Finite Fields and Their Applications 27 (2014) 115–129

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

# Hasse–Weil zeta functions of $SL_2$-character varieties of arithmetic two-bridge link complements

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A R T I C L E I N F O

*Article history:*
Received 23 August 2012
Received in revised form
2 September 2013
Accepted 7 January 2014
Available online 6 February 2014
Communicated by Michael Tsfasman

*MSC:*
11T99
57M25

*Keywords:*
Hasse–Weil zeta function
$SL_2$-character variety
Arithmetic two-bridge link

A B S T R A C T

Hasse–Weil zeta functions of $SL_2$-character varieties of arithmetic two-bridge link groups are determined. Special values of the zeta functions at $s = 0, 1, 2$ are also investigated.



## 0. Introduction

In [6] the Hasse–Weil zeta function of the $SL_2$-character variety of the figure 8 knot, the only arithmetic knot in the 3-sphere, has been determined (see [15] as a reference on arithmetic manifolds). In this note we extend the result to the most basic class of arithmetic 2-component links, the family of arithmetic two-bridge links, $L_0 := 5_1^2 = (8/3)$

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1071-5797/$ – see front matter 
http://dx.doi.org/10.1016/j.ffa.2014.01.001

(Whitehead link), $L_1 := 6^2_2 = (10/3)$ and $L_2 := 6^2_3 = (12/5)$. In fact, for the canonical components (irreducible components containing holonomy representations) of their $\mathrm{SL}_2(\mathbb{C})$-character varieties which are algebraic surfaces defined by

$$\begin{aligned} f_0 &:= z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy, \\ f_1 &:= z^4 - xyz^3 + (x^2 + y^2 - 3)z^2 - xyz + 1, \\ f_2 &:= z^3 - xyz^2 + (x^2 + y^2 - 1)z - xy, \end{aligned}$$

we compute the Hasse–Weil zeta functions

$$\zeta(L_i, s) := \zeta(f_i, s) := \prod_p \exp\left(\sum_{n=1}^{\infty} \frac{\#V(f_i)(\mathbb{F}_{p^n})}{n} (p^{-s})^n\right),$$

where $p$ runs through all the prime numbers and we denote by $V(f_i)(\mathbb{F}_{p^n})$ the set of $\mathbb{F}_{p^n}$-rational points of $f_i$:

$$V(f_i)(\mathbb{F}_{p^n}) := \{(a, b, c) \in (\mathbb{F}_{p^n})^3 \mid f_i(a, b, c) = 0\}$$

for the finite field $\mathbb{F}_{p^n}$ with $p^n$ elements. The result is as follows.

**Theorem.** *The Hasse–Weil zeta functions of the canonical components of* $L_0 = 5^2_1 = (8/3)$, $L_1 = 6^2_2 = (10/3)$ *and* $L_2 = 6^2_3 = (12/5)$ *are the following.*

$$\begin{aligned} \zeta(L_0, s) &= \zeta_{\mathbb{Q}(\sqrt{2})}(s-1)\zeta(s)^2\zeta(s-1)^2\zeta(s-2)(1 - 2^{1-s})^3(1 - 2^{-s}). \\ \zeta(L_1, s) &= \zeta_{\mathbb{Q}(\sqrt{5})}(s-1)^2\zeta(s)^3\zeta(s-2)(1 - 2^{1-s})^3(1 + 2^{1-s})(1 - 2^{-s}). \\ \zeta(L_2, s) &= \zeta(s)^2\zeta(s-2)(1 - 2^{-s}). \end{aligned}$$

Here $\zeta_K(s)$ is the Dedekind zeta function of an algebraic number field $K$ and $\zeta(s) := \zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function. It is shown in [10,5] that the canonical components of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of arithmetic two-bridge links have the structure of conic bundles over the projective line $\mathbb{P}^1$. That makes it easy to compute the number of rational points of the canonical components over finite fields. Note that the fields $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}$ for $5^2_1 = (8/3)$, $6^2_2 = (10/3)$ and $6^2_3 = (12/5)$ appearing in the description of the zeta functions, are obtained from $\mathbb{Q}$ by adjoining all the $\mathbb{P}^1$-coordinates of the degenerate fibers of the canonical components considered as conic bundles over $\mathbb{P}^1$. We also note that the trace fields (which are same as the invariant trace fields) of $5^2_1$, $6^2_2$ and $6^2_3$ are $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$, respectively (cf. [4], §5).

Many researches are made on the special values of the Dedekind zeta functions of algebraic number fields at the integer points. Especially the values at $s = 0, 1$ are interesting, since those values are described by the intrinsic invariants of the number fields (class number formula). Table 1 shows the special values (more precisely, the coefficient

**Table 1**
Special values of the main terms of $\zeta(L_i, s)$ at $s = 0, 1, 2$.

| | $s = 0$ | $s = 1$ | $s = 2$ |
|---|---|---|---|
| $\zeta(L_0, s)$ | zero, order 1 $\frac{-\zeta(3)}{2^{10}3^3\pi^2}$ | pole, order 1 $\frac{R_{\mathbb{Q}(\sqrt{2})}}{2^5 3}$ | pole, order 3 $-\frac{\sqrt{2}\pi^4 R_{\mathbb{Q}(\sqrt{2})}}{2^4 3^2}$ |
| $\zeta(L_1, s)$ | zero, order 1 $\frac{\zeta(3)}{2^7 3^2 5^2 \pi^2}$ | pole, order 1 $-\frac{R^2_{\mathbb{Q}(\sqrt{5})}}{2^4 3}$ | pole, order 2 $-\frac{\pi^6 R^2_{\mathbb{Q}(\sqrt{5})}}{2^2 3^3 5}$ |
| $\zeta(L_2, s)$ | zero, order 1 $-\frac{\zeta(3)}{2^4\pi^2}$ | pole, order 2 $-\frac{1}{2^2 3}$ | $-\frac{\pi^4}{2^3 3^2}$ |

