infty}\frac{\zeta(nq+m,\alpha)}{nq+m}=\frac{1}{q}\sum_{r=1}^{q}\mathrm{e}^{-2\pi irm/q}\log\Gamma(\alpha-\mathrm{e}^{2\pi ir/q})$$

for $2\leqslant m<q$.

Proof. Put $t=\mathrm{e}^{2\pi ir/q}\omega$ in (3.1). Then

$$\sum_{k=2}^{\infty}\frac{1}{k}\zeta(k,\alpha)\mathrm{e}^{2\pi irk/q}\omega^k=\log\Gamma(\alpha-\mathrm{e}^{2\pi ir/q}\omega)-\log\Gamma(\alpha)+\psi(\alpha)\mathrm{e}^{2\pi ir/q}\omega. \tag{3.2}$$

Now sum (3.2) over $r=1,\ldots,q$, and use

$$\sum_{r=1}^{q}\mathrm{e}^{\frac{2\pi irk}{q}}=\begin{cases}0 & \text{if } q\nmid k,\\ q & \text{if } q|k\end{cases}$$

to obtain

$$\sum_{\substack{k=2\\ q|k}}^{\infty}\frac{\omega^k}{k}\zeta(k,\alpha)q=\sum_{r=1}^{q}\log\Gamma(\alpha-\mathrm{e}^{2\pi ir/q}\omega)-q\log\Gamma(\alpha).$$

Since the left-hand side is $\sum_{n=1}^{\infty}(\omega^{nq}/n)\zeta(nq,\alpha)$, this completes the proof of the first formulas. Other formula can be proved similarly. □

Corollary 2 (Generalization of Propositions 3.4 and 3.5, Srivastava and Choi [9, pp. 154–155]). *For* $0 \leqslant n \in \mathbb{Z}$ *and* $\mathrm{Re}\,\alpha > 0$, *we have*

$$\Lambda(n) := \sum_{k=2}^{\infty} k^n z^k \zeta(k,\alpha)$$

$$= \sum_{l=2}^{n+1} S(n+1,l)(l-1)!\zeta(l,\alpha-z)z^l - z(\psi(\alpha-z)-\psi(\alpha))$$

for $|z| < |\alpha|$.

Proof. We substitute the formula (1.3) with n replaced by $n+1$ in the form

$$k^n = 1 + \sum_{l=2}^{\min\{k,n+1\}} S(n+1,l)\,\frac{(k-1)!}{(k-l)!}$$

for k^n in the sum for $\Lambda(n)$ to get

$$\Lambda(n) = \sum_{k=2}^{\infty} z^k \zeta(k,\alpha) + \Lambda_1(n),$$

say, where

$$\Lambda_1(n) = \sum_{k=2}^{\infty} \sum_{l=2}^{\min\{k,n+1\}} S(n+1,l)\,\frac{(k-1)!}{(k-l)!}\,\zeta(k,\alpha)z^k.$$

Changing the order of summation and rewriting the coefficients $(k-1)!/(k-l)!$ as $(l-1)!(\Gamma(l+k)/\Gamma(l)k!)$ by putting k instead of $k-l$, we conclude that

$$\Lambda_1(n) = \sum_{l=2}^{n+1} S(n+1,l)(l-1)!z^l \sum_{k=0}^{\infty} \frac{\Gamma(l+k)}{\Gamma(l)k!}\,\zeta(k+l,\alpha)z^k,$$

whose inner sum is equal to

$$\zeta(l,\alpha-z)$$

due to Wilton [11], which is the Taylor expansion in the second variable (for this interpretation, cf. Klusch [8] and Kanemitsu et al. [4]). □

References

[1] V.S. Adamchik, H.M. Srivastava, Some series of the zeta and related functions, Analysis (Munich) 18 (1998) 131–144.

[2] B.C. Berndt, Ramanujan's Notebooks. Part I, Springer, New York, 1985.

[3] E.R. Hansen, A Table of Series and Products, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[4] S. Kanemitsu, M. Katsurada, M. Yoshimoto, On the Hurwitz–Lerch zeta function, Aequationas Math. 59 (2000) 1–19.

[5] S. Kanemitsu, H. Kumagai, H.M. Srivastava, M. Yoshimoto, Some integral and asymptotic formulas associated with the Hurwitz zeta-function, to be published.

[6] S. Kanemitsu, H. Kumagai, M. Yoshimoto, Sums involving the Hurwitz zeta-function, Ramanujan J. 5 (2001) 5–19.
[7] S. Kanemitsu, H. Kumagai, M. Yoshimoto, On rapidly convergent series expressions for zeta- and L-values, Ramanujan J. 5 (2001) 91–104.
[8] D. Klusch, On the Taylor expansion of the Lerch zeta-function, J. Math. Anal. Appl. 170 (1992) 513–523.
[9] H.M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, 2001.
[10] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge University Press, New York, 1962.
[11] J.R. Wilton, A proof of Burnside's formula for $\log \Gamma(z+1)$ and certain allied properties of Riemann's ζ-function, Messenger Math. 52 (1922/1923) 90–93.