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Trace-order and a distortion theorem for linearly invariant families on the unit ball of a finite dimensional JB*-triple

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ABSTRACT

We give a distortion theorem for linearly invariant families on the unit ball B of a finite dimensional JB*-triple X by using the trace-order. The exponents in the distortion bounds depend on the Bergman metric at 0. Further, we introduce a new definition for the trace-order of a linearly invariant family on B, based on a Jacobian argument. We also construct an example of a linearly invariant family on B which has minimum trace-order and is not a subset of the normalized convex mappings of B for $\dim X \geq 2$. Finally, we prove a regularity theorem for linearly invariant families on B. All four types of classical Cartan domains are the open unit balls of JB*-triples, and the same holds for any finite product of these domains. Thus the unit balls of JB*-triples are natural generalizations of the unit disc in $\mathbb{C}$ and we have a setting in which a large number of bounded symmetric homogeneous domains may be studied simultaneously.

1. Introduction

Linear invariance, introduced by Pommerenke [1,2] has been a powerful tool in extending many ideas of univalent function theory to the study of locally univalent functions on the unit disc. Rudin [3] first considered M-invariant families on the Euclidean unit ball of $\mathbb{C}^n$, i.e. families of locally biholomorphic mappings that are invariant under the group $\mathrm{Aut}(B^n)$ of biholomorphic automorphisms of B^n. Later, Barnard et al. [4] proved an interesting distortion result for linearly invariant families on the Euclidean unit ball of $\mathbb{C}^2$. Generalizations of this result and various contributions in the theory of linearly invariant families in higher dimensions are due to Pfaltzgraff [5], Liu [6], Gong and Zheng [7], Gong and Yu [8], Godula et al. [9], Graham et al. [10], Hamada et al. [11], Hamada and Kohr [12–15], Liczberski and Starkov [16], Pfaltzgraff and Suffridge [17–19], etc.

There are significant differences between the theory in one complex variable and that in several complex variables of linearly invariant families. Pfaltzgraff and Suffridge [17] proved the following unexpected results:

- The n-dimensional analog of Pommerenke's result that a linearly invariant family on the unit disc has order 1 (minimum possible order for a linearly invariant family on the unit disc) if and only if it consists of normalized convex functions on the unit disc does not hold on the Euclidean unit ball B^n of $\mathbb{C}^n$ for $n \geq 2$. Indeed, it was proved by Pfaltzgraff and Suffridge [17] that $\mathrm{ord}\, K(B^n) > (n+1)/2$ for $n \geq 2$, where $K(B^n)$ is the family of normalized biholomorphic convex mappings on B^n. Also, there exist linearly invariant families on B^n of minimum trace-order $(n+1)/2$ which are not subsets of $K(B^n)$ (see [17]).

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- The Cayley transform does not give bounds for the growth of the Jacobian determinant of all normalized convex mappings of the Euclidean unit ball of $\mathbb{C}^n$ for $n \geq 2$ (see [17]). (The family of normalized convex mappings of B^n is a linearly invariant family.)
- In dimension $n \geq 2$, the trace-order of $K(B^n)$ is unknown. However, in the case of the unit polydisc U^n, $\operatorname{ord} K(U^n) = n$ (the minimum possible order of a linearly invariant family on U^n). Also, there exist linearly invariant families on the unit polydisc of minimum order n, which are not subsets of $K(U^n)$ for $n \geq 2$ (see [17]).

Note that the trace-order of a linearly invariant family $\mathcal{F}$ is directly related to estimates of $|J_f(z)|$ for $f \in \mathcal{F}$, where $J_f(z) = \det Df(z)$ for $z \in B^n$. It is an open problem in dimension $n \geq 2$, to find the sharp estimates for $|J_f(z)|$ when f belongs to the linearly invariant family $K(B^n)$ (see e.g. [17,20,21] and the references therein). On the other hand, the family S of normalized univalent functions on the unit disc is a linearly invariant family of order 2. The analog of the family S in higher dimensions is the family $S(B^n)$ of normalized biholomorphic mappings on B^n. It is known that $S(B^n)$ is a linearly invariant family of infinite trace-order for $n \geq 2$ (see [4]), and thus it is of interest to study linearly invariant families of biholomorphic mappings which are proper subsets of $S(B^n)$. For linearly invariant families in several complex variables, see also the books [20,21] and the references therein.

Pfaltzgraff and Suffridge [19] showed a distortion result for mappings that belong to a linearly invariant family on the Euclidean unit ball of $\mathbb{C}^n$ by using the norm-order of a linearly invariant family. Hamada and Kohr generalized the result to the unit polydisc in [13]. Hamada et al. [11] generalized the result to the unit ball of a finite dimensional JB*-triple X as follows.

Theorem 1.1. *Let $\mathcal{F}$ be a linearly invariant family on the unit ball B of a finite dimensional* JB*-*triple X. Let $\|\operatorname{ord}\|_{X,1}\,\mathcal{F}$ denote the norm-order of $\mathcal{F}$, given by*

$$\|\operatorname{ord}\|_{X,1}\mathcal{F} = \sup_{f\in\mathcal{F}}\ \sup_{\|y\|=1}\left\{\frac{1}{2}\|D^2f(0)(y,\cdot)\|\right\}.$$

If $\|\operatorname{ord}\|_{X,1}\,\mathcal{F} = \alpha < \infty$, then

$$\|Df(x)\| \leq \frac{(1+\|x\|)^{\alpha-1}}{(1-\|x\|)^{\alpha+1}}, \quad x \in B$$

for all $f \in \mathcal{F}$.

We remark that the exponent on the right-hand side is independent of the dimension of X.

On the other hand, Barnard et al. [4], Liu [6], Gong and Zheng [7], Pfaltzgraff [5], Pfaltzgraff and Suffridge [17,18], Gong and Yu [8], Godula et al. [9], Liczberski and Starkov [16], Hamada and Kohr [14] (cf. [20, Chapter 5]), have studied linearly invariant families in several complex variables by using the trace-order. Pfaltzgraff [5] proved the following distortion theorem on the Euclidean unit ball of $\mathbb{C}^n$ (cf. [9]).