of the lowest degree in the Laurent expansion when the point is a zero or a pole) of the above zeta functions at $s = 0, 1, 2$. Note that we have only computed the special values of the essential parts of $\zeta(L_i, s)$ (namely, the terms of the products of the Dedekind zeta functions up to rational functions in $p^{-s}$). In the table $R_K$ is the regulator of an algebraic number field $K$. Note that $R_{\mathbb{Q}(\sqrt{2})} = \log(1 + \sqrt{2})$ and $R_{\mathbb{Q}(\sqrt{5})} = \log(\frac{1+\sqrt{5}}{2})$ respectively.

It is shown by Smyth ([1], Appendix or [18], Corollary 2) that the value $\zeta(3)/\pi^2$ is expressed as $2m(1 + x + y + z)/7$, where the (logarithmic) Mahler measure $m(P)$ is defined by

$$m(P) = \int_0^1 \cdots \int_0^1 \log\left|P\left(e^{2\pi t_1\sqrt{-1}}, \ldots, e^{2\pi t_n\sqrt{-1}}\right)\right| dt_1 \cdots dt_n$$

for a multivariable Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Note that this expression is not unique, namely $\zeta(3)/\pi^2$ is also expressed as $m((1+w)(1+x)+(1-w)(1-x)y)/7$ ([8], §2) in terms of the Mahler measure of a 4-variable polynomial. Since the other terms are the logarithms of some algebraic integers, they are expressed as the Mahler measures of those integers. Therefore we see that the special values of $\zeta(L_i, s)$ at $s = 0, 1, 2$ are the products of the Mahler measures of certain polynomials up to algebraic numbers and some positive powers of $\pi$.

There is a relation between the volumes of hyperbolic 3-manifolds and the special values of Dedekind zeta functions and $L$-functions of imaginary quadratic fields (cf. [2,3]). Especially it is known that the volume of the figure 8 knot complement in the 3-sphere is equal to the Mahler measure of the $A$-polynomial of the figure 8 knot. Thus it would be interesting to seek for expressions of the above special values by the Mahler measures of certain link invariants.

The number of conjugacy classes of $\mathrm{SL}_2$-representations of link groups over a fixed finite field is a computable link invariant. The explicit formula for the number of conjugacy classes of $\mathrm{SL}_2$-representations of torus knot groups over finite fields was given by Weiping Li and Liang Xu in 2003–2004 [11,12]. Recently the number of $\mathrm{SL}_2(\mathbb{F}_p)$-conjugacy classes of all the $\mathrm{SL}_2(\mathbb{F}_p)$-representations has been computed for small primes by Kitano and Suzuki [7] for all the knots in the Rolfsen's knot table, which is effective enough to distinguish them. The description of the Hasse–Weil zeta functions in the theorem provides the exact number of $\mathrm{SL}_2(\mathbb{F}_{p^n})$-conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_{p^n})$-representations (in the canonical components) of the arithmetic two-bridge link groups for any prime number $p$ and $n \geqslant 1$.

In §3.1 of her thesis [9] Landes studied the infinite family of hyperbolic two-bridge links (namely hyperbolic two-bridge links with Schubert normal form $(6n + 2/3)$, see also [16] for the hyperbolicity) obtained from the magic 3 manifold by $1/n$ Dehn filling on one cusp, and showed that for $n = 1, 2, 3, 4$ their canonical components of the character varieties have conic bundle structure over $\mathbb{P}^1$. The number of rational points and the local zeta functions of standard conic bundles over finite fields were determined by Tsfasman [19] and Rybakov [17]. Hence it would be interesting to show whether the canonical components of this family have conic bundle structure or not, and to study the Hasse–Weil zeta functions of this family of hyperbolic two-bridge links.

It is well known that the Dehn filling on one cusp of the Whitehead link complement produces infinitely many hyperbolic 3 manifolds with one cusp. In particular we can obtain the twist knot complements from the Whitehead link complement by $1/n$ Dehn filling. Recently Macasieb, Petersen and van Luijk showed [14] that the $\mathrm{SL}_2$ character varieties of the twist knots are hyperelliptic curves. It would be interesting to study the zeta functions of twist knots in terms of that of the Whitehead link complement, which may provide information on the Hasse–Weil zeta functions of certain family of hyperelliptic curves.

The author is grateful to the referees for their careful reading of the original manuscript. Especially he appreciates for pointing out an error in the computation of the number of rational points on the complements of degenerate fibers of the conic bundles, which fixed the description of the zeta functions in theorem.

