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C. R. Acad. Sci. Paris, Ser. I

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Complex Analysis

# Coefficient estimates for a class of meromorphic bi-univalent functions



## *Estimation de coefficients pour une classe de fonctions méromorphes bi-univalentes*

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## ARTICLE INFO

## Article history:

Received 23 April 2013

Accepted after revision 21 May 2013

Available online 30 May 2013

Presented by the Editorial Board

## ABSTRACT

Applying the Faber polynomial coefficient expansions to a class of meromorphic bi-univalent functions, we obtain the general coefficient estimates for such functions and also examine their early coefficient bounds. A function univalent in the open unit disk is said to be bi-univalent if its inverse map is also univalent there. Both the technique and the coefficient bounds presented here are new on their own kind. We hope that this article will generate future interest in applying our approach to other related problems.

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## R É S U M É

Une fonction univalente dans le disque unité ouvert est dite bi-univalente si sa fonction inverse est aussi univalente dans ce domaine. Appliquant le développement à coefficients polynômes de Faber à cette classe de fonctions, nous obtenons des estimations du coefficient général de leur développement de Laurent. Nous examinons également les bornes pour leurs premiers coefficients. Les techniques et les bornes des coefficients présentées ici sont nouvelles dans leur genre. Nous espérons qu'elles susciteront un intérêt pour l'application de notre approche à des problèmes connexes.

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## 1. Introduction

Let  $\Sigma$  be the family of meromorphic functions  $g$  of the form:

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \quad (1.1)$$

that are univalent in  $\Delta := \{z: 1 < |z| < \infty\}$ . The coefficients of  $h = g^{-1}$ , the inverse map of  $g$ , are given by the Faber polynomial:

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n \geq 1} \frac{1}{n} K_{n+1} \frac{1}{w^n},$$

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where  $w \in \Delta$ ,

$$K_{n+1}^n = nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}(b_3 + b_1^2) + \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4 + 3b_1b_2) + \sum_{j \geq 5} b_0^{n-j}V_j$$

and  $V_j$  with  $5 \leq j \leq n$  is a homogeneous polynomial of degree  $j$  in the variables  $b_1, b_2, \dots, b_n$ . (See [1,2] or [11].)

For  $0 \leq \alpha < 1$  and  $\lambda \geq 1$ , let  $B\Sigma(\alpha; \lambda)$  be the class of meromorphic bi-univalent functions  $g \in \Sigma$  so that:

$$\operatorname{Re}\left((1-\lambda)\frac{g(z)}{z} + \lambda g'(z)\right) > \alpha, \quad z \in \Delta$$

and

$$\operatorname{Re}\left((1-\lambda)\frac{h(w)}{w} + \lambda h'(w)\right) > \alpha, \quad w \in \Delta.$$

In 1923, Lowner [9] proved that the inverse of the Koebe function  $k(z) = z/(1-z)^2$  provides the best upper bounds for the coefficients of the inverses of analytic univalent functions. Although the estimates for the coefficients of the inverses of analytic univalent functions have been obtained in a surprisingly straightforward way (e.g., see [7, p. 104]), the case turns out to be a challenge when the bi-univalence condition is imposed on these functions. Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967 (see Lewin [8]). The interest on the bounds for the coefficients of classes of bi-univalent functions picked up by the publications [4,10,6,3], where the estimates for the first two coefficients of certain classes of bi-univalent functions were provided. Not much is known about the higher coefficients of bi-univalent functions as Ali et al. [3] also declared finding the bounds for  $|a_n|; n \geq 4$  an open problem. In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients  $|a_n|$  of meromorphic bi-univalent functions in  $B\Sigma(\alpha; \lambda)$  as well as providing estimates for the early coefficients of these functions. As a result, we are able to prove:

**Theorem 1.1.** *Let  $g$  be given by (1.1). For  $0 \leq \alpha < 1$  and  $\lambda \geq 1$  if  $g \in B\Sigma(\alpha; \lambda)$  and  $b_k = 0; 0 \leq k \leq n-1$ , then:*

$$|b_n| \leq \frac{2(1-\alpha)}{(n+1)\lambda-1}; \quad n \geq 1.$$

**Proof.** For meromorphic functions  $g$  of the form (1.1) we have:

$$(1-\lambda)\frac{g(z)}{z} + \lambda g'(z) = 1 + \sum_{n=0}^{\infty} (1-\lambda(n+1))\frac{b_n}{z^{n+1}}, \tag{1.2}$$

and for its inverse map,  $h = g^{-1}$ , we have:

$$\begin{aligned} (1-\lambda)\frac{h(w)}{w} + \lambda h'(w) &= 1 + \sum_{n=0}^{\infty} (1-\lambda(n+1))\frac{B_n}{w^{n+1}} \\ &= 1 - (1-\lambda)\frac{b_0}{w} - \sum_{n=1}^{\infty} (1-\lambda(n+1))\frac{1}{n}K_{n+1}^n(b_0, b_1, \dots, b_n)\frac{1}{w^{n+1}}. \end{aligned} \tag{1.3}$$