Theorem 1.2. *Let $\mathcal{F}$ be a linearly invariant family on the Euclidean unit ball B^n. Let* $\operatorname{ord}\mathcal{F}$ *denote the trace-order of $\mathcal{F}$. If* $\operatorname{ord}\mathcal{F} = \alpha < \infty$, *then*

$$\frac{(1-\|x\|)^{\alpha-(n+1)/2}}{(1+\|x\|)^{\alpha+(n+1)/2}} \leq |\det Df(x)| \leq \frac{(1+\|x\|)^{\alpha-(n+1)/2}}{(1-\|x\|)^{\alpha+(n+1)/2}}, \quad x \in B^n$$

for all $f \in \mathcal{F}$.

Pfaltzgraff and Suffridge [17] proved the following distortion theorem on the unit polydisc of $\mathbb{C}^n$.

Theorem 1.3. *Let $\mathcal{F}$ be a linearly invariant family on the unit polydisc U^n. If* $\operatorname{ord}\mathcal{F} = \alpha < \infty$, *then*

$$\frac{(1-\|x\|)^{\alpha-n}}{(1+\|x\|)^{\alpha+n}} \leq |\det Df(x)| \leq \frac{(1+\|x\|)^{\alpha}}{(1-\|x\|)^{\alpha}}\prod_{j=1}^{n}(1-|x_j|^2)^{-1}$$

for all $x = (x_1, \ldots, x_n) \in U^n$ and $f \in \mathcal{F}$.

We remark that, in the above theorems, the bounds depend on n. The following natural questions arise.

Question 1.4. *Can we give an explanation for the reason why the exponents in the distortion bounds in Theorems 1.2 and 1.3 are different?*

Question 1.5. *Can we give a distortion theorem for linearly invariant families on other bounded symmetric domains by using the trace-order?*

In this paper, we study the linearly invariant families on the unit ball B of a finite dimensional JB*-triple and give affirmative answers to the above questions. We give a distortion theorem for linearly invariant families on the unit ball of a finite dimensional JB*-triple X by using the trace-order. The exponents in the distortion bounds depend on the Bergman metric at 0. Our result is a generalization of Theorems 1.2 and 1.3 to the unit ball of a finite dimensional JB*-triple X. Further, we introduce a new definition for the trace-order of a linearly invariant family on B, based on a Jacobian argument. We also construct an example of a linearly invariant family on B which has minimum trace-order and is not a subset of the normalized convex mappings of B for $\dim X \geq 2$. These results are generalizations of those in [9,14] to the unit ball of a finite dimensional JB*-triple X. Finally, we prove a regularity theorem for linearly invariant families on B. All four types of classical Cartan domains are the open unit balls of JB*-triples, and the same holds for any finite product of these domains [22]; see also [23,24]. Thus the unit balls of JB*-triples are natural generalizations of the unit disc in $\mathbb{C}$ and we have a setting in which a large number of bounded symmetric homogeneous domains may be studied simultaneously.

2. Preliminaries

Let B be the unit ball of a complex Banach space X. Let Y be a complex Banach space. A holomorphic mapping $f : B \to Y$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. A holomorphic mapping $f : B \to Y$ is said to be biholomorphic if $f(B)$ is a domain in Y, f^{-1} exists and holomorphic on $f(B)$. A biholomorphic mapping $f : B \to Y$ is said to be convex if $f(B)$ is a convex domain. Let $L(X, Y)$ denote the set of continuous linear operators from X into Y. Let I_X be the identity in $L(X, X)$. Let $LS(B)$ denote the family of locally biholomorphic mappings from B to X, normalized by $f(0) = 0$ and $Df(0) = I_X$.

We recall that a JB*-triple is a complex Banach space X together with a continuous mapping (called the Jordan triple product)

$$X \times X \times X \to X \quad (x, y, z) \mapsto \{x, y, z\}$$

such that for all elements in X the following conditions (J$_1$)–(J$_4$) hold, where for every $x, y \in X$, the operator $x\Box y$ on X is defined by $z \mapsto \{x, y, z\}$:

(J$_1$) $\{x, y, z\}$ is symmetric bilinear in the outer variables x, z and conjugate linear in the inner variable y,
(J$_2$) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$, (Jordan triple identity)
(J$_3$) $x \Box x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,
(J$_4$) $\|\{x, x, x\}\| = \|x\|^3$.

It is known [25, p. 523] that in this definition condition (J$_4$) can be replaced by $\|x\Box x\| = \|x\|^2$. Also, we have

$$\|\{x, y, z\}\| \leq \|x\| \cdot \|y\| \cdot \|z\|, \quad \text{for all } x, y, z \in X \tag{2.1}$$

by Friedman and Russo [26, Corollary 3].

Example 2.1. Let H and K be complex Hilbert spaces. Then $L(H, K)$ is a JB*-triple with

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x),$$

where y^* denotes the usual adjoint of y.

For every $a \in X$, let $Q_a : X \to X$ be the conjugate linear operator defined by $Q_a(x) = \{a, x, a\}$. This operator is called the quadratic representation and it satisfies the fundamental formula

$$Q_{Q_a(b)} = Q_a Q_b Q_a$$

for all $a, b \in X$. For every $x, y \in X$, the Bergman operator $B(x, y) \in L(X, X)$ is defined by

$$B(x, y) = I_X - 2x\Box y + Q_x Q_y.$$

From (2.1), we have

$$\|B(x, y)\| \leq (1 + \|x\| \cdot \|y\|)^2, \quad x, y \in X.$$

In the case $\|x\Box y\| < 1$, the spectrum of $B(x, y)$ lies in $\{z \in \mathbb{C} : |z - 1| < 1\}$. In particular, the fractional power $B(x, y)^r \in GL(X)$ exists for every $r \in \mathbb{R}$ in a natural way (cf. [25, p. 517]).

Let B be the unit ball of a JB*-triple X. Then, for each $a \in B$, the Möbius transformation g_a defined by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x\Box a)^{-1}x,$$

is a biholomorphic mapping of B onto itself with $g_a(0) = a$, $g_a(-a) = 0$ and $g_{-a} = g_a^{-1}$. Then we obtain the following lemma.