## 1. Projective models of character varieties of $5_1^2$, $6_2^2$ and $6_3^2$

Let $\mathbb{P}^n := \mathbb{P}^n_k$ be the $n$-dimensional projective space over a field $k$. Let

$$f_0 := z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy,$$
$$f_1 := z^4 - xyz^3 + (x^2 + y^2 - 3)z^2 - xyz + 1,$$
$$f_2 := z^3 - xyz^2 + (x^2 + y^2 - 1)z - xy$$

be the defining polynomial of the canonical component of the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Whitehead link complement $5_1^2 = (8/3)$, $6_2^2 = (10/3)$ and $6_3^2 = (12/5)$ respectively (for details, see [5], 1.1). We remark that the canonical components of arithmetic two-bridge links are hypersurfaces. Thus the defining polynomials are uniquely determined as monic irreducible polynomials. Let

$$F := F_0 := u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3,$$
$$F_1 := u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3 + u^2w^4,$$

$$F_2 := u^2z^3 - xyz^2w + \left(x^2+y^2-u^2\right)zw^2 - xyw^3$$

be the corresponding homogeneous polynomials of $f_i$ in $\mathbb{P}^2 \times \mathbb{P}^1$, where

$$\mathbb{P}^2 \times \mathbb{P}^1 := \left\{(x:y:u, z:w) \mid (x:y:u) \in \mathbb{P}^2, (z:w) \in \mathbb{P}^1\right\}.$$

Let $S_i := V(F_i) \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the projective (singular) surface defined by $F_i$ over the field $k$. For the singularities of $S_i$ we can show the following lemma by the Jacobian criterion (we will not use this lemma in what follows in this note).

**Lemma 1.1.** *Let $k$ be an algebraically closed field with characteristic $\operatorname{Char} k = p \geqslant 0$.*

(1) *When $\operatorname{Char} k \neq 2$, the surface $S_0$ has the following 4 singular points*

$$(1:0:0,1:0), \quad (0:1:0,1:0), \quad (1:1:0,1:1), \quad (1:-1:0,1:-1).$$

*When $\operatorname{Char} k = 2$, in addition to the above 3 points (note that $(1:1:0,1:1)$, $(1:-1:0,1:-1)$ are same in $\operatorname{Char} k = 2$), the other singular points of $S_0$ are as follows*:

$$(x:1:x+1,1:1), \quad (x:x+1:1,1:1), \quad (1:y:y+1,1:1), \quad (0:0:1,0:1),$$

*where $x,\ y \in k$.*

(2) *When $\operatorname{Char} k \neq 2,5$, the surface $S_1$ has the following 6 singular points:*

$$\begin{array}{lll}(1:0:0,1:0), & (0:1:0,1:0), & (1:0:0,0:1),\\ (0:1:0,0:1), & (1:1:0,1:1), & (1:-1:0,1:-1).\end{array}$$

*When $\operatorname{Char} k = 5$, in addition to the above 6 points in $\operatorname{Char} k \neq 2,5$ case, $S_1$ has 2 more singular points $(0:0:1,1:\pm 2)$.*
*When $\operatorname{Char} k = 2$, in addition to the 5 points in $\operatorname{Char} k \neq 2,5$ case, $S_1$ has the following singular points:*

$$(x:x+1:1,1:1), \quad (x:1:x+1,1:1), \quad (1:y:y+1,1:1), \quad (0:0:1,1:w),$$

*where $x,\ y \in k$ and $w \in k$ satisfies $w^2+w+1=0$.*

(3) *When $\operatorname{Char} k \neq 2$, the projective surface $S_2 = V(F_2)$ has the following 4 singular points*:

$$(1:0:0,1:0), \quad (0:1:0,1:0), \quad (1:1:0,1:1), \quad (1:-1:0,1:-1).$$

*When $\operatorname{Char} k = 2$, in addition to the above 3 points, $S_2$ has the following singularities*:

$$(x : x : 1, 1 : 1), \quad (1 : 1 : u, 1 : 1)$$

*where* $x, u \in k$.

Consider the projection $\phi_i : S_i \to \mathbb{P}^1$ which is defined by $(x : y : u, z : w) \mapsto (z : w)$. Then the surface $S_i$ can be considered as a (singular) conic bundle over $\mathbb{P}^1$, that is, each fiber is a (singular) projective curve of degree 2.

## 2. Number of rational points on $S_i$

In what follows we consider the surfaces $S_i$ over the finite field $\mathbb{F}_p$ and compute the number of their $k$-rational points, where $k = \mathbb{F}_q$ is the finite field with $q = p^n$ elements. For this purpose we regard the set of $k$-rational points of $S_i$ as the union of fibers at the $k$-rational points of $\mathbb{P}^1$ and study its degenerate fibers. Then we compute the local and global zeta functions of them.

### *2.1.* $5_1^2$ *case*

When $p \neq 2$ we see directly from the defining polynomial by the Jacobian criterion that the surface $S_0$ has 6 degenerate fibers at $(1 : 0)$, $(0 : 1)$, $(1 : \pm 1)$, $(1 : \pm \frac{1}{\sqrt{2}})$. In fact, the degenerate fibers of $\phi_0 : S_0 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned}
\phi_0^{-1}(1:0) &= \bigl\{(x : y : u, 1 : 0) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| u^2 = 0\bigr\}, \\
\phi_0^{-1}(0:1) &= \bigl\{(x : y : u, 0 : 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| xy = 0\bigr\}, \\
\phi_0^{-1}\left(1 : \pm\frac{1}{\sqrt{2}}\right) &= \left\{\left(x : y : u, 1 : \pm\frac{1}{\sqrt{2}}\right) \in \mathbb{P}^2 \times \mathbb{P}^1 \biggm| \frac{1}{2}(x \mp \sqrt{2}y)\left(x \mp \frac{1}{\sqrt{2}}y\right) = 0\right\}, \\
\phi_0^{-1}(1 : \pm 1) &= \bigl\{(x : y : u, 1 : \pm 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| (x \mp y) - u)\bigl((x \mp y) + u\bigr) = 0\bigr\}.
\end{aligned}$$