On the other hand, since  $g \in B\Sigma(\alpha; \lambda)$ , by definition, there exist two positive real-part functions  $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n}$  and  $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^{-n}$ , where  $\operatorname{Re}p(z) > 0$  and  $\operatorname{Re}q(w) > 0$  in  $\Delta$  so that:

$$(1-\lambda)\frac{g(z)}{z} + \lambda g'(z) = 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(c_1, c_2, \dots, c_n)\frac{1}{z^n}, \tag{1.4}$$

and

$$(1-\lambda)\frac{h(w)}{w} + \lambda h'(w) = 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(d_1, d_2, \dots, d_n)\frac{1}{w^n}. \tag{1.5}$$

Note that, according to the Caratheodory Lemma (see Duren [5, p. 41]),  $|c_n| \leq 2$  and  $|d_n| \leq 2$  for  $n = 1, 2, 3, \dots$ . Comparing the corresponding coefficients of (1.2) and (1.4) yields:

$$(1-\lambda(n+1))b_n = (1-\alpha)K_{n+1}^1(c_1, c_2, \dots, c_{n+1})$$

and, similarly, from (1.3) and (1.5) we obtain:

$$\begin{cases} -(1-\lambda)b_0 = (1-\alpha)d_1, \\ -\frac{(1-\lambda(n+1))}{n}K_{n+1}^n(b_0, b_1, \dots, b_n) = (1-\alpha)K_{n+1}^1(d_1, d_2, \dots, d_{n+1}). \end{cases}$$

Note that for  $b_k = 0; 0 \leq k \leq n-1$  we have  $B_n = -b_n$  and so:

$$\begin{cases} (1-\lambda(n+1))b_n = (1-\alpha)c_{n+1}, \\ -(1-\lambda(n+1))b_n = (1-\alpha)d_{n+1}. \end{cases}$$

Now taking the absolute values of either of the above two equations and applying the Caratheodory Lemma, we obtain:

$$|b_n| = \frac{(1-\alpha)|c_{n+1}|}{|1-\lambda(n+1)|} = \frac{(1-\alpha)|d_{n+1}|}{|1-\lambda(n+1)|} \leq \frac{2(1-\alpha)}{(n+1)\lambda-1}. \quad \square$$

Relaxing the coefficient restrictions imposed on Theorem 1.1, in the following theorem, we obtain estimates for the early coefficients of functions  $g$  in  $B\Sigma(\alpha; \lambda)$  as well as a bound for the coefficient body  $(b_2 + b_0b_1)$ .

**Theorem 1.2.** *Let  $g$  be given by (1.1). For  $0 \leq \alpha < 1$  and  $\lambda \geq 1$  if  $g \in B\Sigma(\alpha; \lambda)$ , then*

- i)  $|b_0| \leq \frac{2(1-\alpha)}{\lambda-1}$ ,
- ii)  $|b_1| \leq \frac{2(1-\alpha)}{2\lambda-1}$ ,
- iii)  $|b_2| \leq \frac{2(1-\alpha)}{3\lambda-1}$ ,
- iv)  $|b_2 + b_0b_1| \leq \frac{2(1-\alpha)}{3\lambda-1}$ .

**Proof.** Comparing Eqs. (1.2) and (1.4) for  $n = 0, 1, 2$ , we obtain:

$$(1-\lambda)b_0 = (1-\alpha)c_1, \tag{1.6}$$

$$(1-2\lambda)b_1 = (1-\alpha)c_2, \tag{1.7}$$

and

$$(1-3\lambda)b_2 = (1-\alpha)c_3. \tag{1.8}$$

On the other hand, from (1.3) and (1.5), for  $n = 2$ , we obtain:

$$-(1-3\lambda)(b_0b_1 + b_2) = (1-\alpha)d_3. \tag{1.9}$$

Solving Eqs. (1.6), (1.7), (1.8) and (1.9) for  $b_0, b_1, b_2$  and  $(b_2 + b_0b_1)$ , respectively, taking their absolute vales and then applying the Caratheodory Lemma, we obtain:

$$|b_0| \leq \frac{(1-\alpha)|c_1|}{|1-\lambda|} \leq \frac{2(1-\alpha)}{\lambda-1},$$

$$|b_1| \leq \frac{(1-\alpha)|c_2|}{|1-2\lambda|} \leq \frac{2(1-\alpha)}{2\lambda-1},$$

$$|b_2| \leq \frac{(1-\alpha)|c_3|}{|1-3\lambda|} \leq \frac{2(1-\alpha)}{3\lambda-1},$$

and

$$|b_2 + b_0b_1| \leq \frac{(1-\alpha)|d_3|}{|1-3\lambda|} \leq \frac{2(1-\alpha)}{3\lambda-1}. \quad \square$$

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