Lemma 2.2. *Let g_a be as above. Then for any $a \in B$, we have*

$$[Dg_a(0)]^{-1}D^2g_a(0)(x, y) = -2\{x, a, y\}, \tag{2.2}$$
$$Dg_a(0)a = a - \{a, a, a\}. \tag{2.3}$$

Proof. Eq. (2.2) is proved in [11, Proposition 2.2]. Eq. (2.3) is proved in [27, p.620]. This completes the proof. □

An element $u \in X$ is called a tripotent if $\{u, u, u\} = u$. Two tripotents u and v are said to be orthogonal if $D(u, v) = 0$, where $D(u, v) = 2u\Box v$. Orthogonality is a symmetric relation. A tripotent u is said to be maximal if the only tripotent which is orthogonal to u is 0. A tripotent u is said to be minimal if it cannot be written as a sum of two non-zero orthogonal tripotents. A frame is a maximal family of pairwise orthogonal, minimal tripotents. The cardinality of all frames is the same, and is called the rank r of X. A subspace I of X is called a triple ideal if $\{X, X, I\} + \{X, I, X\} \subset I$. A JB*-triple is simple if it has no non-trivial (norm) closed triple ideals.

Assume that $\dim X < \infty$. A point $u \in \overline{B}$ is said to be an extreme point of $\overline{B}$ if the only $x \in X$ satisfying $\|u + \lambda x\| \le 1$ for all real numbers λ with $|\lambda| \le 1$ is $x = 0$. Let $\mathcal{E}$ be the set of all extreme points of $\overline{B}$. By the Krein–Milman theorem (see e.g. [28, Chapter 4]), $\mathcal{E}$ is nonempty, since $\overline{B}$ is a compact subset of X. A subset Γ of $\overline{B}$ is called the Bergmann–Shilov boundary of B if Γ is the smallest closed subset of $\overline{B}$ where every continuous function on $\overline{B}$ which is holomorphic on B attains its maximum absolute value.

Let μ be a Haar measure on the additive group of X, and let $H^2(B) = \mathrm{Hol}\,(B) \cap L^2(B)$ be the set of square-integrable (with respect to μ) holomorphic functions on B. Let $k(z, \overline{w})$ be the Bergman kernel of B, that is, the reproducing kernel of the Hilbert space $H^2(B)$. The Bergman metric at $x \in B$ is defined by

$$h_x(u, v) = \partial_u \overline{\partial}_v \log k(x, \overline{x}).$$

For $x \in X$, $h_0(x, x)^{1/2}$ is called the Euclidean norm on X.

The following result is obtained in [29, Theorem 6.5] (cf. [11, Proposition 2.4], [22, Corollary 9] and [30, Proposition 3.5]).

Proposition 2.3. *Let B be the unit ball of a finite dimensional* JB*-*triple X. Then the Bergmann–Shilov boundary Γ of B coincides with each of the following sets:*

(i) *The set of maximal tripotents of X;*
(ii) *the set of extreme points of $\overline{B}$;*
(iii) *the set of points of maximum Euclidean norm in $\overline{B}$.*

3. Linear invariance in *X*

We define the notion of linearly invariant families and the trace-order on the unit ball B of a complex Banach space X.

Definition 3.1. Let B be the unit ball of a complex Banach space X. Then a family $\mathcal{F}$ is called a linearly invariant family if:

(i) $\mathcal{F} \subset LS(B)$,

and

(ii) $\Lambda_\phi(f) \in \mathcal{F}$, for all $f \in \mathcal{F}$ and $\phi \in \mathrm{Aut}\,(B)$,

where $\mathrm{Aut}\,(B)$ denotes the set of biholomorphic automorphisms of B, and $\Lambda_\phi(f)$ is the Koebe transform

$$\Lambda_\phi(f)(x) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}(f(\phi(x)) - f(\phi(0))), \tag{3.1}$$

for all $x \in B$.

Note that the Koebe transform has the group property $\Lambda_\psi \circ \Lambda_\phi = \Lambda_{\phi\circ\psi}$.

If $\mathcal{F}$ is a linearly invariant family on the unit ball of a finite dimensional complex Banach space X, we define the trace-order of $\mathcal{F}$ (cf. [5]), given by

$$\mathrm{ord}\,\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|y\|=1} \left\{ \frac{1}{2} \left| \mathrm{tr}\, \left[D^2 f(0)(y, \cdot) \right] \right| \right\}.$$

Since the trace is a similarity invariant, the above definition is well-defined. Also, since $|\mathrm{tr}\,(A)| \le n\|A\|$ for all $A \in L(X, X)$, where $n = \dim X$, we have

$$\mathrm{ord}\,\mathcal{F} \le n\|\mathrm{ord}\,\|_{X,1}\mathcal{F}.$$

We now give some examples of linearly invariant families on the unit ball B of a complex Banach space X.

Example 3.2. (i) $K(B)$, the set of convex mappings in $LS(B)$.

(ii) $S(B)$, the set of all biholomorphic mappings in $LS(B)$. If X is a complex Hilbert space of dimension n, where $n > 1$, the linearly invariant family $S(B)$ does not have finite trace-order (see [4], cf. [5]).

(iii) $\mathcal{U}_\alpha(B)$, the union of all linearly invariant families contained in $LS(B)$ with trace-order not greater than α. This is a generalization of the universal linearly invariant families $\mathcal{U}_\alpha = \mathcal{U}_\alpha(\Delta)$ considered in [1].

(iv) If $\mathcal{G}$ is a nonempty subset of $LS(B)$, then the linearly invariant family generated by $\mathcal{G}$ is the family

$$\Lambda[\mathcal{G}] = \{\Lambda_\phi(g) : g \in \mathcal{G}, \ \phi \in \mathrm{Aut}\,(B)\}.$$

The linear invariance is a consequence of the group property of the Koebe transform. Obviously, $\Lambda[\mathcal{G}] = \mathcal{G}$ if and only if $\mathcal{G}$ is a linearly invariant family. In the case of the unit Euclidean ball and the unit polydisc of $\mathbb{C}^n$, this example provided a useful technique for generating many interesting mappings (see [5,17,18]). For example, we can use a single mapping f from $LS(B)$ to generate the linearly invariant family $\Lambda[\{f\}]$. The family $\Lambda[\{i\}]$, generated by the identity mapping $i(x) = x$, consists of all the Koebe transforms of $i(x)$.

As in the proof of [5, Lemma 4.2], we obtain the following lemma.