Then the number of $k = \mathbb{F}_q$-rational points on each fiber is as follows:

$$\bigl|\phi_0^{-1}(1:0)\bigr| = q+1, \qquad \bigl|\phi_0^{-1}(0:1)\bigr| = 2q+1, \qquad \bigl|\phi_0^{-1}(1:\pm 1)\bigr| = 2q+1,$$

$$\left|\phi_0^{-1}\left(1 : \pm\frac{1}{\sqrt{2}}\right)\right| = \begin{cases} 2q+1 & \text{if } (\frac{2}{p}) = 1, \\ (2q+1)\frac{1+(-1)^n}{2} & \text{if } (\frac{2}{p}) = -1. \end{cases}$$

Here $(\frac{2}{p})$ is the Legendre symbol of 2 modulo $p$. Since any other fiber at $(1 : w)$ except these degenerate fibers have the $k$-rational point $(1 : w : 0, 1 : w)$, they are isomorphic to $\mathbb{P}^1$ over $k$ (cf. [13], Chap. 7, Prop. 4.1). Therefore, for $q = p^n$, the number $N_{n,0}$ of all the $k = \mathbb{F}_q$-rational points of $S_0$ is

$$N_{n,0} = \begin{cases} (q+1)(q-5) \\ (q+1)(q-3-(1+(-1)^n)) \end{cases} + (q+1) + 3(2q+1)$$

$$+ 2\begin{cases} 2q+1 & \text{if } (\frac{2}{p}) = 1, \\ (2q+1)\frac{1+(-1)^n}{2} & \text{if } (\frac{2}{p}) = -1 \end{cases}$$

$$= \begin{cases} q^2+7q+1 & \text{if } (\frac{2}{p}) = 1, \\ q^2+5q+1+q(1+(-1)^n) & \text{if } (\frac{2}{p}) = -1. \end{cases}$$

Therefore, for an arbitrary odd prime $p$, the local zeta function $Z_0(p,T)$ is computed as follows:

$$\begin{aligned} Z_0(p,T) &:= \exp\left(\sum_{n=1}^{\infty} \frac{N_{n,0}}{n} T^n\right) \\ &= \left(1-p^2T\right)^{-1}(1-pT)^{-5}(1-T)^{-1} \times \begin{cases} (1-pT)^{-2} & \text{if } (\frac{2}{p}) = 1, \\ \exp(\sum_{n=1}^{\infty} \frac{2p^{2n}}{2n} T^{2n}) & \text{if } (\frac{2}{p}) = -1 \end{cases} \\ &= \begin{cases} (1-p^2T)^{-1}(1-pT)^{-7}(1-T)^{-1} & \text{if } (\frac{2}{p}) = 1, \\ (1-p^2T)^{-1}(1-pT)^{-6}(1-T)^{-1}(1+pT)^{-1} & \text{if } (\frac{2}{p}) = -1. \end{cases} \end{aligned}$$

When $p = 2$, the surface $S_0$ has 3 degenerate fibers at $(1:0)$, $(0:1)$, $(1:1)$. In fact the degenerate fibers of $\phi_0 : S_0 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned} \phi_0^{-1}(1:0) &= \left\{(x:y:u, 1:0) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid u^2 = 0\right\}, \\ \phi_0^{-1}(0:1) &= \left\{(x:y:u, 0:1) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid xy = 0\right\}, \\ \phi_0^{-1}(1:1) &= \left\{(x:y:u, 1:1) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid (x+y+u)^2 = 0\right\}. \end{aligned}$$

Hence we can compute the number of $k$-rational points on each degenerate fiber. The result is

$$\left|\phi_0^{-1}(1:0)\right| = q+1, \qquad \left|\phi_0^{-1}(0:1)\right| = 2q+1, \qquad \left|\phi_0^{-1}(1:1)\right| = q+1.$$

Since any other fiber except these degenerate fibers is isomorphic to $\mathbb{P}^1$ over $k$, the number $N_{n,0}$ of all the $k = \mathbb{F}_q$-rational points of $S_0$ for $p = 2$ is

$$\begin{aligned} N_{n,0} &= (q+1)(q-2) + 2(q+1) + (2q+1) \\ &= q^2 + 3q + 1. \end{aligned}$$

Thus the local zeta function $Z_0(2,T)$ is

$$Z_0(2,T) = \exp\left(\sum_{n=1}^{\infty} \frac{N_{n,0}}{n} T^n\right) = \left(1-2^2T\right)^{-1}(1-2T)^{-3}(1-T)^{-1}.$$

Thus the Hasse–Weil zeta function

$$\zeta(S_0, s) := \prod_p Z_0\bigl(p, p^{-s}\bigr)$$

is written as follows:

$$\begin{aligned}\zeta(S_0, s) &= \Bigl(\prod_p \bigl(1-p^{2-s}\bigr)^{-1}\bigl(1-p^{1-s}\bigr)^{-6}\bigl(1-p^{-s}\bigr)^{-1}\Bigr)\bigl(1-2^{1-s}\bigr)^3 \\ &\quad \times \prod_{(\frac{2}{p})=1}\bigl(1-p^{1-s}\bigr)^{-1} \times \prod_{(\frac{2}{p})=-1}\bigl(1+p^{1-s}\bigr)^{-1} \\ &= \zeta(s-2)\zeta(s-1)^6\zeta(s)L\Bigl(\Bigl(\frac{2}{\cdot}\Bigr), s-1\Bigr)\bigl(1-2^{1-s}\bigr)^3.\end{aligned}$$