Lemma 3.3. *Let B be the unit ball of a finite dimensional complex Banach space X. Let $f \in LS(B)$ and $\phi \in \mathrm{Aut}\,(B)$. Let $F(w) = \Lambda_\phi(f)(w)$. Then*

$$\mathrm{tr}\,\{D^2F(0)(y,\cdot)\} = \mathrm{tr}\,\{[D\phi(0)]^{-1}D^2\phi(0)(y,\cdot)\} + \mathrm{tr}\,\{[Df(\phi(0))]^{-1}D^2f(\phi(0))(D\phi(0)y,\cdot)\}. \tag{3.2}$$

Proof. If we differentiate twice the mapping $F = \Lambda_\phi(f)$, given by (3.1), we obtain that

$$DF(w) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}Df(\phi(w))D\phi(w), \quad w \in B,$$

and

$$D^2F(w)(y,z) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}\{D^2f(\phi(w))(D\phi(w)y, D\phi(w)z) + Df(\phi(w))D^2\phi(w)(y,z)\}, \quad y,z \in X.$$

Evaluating at $w = 0$, we obtain that

$$D^2F(0)(y,z) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}D^2f(\phi(0))(D\phi(0)y, D\phi(0)z) + [D\phi(0)]^{-1}D^2\phi(0)(y,z).$$

Taking the trace and noting that the trace is a similarity invariant, we obtain (3.2). This completes the proof. □

4. Distortion bounds

In this section, we will prove the distortion theorem for linearly invariant families on the unit ball B of a finite dimensional JB*-triple X. We remark that $\det Df(x)$ is well-defined, because the determinant is a similarity invariant. This theorem is a generalization of [5, Theorem 5.1] to the unit ball of a finite dimensional JB*-triple. For other distortion theorems of normalized locally biholomorphic mappings on the unit ball of a JB*-triple, see [31,11].

Let h_0 be the Bergman metric on X at 0 and let

$$c(B) = \frac{1}{2}\sup_{x,y\in B}|h_0(x,y)|.$$

By Proposition 2.3, we have

$$c(B) = \frac{1}{2}h_0(e,e), \tag{4.1}$$

where e is an arbitrary maximal tripotent in X.

Theorem 4.1. *Let $\mathcal{F}$ be a linearly invariant family on the unit ball B of a finite dimensional* JB*-*triple X. If $\mathrm{ord}\,\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|x\|)^{\alpha-c(B)}}{(1+\|x\|)^{\alpha+c(B)}} \le |\det Df(x)| \le \frac{(1+\|x\|)^{\alpha-c(B)}}{(1-\|x\|)^{\alpha+c(B)}}, \quad x \in B \tag{4.2}$$

for all $f \in \mathcal{F}$. If B is the Euclidean unit ball or the unit polydisc of $\mathbb{C}^n$, then the above estimates are sharp.

Proof. Sharpness of (4.2) in the case of the Euclidean unit ball or the unit polydisc of $\mathbb{C}^n$ is proved in [9,5,17]. Since $\det Df(x)$ is a non-vanishing holomorphic function on B, it suffices to show (4.2) for x such that $u = x/\|x\|$ is a maximal tripotent in view of Proposition 2.3.

Let $f \in \mathcal{F}$, $\phi = g_a$, where $a = \rho x$ with $0 < \rho \le 1$. Let $F(w) = \Lambda_\phi(f)(w)$. Then, by (2.2) and Lemma 3.3, we have

$$\mathrm{tr}\,\{D^2F(0)(y,\cdot)\} = \mathrm{tr}\,\{-2\{y,a,\cdot\}\} + \mathrm{tr}\,\{[Df(a)]^{-1}D^2f(a)(Dg_a(0)y,\cdot)\}. \tag{4.3}$$

By (2.3), we have

$$Dg_a(0)a = \|a\|u - \|a\|^3\{u,u,u\} = (1-\|a\|^2)\|a\|u = (1-\|a\|^2)a.$$

Also, $\operatorname{tr}\{2\{y, a, \cdot\}\} = h_0(y, a)$ by Loos [29, Theorem 2.10]. Therefore, putting $y = [Dg_a(0)]^{-1}a = a/(1 - \|a\|^2)$ in (4.3), we have

$$\operatorname{tr}\left\{D^2F(0)\left(\frac{a}{1-\|a\|^2}, \cdot\right)\right\} = -h_0\left(\frac{a}{1-\|a\|^2}, a\right) + \operatorname{tr}\{[Df(a)]^{-1}D^2f(a)(a, \cdot)\}.$$

Since $a = \rho x$, we have

$$\operatorname{tr}\left\{D^2F(0)\left(\frac{\rho x}{1-\rho^2\|x\|^2}, \cdot\right)\right\} = -\frac{\rho^2}{1-\rho^2\|x\|^2}h_0(x, x) + \rho\frac{d}{d\rho}\log(\det Df(\rho x)) \tag{4.4}$$

by taking into account the following trace formula (see [5, p. 243]):

$$\frac{d}{d\rho}\log(\det Df(\rho x)) = \operatorname{tr}\{[Df(\rho x)]^{-1}D^2f(\rho x)(x, \cdot)\}. \tag{4.5}$$

Substituting

$$\frac{1}{1-\rho^2\|x\|^2} = \frac{1}{2}\frac{1}{\|x\|}\frac{d}{d\rho}\log\left(\frac{1+\rho\|x\|}{1-\rho\|x\|}\right)$$

and

$$-\frac{\rho}{1-\rho^2\|x\|^2} = \frac{1}{2}\frac{1}{\|x\|^2}\frac{d}{d\rho}\log(1-\rho^2\|x\|^2)$$

into (4.4) and taking the real part, we have

$$\Re\left\{\operatorname{tr}\left[\frac{1}{2}D^2F(0)(u, \cdot)\right]\right\}\frac{d}{d\rho}\log\left(\frac{1+\rho\|x\|}{1-\rho\|x\|}\right) = \frac{1}{2}h_0(u, u)\frac{d}{d\rho}\log(1-\rho^2\|x\|^2) + \frac{d}{d\rho}\log|\det Df(\rho x)|.$$

By the definition of the trace-order and the relation (4.1), we have

$$\begin{aligned} -\alpha\frac{d}{d\rho}\log\left(\frac{1+\rho\|x\|}{1-\rho\|x\|}\right) &\le c(B)\frac{d}{d\rho}\log(1-\rho^2\|x\|^2) + \frac{d}{d\rho}\log|\det Df(\rho x)| \\ &\le \alpha\frac{d}{d\rho}\log\left(\frac{1+\rho\|x\|}{1-\rho\|x\|}\right). \end{aligned} \tag{4.6}$$

Integrating these inequalities on $0 \le \rho \le 1$, we have

$$\left|c(B)\log(1-\|x\|^2) + \log|\det Df(x)|\right| \le \alpha\log\left(\frac{1+\|x\|}{1-\|x\|}\right).$$

Thus we obtain the theorem. This completes the proof. □

Now, we can give the following conjecture.

Conjecture 4.2. *The bounds in (4.2) are sharp in the general case of finite dimensional* JB*-*triples.*

By Theorem 4.1 and the maximum principle for holomorphic functions, we obtain the following corollary (cf. [5, p. 241], [17, p. 38]).

Corollary 4.3. *Let $\mathcal{F}$ be a linearly invariant family on the unit ball B of a finite dimensional* JB*-*triple X. Then* $\operatorname{ord}\mathcal{F} \ge c(B)$.

Proof. Assume that $\alpha = \operatorname{ord}\mathcal{F} < c(B) < \infty$. By Theorem 4.1, we have

$$\left|[\det Df(x)]^{-1}\right| \le \frac{(1+\|x\|)^{\alpha+c(B)}}{(1-\|x\|)^{\alpha-c(B)}}, \quad x \in B.$$

Therefore, there exists $r \in (0, 1)$ such that $\left|[\det Df(x)]^{-1}\right| < 1$ for $\|x\| = r$. However, since $\det Df(0) = 1$, this is a contradiction by the maximum principle. This completes the proof. □

As in the proof of [9, Theorem 1], [14, Theorem 3.2], we obtain the following theorem.

Theorem 4.4. *Let $\mathcal{F}$ be a linearly invariant family on the unit ball B of a finite dimensional* JB*-*triple X. If* $\operatorname{ord}\mathcal{F} = \alpha < \infty$, *then α is the smallest positive number for which (4.2) holds for all $f \in \mathcal{F}$ and $x \in B$.*

Proof. Let $\beta > 0$ be such that

$$\frac{(1-\|x\|)^{\beta-c(B)}}{(1+\|x\|)^{\beta+c(B)}} \le |\det Df(x)| \le \frac{(1+\|x\|)^{\beta-c(B)}}{(1-\|x\|)^{\beta+c(B)}}, \quad f \in \mathcal{F}, x \in B. \tag{4.7}$$

By the proof of Corollary 4.3, we must have $\beta \ge c(B)$. Let $f \in \mathcal{F}$. From (4.7), we have

$$\left| c(B) \log(1-\|x\|^2) + \log|\det Df(x)| \right| \le \beta \log\left(\frac{1+\|x\|}{1-\|x\|}\right), \quad x \in B.$$

Substituting $x = \rho y$ with $0 < \rho < 1$ and $\|y\| = 1$ into these inequalities, we have

$$\left| c(B) \log(1-\rho^2) + \log|\det Df(\rho y)| \right| \le \beta \log\left(\frac{1+\rho}{1-\rho}\right).$$

Dividing by ρ and letting $\rho \to +0$, we have

$$\left| \Re\left\{ \frac{d}{d\rho} \log \det Df(\rho y) \right\} \Big|_{\rho=0} \right| \le 2\beta.$$

By the trace formula (4.5), we have

$$\left| \frac{1}{2} \Re\{\mathrm{tr}\,\{D^2 f(0)(y,\cdot)\}\} \right| \le \beta.$$

Since $f \in \mathcal{F}$ and $y \in \partial B$ are arbitrary, if we multiply y by a suitable complex number with modulus 1, we have

$$\alpha = \sup_{f\in\mathcal{F}} \sup_{\|y\|=1} \left| \frac{1}{2} \{\mathrm{tr}\,\{D^2 f(0)(y,\cdot)\}\} \right| \le \beta.$$

This completes the proof. □

Remark 4.5. Let B be the unit ball of a finite dimensional JB*-triple X and let $\mathcal{F}$ be a linearly invariant family on B of finite trace-order. From Theorem 4.4, we can deduce another equivalent definition for the trace-order of $\mathcal{F}$:

$$\mathrm{ord}\,\mathcal{F} = \inf\left\{ \alpha : \frac{(1-\|x\|)^{\alpha-c(B)}}{(1+\|x\|)^{\alpha+c(B)}} \le |\det Df(x)| \le \frac{(1+\|x\|)^{\alpha-c(B)}}{(1-\|x\|)^{\alpha+c(B)}}, \ f \in \mathcal{F}, x \in B \right\}.$$

From the above remark, we obtain the following corollary (cf. [9, Proposition 1], [14, Corollary 3.4]). For this purpose, given $f \in LS(B)$, we let $\mathrm{ord}\, f = \mathrm{ord}\, \Lambda[\{f\}]$ be the trace-order of the linearly invariant family $\Lambda[\{f\}]$ generated by f.

Corollary 4.6. *Let $f_1, f_2 \in LS(B)$ with $|\det Df_1(x)| = |\det Df_2(x)|$ for all $x \in B$. Then $\mathrm{ord}\, f_1 = \mathrm{ord}\, f_2$.*

Proof. Let $\phi \in \mathrm{Aut}\,(B)$ and $g_i = \Lambda_\phi(f_i)$ for $i = 1, 2$. Since

$$\det Dg_i(x) = \frac{\det Df_i(\phi(x)) \det D\phi(x)}{\det Df_i(\phi(0)) \det D\phi(0)},$$

we have $|\det Dg_1(x)| = |\det Dg_2(x)|$ for all $x \in B$. Taking into account Remark 4.5, the conclusion follows. This completes the proof. □

In [17, Example 2.2], it is proved that the trace-order of $K(B^n)$ is strictly greater than $c(B^n) = (n+1)/2$ for $n \ge 2$. However, the trace-order of $K(U^n)$ is $c(U^n) = n$. Now, we can give the following open problem.

Open Problem 4.7. *What can we say about the trace-order of $K(B)$ in the general case of finite dimensional JB*-triples?*

We give a partial answer to this open problem. The following theorem implies that if $c(B) = n$, then the trace-order of $K(B)$ is $c(B) = n$. We will see in Theorem 4.10 that $c(B) = n$ when X is a simple JB*-triple of type II(n) (n even), III(n), IV(n) and VI.