Here

$$\Bigl(\frac{2}{\cdot}\Bigr) : (\mathbb{Z}/8\mathbb{Z})^\times \to \mathbb{C}^\times; \quad n \mapsto \Bigl(\frac{2}{n}\Bigr)$$

is the Dirichlet character associated with the Legendre symbol $(\frac{2}{\cdot})$ and

$$L\Bigl(\Bigl(\frac{2}{\cdot}\Bigr), s\Bigr) = \sum_{n \geqslant 1} \frac{(\frac{2}{n})}{n^s} = \prod_p \Bigl(1 - \Bigl(\frac{2}{p}\Bigr)p^{-s}\Bigr)^{-1}$$

is the Dirichlet $L$-function of $(\frac{2}{\cdot})$.

*2.1.1. Non-affine part*

To have the description of the Hasse–Weil zeta function $\zeta(f_0, s)$ here we compute the number of $k$-rational points of the non-affine part $V(F_0) \setminus V(f_0)$ and its zeta function. We see that the set $V(F_0) \setminus V(f_0)$ consists of points which have coordinates $(x : y : 0, 1 : 0)$ or $(x : y : 0, z : 1)$. Indeed it consists of three subsets

$$\begin{aligned}V(F_0) \setminus V(f_0)(k) = &\bigl\{(x : y : 0, 1 : 0) \bigm| (x : y) \in \mathbb{P}^1_k\bigr\} \\ &\cup \bigl\{(z : w : 0, z : w) \bigm| (z : w) \in \mathbb{P}^1_k\bigr\} \\ &\cup \bigl\{(w : z : 0, z : w) \bigm| (z : w) \in \mathbb{P}^1_k\bigr\}.\end{aligned}$$

Then it is easy to compute the number $N'_{n,0}$ of $k$-rational points of $V(F_0) \setminus V(f_0)$ and its zeta functions.

$$\begin{aligned}N'_{n,0} &= \begin{cases} 3q-1, & \text{if } p \neq 2, \\ 3q, & \text{if } p = 2.\end{cases} \\ Z'_0(p, T) &= \exp\Bigl(\sum_{n=1}^{\infty} \frac{N'_{n,0}}{n} T^n\Bigr) = \begin{cases} (1-pT)^{-3}(1-T), & \text{if } p \neq 2, \\ (1-2T)^{-3}, & \text{if } p = 2.\end{cases} \\ \tilde{\zeta}(S_0, s) &:= \prod_p Z'_0\bigl(p, p^{-s}\bigr) = \zeta(s-1)^3\zeta(s)^{-1}\bigl(1-2^{-s}\bigr)^{-1}.\end{aligned}$$

Thus we have the description of the Hasse–Weil zeta function of the canonical component $V(f_0)$ of $5_1^2$.

$$\begin{aligned}\zeta\big(5_1^2, s\big) := \zeta(f_0, s) &:= \prod_p \exp\left(\sum_{n=1}^{\infty} \frac{\#V(f_0)(\mathbb{F}_{p^n})}{n}\big(p^{-s}\big)^n\right) = \zeta(S_0, s)/\tilde{\zeta}(S_0, s) \\ &= \zeta(s-2)\zeta(s-1)^3\zeta(s)^2 L\left(\left(\frac{2}{\cdot}\right), s-1\right)\big(1-2^{1-s}\big)^3\big(1-2^{-s}\big) \\ &= \zeta(s-2)\zeta(s-1)^2\zeta(s)^2\zeta_{\mathbb{Q}(\sqrt{2})}(s-1)\big(1-2^{1-s}\big)^3\big(1-2^{-s}\big).\end{aligned}$$

### *2.2. $6_3^2$ case*

Since $6_3^2$ case is similar to the previous $5_1^2$ case and the computation is rather simple, we treat this case next and summarize the result without much detail. When $p \neq 2$ the surface $S_2$ has 4 degenerate fibers at $(1:0)$, $(0:1)$, $(1:\pm 1)$. In fact, the degenerate fibers of $\phi_2 : S_2 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned}\phi_2^{-1}(1:0) &= \big\{(x:y:u, 1:0) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| u^2 = 0\big\} = \big\{(x:y:0, 1:0)\big\}, \\ \phi_2^{-1}(0:1) &= \big\{(x:y:u, 0:1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| xy = 0\big\} \\ &= \big\{(x:0:u, 0:1)\big\} \vee \big\{(0:y:u, 0:1)\big\}, \\ \phi_2^{-1}(1:\pm 1) &= \big\{(x:y:u, 1:\pm 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| (x \mp y)^2 = 0\big\} \\ &= \big\{(x:\pm x:u, 1:\pm 1) \in \mathbb{P}^2 \times \mathbb{P}^1\big\}.\end{aligned}$$

Hence the number of $k = \mathbb{F}_q$-rational points on each fiber is as follows:

$$\begin{aligned}\big|\phi_2^{-1}(1:0)\big| &= q+1, \\ \big|\phi_2^{-1}(0:1)\big| &= 2q+1, \\ \big|\phi_2^{-1}(1:\pm 1)\big| &= q+1.\end{aligned}$$

Since any other fiber except these degenerate fibers are isomorphic to $\mathbb{P}^1$ over $k$, the number $N_{n,2}$ of all the $k$-rational points of $S_2$ is

$$\begin{aligned}N_{n,2} &= (q+1)(q-3) + (q+1) + (2q+1) + 2(q+1) \\ &= q^2 + 3q + 1.\end{aligned}$$

For the surface $S_2$ we can see that the number of $k$-rational points for $p = 2$ is same as the odd prime case. In fact, when $p = 2$, the surface $S_2$ has 3 degenerate fibers at $(1:0)$, $(0:1)$, $(1:1)$. The fibers at these points are same as the ones in the $p \neq 2$ case. Hence we can compute the number of $k$-rational points on each degenerate fiber. The result is

$$\left|\phi_2^{-1}(1:0)\right| = q+1, \qquad \left|\phi_2^{-1}(0:1)\right| = 2q+1, \qquad \left|\phi_2^{-1}(1:1)\right| = q+1.$$

Since any other fiber except these degenerate fibers is isomorphic to $\mathbb{P}^1$ over $k$, the number $N_{n,2}$ of all the $k$-rational points of $S_2$ for $p = 2$ is

$$\begin{aligned} N_{n,2} &= (q+1)(q-2) + 2(q+1) + (2q+1) \\ &= q^2 + 3q + 1. \end{aligned}$$

Therefore, for an arbitrary prime $p$ (including 2), the local zeta function $Z_2(p,T)$ is computed as follows:

$$Z_2(p,T) := \exp\left(\sum_{n=1}^{\infty} \frac{N_{n,2}}{n} T^n\right) = \left(1 - p^2 T\right)^{-1} (1 - pT)^{-3} (1 - T)^{-1}.$$

Thus the Hasse–Weil zeta function

$$\zeta(S_2, s) := \prod_p Z_2\left(p, p^{-s}\right)$$

is written as follows:

$$\begin{aligned} \zeta(S_2, s) &= \prod_p \left(1 - p^{2-s}\right)^{-1} \left(1 - p^{1-s}\right)^{-3} \left(1 - p^{-s}\right)^{-1} \\ &= \zeta(s-2)\zeta(s-1)^3\zeta(s). \end{aligned}$$

*2.2.1. Non-affine part*

The set $V(F_2) \setminus V(f_2)$ consists of points which have coordinates $(x : y : 0, 1 : 0)$ or $(x : y : 0, z : 1)$. Indeed it consists of three subsets

$$\begin{aligned} V(F_2) \setminus V(f_2)(k) = &\left\{(x : y : 0, 1 : 0) \;\middle|\; (x : y) \in \mathbb{P}_k^1\right\} \\ &\cup \left\{(z : w : 0, z : w) \;\middle|\; (z : w) \in \mathbb{P}_k^1\right\} \\ &\cup \left\{(w : z : 0, z : w) \;\middle|\; (z : w) \in \mathbb{P}_k^1\right\}. \end{aligned}$$

Then it is easy to compute the number $N'_{n,2}$ of $k$-rational points of $V(F_2) \setminus V(f_2)$ and its zeta functions.

$$\begin{aligned} N'_{n,2} &= \begin{cases} 3q - 1, & \text{if } p \neq 2, \\ 3q, & \text{if } p = 2. \end{cases} \\ Z'_2(p,T) &= \exp\left(\sum_{n=1}^{\infty} \frac{N'_{n,2}}{n} T^n\right) = \begin{cases} (1 - pT)^{-3}(1 - T), & \text{if } p \neq 2, \\ (1 - 2T)^{-3}, & \text{if } p = 2. \end{cases} \\ \tilde{\zeta}(S_2, s) &:= \prod_p Z'_2\left(p, p^{-s}\right) = \zeta(s-1)^3 \zeta(s)^{-1} \left(1 - 2^{-s}\right)^{-1}. \end{aligned}$$

Thus we have the description of the Hasse–Weil zeta function of the canonical component $V(f_2)$ of $6_3^2$.

$$\zeta\bigl(6_3^2, s\bigr) := \zeta(f_2, s) := \prod_p \exp\left(\sum_{n=1}^{\infty} \frac{\#V(f_2)(\mathbb{F}_{p^n})}{n}\bigl(p^{-s}\bigr)^n\right) = \zeta(S_2, s)/\tilde{\zeta}(S_2, s)$$
$$= \zeta(s-2)\zeta(s)^2\bigl(1-2^{-s}\bigr).$$

*2.3.* $6_2^2$ *case*

In this case we first assume Char $k \neq 2, 5$. Then the surface $S_1$ has 8 degenerate fibers at $(1:0)$, $(0:1)$, $(1:\pm 1)$, $(1:\pm\frac{\sqrt{5}\pm 1}{2})$.

$$\phi_1^{-1}(1:0) = \bigl\{(x:y:u, 1:0) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| u^2 = 0\bigr\},$$
$$\phi_1^{-1}(0:1) = \bigl\{(x:y:u, 0:1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| u^2 = 0\bigr\},$$
$$\phi_1^{-1}\left(1:\frac{\sqrt{5}\pm 1}{2}\right) = \left\{\left(x:y:u, 1:\frac{\sqrt{5}\pm 1}{2}\right) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; \frac{3\pm\sqrt{5}}{2}\left(x - \frac{\sqrt{5}+1}{2}y\right)\left(x - \frac{\sqrt{5}-1}{2}y\right) = 0\right\},$$
$$\phi_1^{-1}\left(1:-\frac{\sqrt{5}\pm 1}{2}\right) = \left\{\left(x:y:u, 1:-\frac{\sqrt{5}\pm 1}{2}\right) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; \frac{3\pm\sqrt{5}}{2}\left(x + \frac{\sqrt{5}+1}{2}y\right)\left(x + \frac{\sqrt{5}-1}{2}y\right) = 0\right\},$$
$$\phi_1^{-1}(1:\pm 1) = \bigl\{(x:y:u, 1:\pm 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| \bigl((x\mp y) - u\bigr)\bigl((x\mp y) + u\bigr) = 0\bigr\}.$$