Theorem 4.8. *Let B be the unit ball of an n-dimensional JB*-triple X. Then*

(i)

$$|\det Df(x)| \le \frac{1}{(1-\|x\|)^{2n}}, \quad x \in B, \ f \in K(B).$$

(ii) $c(B) \le \mathrm{ord}\, K(B) \le n.$

Proof. (i) Let $f \in K(B)$. Then by Chu et al. [31, Theorem 1.1], we have

$$\|Df(x)\| \leq \frac{1}{(1-\|x\|)^2}, \quad x \in B.$$

Therefore, we have

$$|\det Df(x)| \leq \frac{1}{(1-\|x\|)^{2n}}, \quad x \in B.$$

(ii) Let $f \in K(B)$. From (i), we have

$$\log|\det Df(x)| \leq -2n \log(1-\|x\|), \quad x \in B.$$

Substituting $x = \rho y$ with $0 < \rho < 1$ and $\|y\| = 1$ into these inequalities, we have

$$\log|\det Df(\rho y)| \leq -2n \log(1-\rho).$$

Dividing by ρ and letting $\rho \to +0$, we have

$$\Re\left\{\frac{d}{d\rho}\log\det Df(\rho y)\right\}\Big|_{\rho=0} \leq 2n.$$

By the trace formula (4.5), we have

$$\frac{1}{2}\Re\{\mathrm{tr}\,\{D^2 f(0)(y,\cdot)\}\} \leq n.$$

Since $f \in K(B)$ and $y \in \partial B$ are arbitrary, if we multiply y by a suitable complex number with modulus 1, we have

$$\sup_{f\in K(B)} \sup_{\|y\|=1} \left|\frac{1}{2}\{\mathrm{tr}\,\{D^2 f(0)(y,\cdot)\}\}\right| \leq n.$$

This completes the proof. □

Next, we give an example of a linearly invariant family on B with minimum trace-order which is not a subset of $K(B)$, when $\dim X \geq 2$. This result is a generalization of [17, Corollary 2.1] (in the Euclidean unit ball case) and [17, Theorem 3.3] (in the unit polydisc case).

Theorem 4.9. *Let B be the unit ball of a finite dimensional* JB*-*triple X and let $\mathcal{F} = \{F \in LS(B) : \det DF(x) \equiv 1\}$. Then*

(i) $\mathrm{ord}\,\Lambda[\mathcal{F}] = c(B)$.

(ii) $\Lambda[\mathcal{F}] \not\subseteq K(B)$ *when* $\dim X \geq 2$.

Proof. (i) Let $f(x) = x$. Then $f \in \mathcal{F}$ and by Lemma 3.3 and (2.2), we have

$$\mathrm{ord}\,f = \frac{1}{2}\sup_{a\in B}\sup_{\|y\|=1}|h_0(y,a)| = c(B).$$

Therefore, we have

$$\frac{1}{(1+\|x\|)^{2c(B)}} \leq \left|\det([D\phi(0)]^{-1}D\phi(x))\right| \leq \frac{1}{(1-\|x\|)^{2c(B)}} \tag{4.8}$$

for all $\phi \in \mathrm{Aut}\,(B)$, $x \in B$ by Theorem 4.1. Let $G \in \Lambda[\mathcal{F}]$. Then there exist $F \in \mathcal{F}$ and $\phi \in \mathrm{Aut}\,(B)$ such that $G = \Lambda_\phi(F)$. Since

$$\det DG(x) = \frac{\det DF(\phi(x))\det D\phi(x)}{\det DF(\phi(0))\det D\phi(0)} = \frac{\det D\phi(x)}{\det D\phi(0)},$$

we have

$$\frac{1}{(1+\|x\|)^{2c(B)}} \leq |\det DG(x)| \leq \frac{1}{(1-\|x\|)^{2c(B)}}$$

for all $x \in B$ by (4.8). By Corollary 4.3 and Remark 4.5, $\mathrm{ord}\,\Lambda[\mathcal{F}] = c(B)$.

(ii) Let $n = \dim X \geq 2$ and $e_1, \ldots, e_n$ be a basis of X. Let

$$F(x) = x + bx_1^2 e_2,$$

where $x = x_1e_1 + \cdots + x_ne_n$ and $b \in \mathbb{C}$. Then $F \in LS(B)$ and $\det DF(x) \equiv 1$. Thus $F \in \mathcal{F}$. However, $F \notin K(B)$ for b with sufficiently large modulus, because F does not satisfy the following growth theorem of normalized convex mappings on the unit ball of a complex Banach space [32]:

$$\frac{\|x\|}{1+\|x\|} \leq \|f(x)\| \leq \frac{\|x\|}{1-\|x\|}, \quad x \in B.$$

This completes the proof. □

We will compute the value of $c(B)$. By (4.1), we have

$$c(B) = \frac{1}{2} h_0(e, e),$$

where h_0 is the Bergman metric on X at 0 and e is an arbitrary maximal tripotent in X. Also, since $\operatorname{tr} D(y, a) = h_0(y, a)$ by Loos [29, Theorem 2.10], where $D(y, a) = 2\{y, a, \cdot\}$, we have

$$c(B) = \frac{1}{2} \operatorname{tr} D(e, e),$$

where e is an arbitrary maximal tripotent in X. Let

$$X = V_0(e) \oplus V_1(e) \oplus V_2(e)$$

be the Peirce decomposition of X, where $V_j(e)$ is the eigenspace of $D(e, e)$ with the eigenvalue j for $j = 0, 1, 2$. Then we have

$$c(B) = \frac{1}{2}(\dim V_1(e) + 2 \dim V_2(e)),$$

where e is an arbitrary maximal tripotent in X. Since $V_0(e) = 0$ by Roos [33, Proposition VI.2.4(iii)], we have

$$c(B) = \dim X - \frac{1}{2} \dim V_1(e), \tag{4.9}$$

where e is an arbitrary maximal tripotent in X.

Let $\mathbf{e} = (e_1, \ldots, e_r)$ be a frame of X. By Roos [33, Proposition VI.2.6], the number r of elements of a frame is equal to the rank of X. Also, $e = e_1 + \cdots + e_r$ is a maximal tripotent in X. Let

$$X = \bigoplus_{0 \le i \le j \le r} V_{ij}(\mathbf{e})$$

be the Peirce decomposition with respect to $\mathbf{e}$, where

$$V_{ij}(\mathbf{e}) = \{v \in X : D(e_l, e_l)v = (\delta_{li} + \delta_{lj})v,\ 1 \le l \le r\},$$

for $(i, j) \ne (0, 0)$ and $V_{00}(\mathbf{e}) = \{0\}$. Then by Roos [33, p. 504], we have

$$V_1(e) = \bigoplus_{1 \le j \le r} V_{0j}(\mathbf{e}).$$

From now on, we assume that X is simple. Then $V_{0j}(\mathbf{e})(1 \le j \le r)$ have the same dimension b by Roos [33, Theorem VI.3.5]. Therefore, we have

$$\dim V_1(e) = br. \tag{4.10}$$

The following is a list of all the finite dimensional simple JB*-triples (known as the classical Cartan factors).