Hence the number of $k = \mathbb{F}_q$-rational points on each fiber for $p \neq 2, 5$ is as follows:

$$\bigl|\phi_1^{-1}(1:0)\bigr| = q+1, \qquad \bigl|\phi_1^{-1}(0:1)\bigr| = q+1, \qquad \bigl|\phi_1^{-1}(1:\pm 1)\bigr| = 2q+1,$$
$$\left|\phi_1^{-1}\left(1:\pm\frac{\sqrt{5}\pm 1}{2}\right)\right| = \begin{cases} 2q+1 & \text{if } (\frac{5}{p}) = 1, \\ (2q+1)\frac{1+(-1)^n}{2} & \text{if } (\frac{5}{p}) = -1. \end{cases}$$

Here $(\frac{5}{p})$ is the Legendre symbol of 5 modulo $p$. Since any other fiber except these degenerate fibers are isomorphic to $\mathbb{P}^1$ over $k$, the number $N_{n,1}$ of all the $k$-rational points of $S_1$ is

$$N_{n,1} = \begin{cases} (q+1)(q-7) \\ (q+1)(q-3-2(1+(-1)^n)) \end{cases} + 2(q+1) + 2(2q+1)$$
$$+ 4\begin{cases} 2q+1 & \text{if } (\frac{5}{p}) = 1, \\ (2q+1)\frac{1+(-1)^n}{2} & \text{if } (\frac{5}{p}) = -1 \end{cases}$$

$$= \begin{cases} q^2 + 8q + 1 & \text{if } (\frac{5}{p}) = 1, \\ q^2 + 4q + 1 + 2q(1 + (-1)^n) & \text{if } (\frac{5}{p}) = -1. \end{cases}$$

Therefore, for an arbitrary prime $p \neq 2, 5$, the local zeta function $Z_1(p, T)$ is computed as follows:

$$\begin{aligned} Z_1(p, T) &:= \exp\left(\sum_{n=1}^{\infty} \frac{N_{n,1}}{n} T^n\right) \\ &= \left(1 - p^2 T\right)^{-1} (1 - pT)^{-4} (1 - T)^{-1} \times \begin{cases} (1 - pT)^{-4} & \text{if } (\frac{5}{p}) = 1, \\ \exp(\sum_{n=1}^{\infty} \frac{4p^{2n}}{2n} T^{2n}) & \text{if } (\frac{5}{p}) = -1 \end{cases} \\ &= \begin{cases} (1 - p^2 T)^{-1} (1 - pT)^{-8} (1 - T)^{-1} & \text{if } (\frac{5}{p}) = 1, \\ (1 - p^2 T)^{-1} (1 - pT)^{-4} (1 - T)^{-1} (1 - p^2 T^2)^{-2} & \text{if } (\frac{5}{p}) = -1. \end{cases} \end{aligned}$$

When $p = 2$, the surface $S_1$ has 5 degenerate fibers at $(1 : 0)$, $(0 : 1)$, $(1 : 1)$, $(1 : \alpha^{\pm 1})$, where $\alpha$ is a root of $T^2 + T + 1 = 0$. In fact, the degenerate fibers of $\phi_1 : S_1 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned} \phi_1^{-1}(1 : 0) &= \left\{(x : y : u, 1 : 0) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; u^2 = 0\right\}, \\ \phi_1^{-1}(0 : 1) &= \left\{(x : y : u, 0 : 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; xy = 0\right\}, \\ \phi_1^{-1}(1 : 1) &= \left\{(x : y : u, 1 : 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; (x + y + u)^2 = 0\right\}, \\ \phi_1^{-1}\left(1 : \alpha^{\pm 1}\right) &= \left\{\left(x : y : u, 1 : \alpha^{\pm 1}\right) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; (x - \alpha y)\left(x - \alpha^{-1} y\right) = 0\right\}. \end{aligned}$$

Hence we can compute the number of $k$-rational points on each degenerate fiber. The result is

$$\begin{aligned} \left|\phi_1^{-1}(1 : 0)\right| &= q + 1, & \left|\phi_1^{-1}(0 : 1)\right| &= q + 1, \\ \left|\phi_1^{-1}(1 : 1)\right| &= q + 1, & \left|\phi_1^{-1}\left(1 : \alpha^{\pm 1}\right)\right| &= (2q + 1)\frac{1 + (-1)^n}{2}. \end{aligned}$$

Since any other fiber except these degenerate fibers is isomorphic to $\mathbb{P}^1$ over $k$, the number $N_{n,1}$ of all the $k$-rational points of $S_1$ for $p = 2$ is

$$\begin{aligned} N_{n,1} &= (q + 1)\left(q - 2 - \left(1 + (-1)^n\right)\right) + 3(q + 1) + 2(2q + 1)\frac{1 + (-1)^n}{2} \\ &= q^2 + 2q + 1 + q\left(1 + (-1)^n\right). \end{aligned}$$

Thus the local zeta function $Z_1(2, T)$ is

$$Z_1(2, T) = \exp\left(\sum_{n=1}^{\infty} \frac{N_{n,1}}{n} T^n\right) = \left(1 - 2^2 T\right)^{-1} (1 - 2T)^{-2} (1 - T)^{-1} \left(1 - 2^2 T^2\right)^{-1}.$$