Type I$(p, q)(p \le q)$: $X = M_{p,q}(\mathbb{C})$;
Type II(n): $X = \{A \in M_{n,n}(\mathbb{C}) : {}^tA = -A\}$;
Type III(n): $X = \{A \in M_{n,n}(\mathbb{C}) : {}^tA = A\}$;
Type IV$(n)(n > 2)$: $X = \mathbb{C}^n$ (spin factor);
Type V: $X = M_{1,2}(\mathbb{O}_{\mathbb{C}})$;
Type VI: $X = \{A \in M_{3,3}(\mathbb{O}_{\mathbb{C}}) : {}^t\tilde{A} = A\}$.

Here, $\mathbb{O}_{\mathbb{C}}$ is the 8-dimensional Cayley algebra, and $\tilde{A}$ denotes the conjugate of A. The triple product for Type I–III domains is

$$\{x, y, z\} = \frac{1}{2}\left(xy^*z + zy^*x\right),$$

where y^* is the adjoint of the matrix y. A spin factor is a Banach space that is equipped with a complete inner product $\langle \cdot, \cdot \rangle$ and a conjugation j on the resulting Hilbert space, with triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x - \langle x, jz\rangle jy)$$

such that the given norm and the Hilbert space norm are equivalent. The triple product for $M_{1,2}(\mathbb{O}_{\mathbb{C}})$ is

$$\{x, y, z\} = \frac{1}{2}\left\{x({}^t\tilde{y}z) + z({}^t\tilde{y}x)\right\}.$$

The triple product for Type VI is

$$\{x, y, z\} = (x \circ {}^t\tilde{y}) \circ z + (z \circ {}^t\tilde{y}) \circ x - (x \circ z) \circ {}^t\tilde{y},$$

where $x \circ y = (xy + yx)/2$.

Since

$$(b,r)=\begin{cases}(q-p,p) & \text{Type I}(p,q)\\ \left(0,\dfrac{n}{2}\right) & \text{Type II}(n),\ n:\text{even}\\ \left(2,\dfrac{n-1}{2}\right) & \text{Type II}(n),\ n:\text{odd}\\ (0,n) & \text{Type III}(n),\\ (0,2) & \text{Type IV}(n),\\ (4,2) & \text{Type V},\\ (0,3) & \text{Type VI},\end{cases}$$

we obtain the following theorem from (4.9) and (4.10).

Theorem 4.10.

$$c(B)=\begin{cases}\dfrac{p(p+q)}{2} & \text{Type I}(p,q)\\ \dim X & \text{Type II}(n),\ n:\text{even}\\ \dim X-\dfrac{n-1}{2} & \text{Type II}(n),\ n:\text{odd}\\ \dim X & \text{Type III}(n),\\ \dim X & \text{Type IV}(n),\\ 12 & \text{Type V},\\ 27(=\dim X) & \text{Type VI}.\end{cases}$$

Let B^n be the Euclidean unit ball of $\mathbb{C}^n$ (that is, the Type I$(1,n)$JB*-triple). Then $c(B^n)=(n+1)/2$. Therefore, we obtain the following results as corollaries to the above results (cf. [14,5]).

Theorem 4.11. *Let $\mathcal{F}$ be a linearly invariant family on the Euclidean unit ball B^n.*

(i) *If* $\operatorname{ord}\mathcal{F}=\alpha<\infty$, *then*

$$\frac{(1-\|x\|)^{\alpha-(n+1)/2}}{(1+\|x\|)^{\alpha+(n+1)/2}}\le|\det Df(x)|\le\frac{(1+\|x\|)^{\alpha-(n+1)/2}}{(1-\|x\|)^{\alpha+(n+1)/2}},\quad x\in B^n$$

for all $f\in\mathcal{F}$. These estimates are sharp;

(ii) $\operatorname{ord}\mathcal{F}\ge(n+1)/2$;

(iii) *if* $\operatorname{ord}\mathcal{F}=\alpha<\infty$, *then*

$$\operatorname{ord}\mathcal{F}=\inf\left\{\alpha:\frac{(1-\|x\|)^{\alpha-(n+1)/2}}{(1+\|x\|)^{\alpha+(n+1)/2}}\le|\det Df(x)|\le\frac{(1+\|x\|)^{\alpha-(n+1)/2}}{(1-\|x\|)^{\alpha+(n+1)/2}},\ x\in B^n,f\in\mathcal{F}\right\};$$

(iv) *in addition, if* $\mathcal{F}=\{F\in LS(B^n):\det DF(x)\equiv1\}$, *then*
 1. $\operatorname{ord}\Lambda[\mathcal{F}]=(n+1)/2$;
 2. $\Lambda[\mathcal{F}]\not\subseteq K(B^n)$ *when* $n\ge2$.

Let U^n be the unit polydisc of $\mathbb{C}^n$. The Bergman kernel of U^n is as follows:

$$k_{U^n}(z,\overline{w})=\frac{1}{\pi^n}\prod_{j=1}^{n}\frac{1}{(1-z_j\overline{w}_j)^2}.$$

Then the Bergman metric at 0 is

$$h_0(u,v)=2\sum_{j=1}^{n}u_j\overline{v}_j.$$

Thus $c(U^n)=n$. Therefore, we obtain the following results as corollaries to the above results (cf. [14,17]).

Theorem 4.12. *Let $\mathcal{F}$ be a linearly invariant family on the unit polydisc U^n.*

(i) *If* $\operatorname{ord}\mathcal{F}=\alpha<\infty$, *then*

$$\frac{(1-\|x\|)^{\alpha-n}}{(1+\|x\|)^{\alpha+n}}\le|\det Df(x)|\le\frac{(1+\|x\|)^{\alpha-n}}{(1-\|x\|)^{\alpha+n}},\quad x\in U^n$$

for all $f\in\mathcal{F}$. These estimates are sharp;

(ii) $\operatorname{ord}\mathcal{F}\ge n$;

(iii) *if* $\operatorname{ord}\mathcal{F} = \alpha < \infty$, *then*

$$\operatorname{ord}\mathcal{F} = \inf\left\{\alpha : \frac{(1-\|x\|)^{\alpha-n}}{(1+\|x\|)^{\alpha+n}} \le |\det Df(x)| \le \frac{(1+\|x\|)^{\alpha-n}}{(1-\|x\|)^{\alpha+n}},\ x\in U^n, f\in\mathcal{F}\right\};$$

(iv) *in addition, if* $\mathcal{F} = \{F \in LS(U^n) : \det DF(x) \equiv 1\}$, *then*

1. $\operatorname{ord}\Lambda[\mathcal{F}] = n$;
2. $\Lambda[\mathcal{F}] \not\subseteq K(U^n)$ *when* $n \ge 2$.