When $p=5$, the surface $S_1$ has 6 degenerate fibers at $(1:0)$, $(0:1)$, $(1:\pm 1)$, $(1:\pm 2)$. In fact, the degenerate fibers of $\phi_1 : S_1 \to \mathbb{P}^1$ are expressed as follows:

$$\begin{aligned}
\phi_1^{-1}(1:0) &= \bigl\{(x:y:u,1:0) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| u^2 = 0\bigr\},\\
\phi_1^{-1}(0:1) &= \bigl\{(x:y:u,0:1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| xy = 0\bigr\},\\
\phi_1^{-1}(1:\pm 1) &= \bigl\{(x:y:u,1:\pm 1) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| \bigl((x\mp y)-u\bigr)\bigl((x\mp y)+u\bigr) = 0\bigr\},\\
\phi_1^{-1}(1:\pm 2) &= \bigl\{(x:y:u,1:\pm 2) \in \mathbb{P}^2 \times \mathbb{P}^1 \bigm| (x-2y)(x+2y) = 0\bigr\}.
\end{aligned}$$

Hence we can compute the number of $k$-rational points on each degenerate fiber. The result is

$$\bigl|\phi_1^{-1}(1:0)\bigr| = q+1, \qquad \bigl|\phi_1^{-1}(0:1)\bigr| = q+1, \qquad \bigl|\phi_1^{-1}(1:\pm 1)\bigr| = 2q+1,$$
$$\bigl|\phi_1^{-1}(1:\pm 2)\bigr| = 2q+1.$$

Since any other fiber except these degenerate fibers is isomorphic to $\mathbb{P}^1$, the number $N_{n,1}$ of all the $k$-rational points of $S_1$ for $p=5$ is

$$N_{n,1} = (q+1)(q-5) + 2(q+1) + 4(2q+1) = q^2 + 6q + 1.$$

Thus the local zeta function $Z_1(5,T)$ is

$$Z_1(5,T) = \exp\left(\sum_{n=1}^{\infty} \frac{N_{n,1}}{n} T^n\right) = \bigl(1-5^2T\bigr)^{-1}(1-5T)^{-6}(1-T)^{-1}.$$

Hence the Hasse–Weil zeta function

$$\zeta(S_1,s) := \prod_p Z_1\bigl(p,p^{-s}\bigr)$$

is written as follows:

$$\zeta(S_1,s) = \zeta(s-2)\zeta(s-1)^6\zeta(s)L\left(\left(\frac{5}{\cdot}\right), s-1\right)^2 \times \bigl(1-2^{1-s}\bigr)^3\bigl(1+2^{1-s}\bigr).$$

*2.3.1. Non-affine part*

The set $V(F_1) \smallsetminus V(f_1)$ consists of four subsets

$$\begin{aligned}
V(F_1) \smallsetminus V(f_1)(k) = {}& \bigl\{(x:y:0,1:0) \bigm| (x:y) \in \mathbb{P}^1_k\bigr\} \cup \bigl\{(x:y:0,0:1) \bigm| (x:y) \in \mathbb{P}^1_k\bigr\}\\
&\cup \bigl\{(z:w:0,z:w) \bigm| (z:w) \in \mathbb{P}^1_k\bigr\}\\
&\cup \bigl\{(w:z:0,z:w) \bigm| (z:w) \in \mathbb{P}^1_k\bigr\}.
\end{aligned}$$

Then it is easy to compute the number $N'_{n,1}$ of $k$-rational points of $V(F_1) \smallsetminus V(f_1)$ and its zeta functions.

$$\begin{aligned} N'_{n,1} &= 2(q+1) + (q-1) + \begin{cases} q-3, & \text{if } p \neq 2, \\ q-2, & \text{if } p = 2 \end{cases} \\ &= 4q - \begin{cases} 2, & \text{if } p \neq 2, \\ 1, & \text{if } p = 2. \end{cases} \\ Z'_1(p,T) &= \begin{cases} (1-pT)^{-4}(1-T)^2, & \text{if } p \neq 2, \\ (1-2T)^{-4}(1-T), & \text{if } p = 2. \end{cases} \\ \tilde{\zeta}(S_1,s) := \prod_p Z'_1\left(p,p^{-s}\right) &= \zeta(s-1)^4\zeta(s)^{-2}\left(1-2^{-s}\right)^{-1}. \end{aligned}$$

Thus we have the description of the Hasse–Weil zeta function of the canonical component $V(f_1)$ of $6_2^2$.

$$\begin{aligned} \zeta\left(6_2^2,s\right) := \zeta(f_1,s) := \prod_p \exp\left(\sum_{n=1}^{\infty} \frac{\#V(f_1)(\mathbb{F}_{p^n})}{n}\left(p^{-s}\right)^n\right) &= \zeta(S_1,s)/\tilde{\zeta}(S_1,s) \\ &= \zeta(s-2)\zeta(s-1)^2\zeta(s)^3 L\left(\left(\frac{5}{\cdot}\right), s-1\right)^2 \\ &\quad \times \left(1-2^{1-s}\right)^3\left(1+2^{1-s}\right)\left(1-2^{-s}\right) \\ &= \zeta_{\mathbb{Q}(\sqrt{5})}(s-1)^2\zeta(s-2)\zeta(s)^3 \times \left(1-2^{1-s}\right)^3\left(1+2^{1-s}\right)\left(1-2^{-s}\right). \end{aligned}$$

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