5. Regularity theorem

In this section, we will apply Theorem 4.1 and its proof to prove a regularity theorem for linearly invariant families on the unit ball B of a finite dimensional JB*-triple X. For every continuous function $g : B \to \mathbb{C}$ and $r \in [0, 1)$, let

$$M(r, g) = \max_{\|x\|=r} |g(x)|.$$

Using a similar reasoning to that in the proof of [16, Theorem 2], we obtain the following theorem. In the case of one complex variable, this result reduces to the well known regularity theorem (see e.g. [34,35]).

Theorem 5.1. *If* $f \in \mathcal{U}_\alpha(B)$, *then*

(i) *for every maximal tripotent* $u \in \partial B$,

$$|\det Df(ru)|\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}}$$

is a non-increasing function of r *on* $[0, 1)$;

(ii)

$$M(r, \det Df)\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}}$$

is a non-increasing function of r *on* $[0, 1)$;

(iii) *there exist a maximal tripotent* $u_0 \in \partial B$ *and a number* $\delta_0 = \delta_0(f) \in [0, 1]$ *such that*

$$\delta_0 = \lim_{r\to 1^-} M(r, \det Df)\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}} = \lim_{r\to 1^-} |\det Df(ru_0)|\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}} \tag{5.1}$$

and

$$\delta_0 = \lim_{r\to 1^-}\max_{\|u\|=1}\left|\frac{d}{dr}\det Df(ru)\right|\frac{(1-r)^{\alpha+c(B)+1}}{2(c(B)r+\alpha)(1+r)^{\alpha-c(B)-1}} = \limsup_{r\to 1^-}\left|\frac{d}{dr}\det Df(ru_0)\right|\frac{(1-r)^{\alpha+c(B)+1}}{2(c(B)r+\alpha)(1+r)^{\alpha-c(B)-1}}. \tag{5.2}$$

Proof. From (4.6), we have

$$\frac{d}{dr}\log\left\{(1-r^2)^{c(B)}|\det Df(ru)|\frac{(1-r)^\alpha}{(1+r)^\alpha}\right\} \le 0$$

for $r \in (0, 1)$. This implies (i).

Let $r_1, r_2 \in [0, 1)$ be arbitrary fixed numbers with $r_1 < r_2$. By Proposition 2.3, there exists a maximal tripotent $u_2 \in \partial B$ such that $M(r_2, \det Df) = |\det Df(r_2u_2)|$. Using (i), we have

$$\begin{aligned} M(r_1, \det Df)\frac{(1-r_1)^{\alpha+c(B)}}{(1+r_1)^{\alpha-c(B)}} &\ge |\det Df(r_1u_2)|\frac{(1-r_1)^{\alpha+c(B)}}{(1+r_1)^{\alpha-c(B)}} \\ &\ge |\det Df(r_2u_2)|\frac{(1-r_2)^{\alpha+c(B)}}{(1+r_2)^{\alpha-c(B)}} \\ &= M(r_2, \det Df)|\frac{(1-r_2)^{\alpha+c(B)}}{(1+r_2)^{\alpha-c(B)}}. \end{aligned}$$

This implies (ii).

(i) and (ii) imply that both limits in (5.1) exist. If we denote the first limit by δ_0 and the second limit by δ_1, then $\delta_0, \delta_1 \in [0, 1]$, because $M(0, \det Df) = |\det Df(0)| = 1$. It suffices to show that $\delta_0 = \delta_1$ for some maximal tripotent $u_0 \in \partial B$. For every $r \in (0, 1)$, there exists a maximal tripotent $u(r) \in \partial B$ such that $M(r, \det Df) = |\det Df(ru(r))|$. Let $(r_\nu) \subset (0, 1)$ be an increasing sequence which converges to 1. We may assume that $u(r_\nu)$ converges to a maximal tripotent $u_0 \in \partial B$ by Proposition 2.3. Let $r \in (0, 1)$ be arbitrarily fixed. Then $r < r_\nu$ for sufficiently large ν and we have

$$\begin{aligned} M(r, \det Df)\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}} &\geq |\det Df(ru(r_\nu))|\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}} \\ &\geq |\det Df(r_\nu u(r_\nu))|\frac{(1-r_\nu)^{\alpha+c(B)}}{(1+r_\nu)^{\alpha-c(B)}} \\ &= M(r_\nu, \det Df)\frac{(1-r_\nu)^{\alpha+c(B)}}{(1+r_\nu)^{\alpha-c(B)}}. \end{aligned}$$

Letting $\nu \to \infty$ in the above inequalities, we have

$$M(r, \det Df)\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}} \geq |\det Df(ru_0)|\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}} \geq \delta_0.$$

Letting $r \to 1^-$ in the above inequalities, we have $\delta_0 \geq \delta_1 \geq \delta_0$. Therefore, we have $\delta_0 = \delta_1$.

Now, we prove the equalities (5.2). Putting $\rho x = ru$ in (4.4) and multiplying by e^{it}, where $r \in (0, 1)$, u is a maximal tripotent and $t \in [0, 2\pi]$, we have

$$\operatorname{tr}\left\{D^2F(0)\left(\frac{re^{it}u}{1-r^2}, \cdot\right)\right\} = -2\frac{r^2e^{it}}{1-r^2}c(B) + re^{it}\frac{d}{dr}\log(\det Df(ru)). \tag{5.3}$$

Let t be such that $e^{it}\frac{d}{dr}\log(\det Df(ru))$ is real and positive. Then, by taking the real part of (5.3), we have

$$\left|\frac{\frac{d}{dr}\det Df(ru)}{\det Df(ru)}\right| \leq \frac{2c(B)r + 2\alpha}{1-r^2}, \quad r \in (0, 1).$$

Therefore, from (4.2), we obtain

$$\begin{aligned} \left|\frac{d}{dr}\det Df(ru)\right| &\leq \frac{2c(B)r + 2\alpha}{1-r^2}|\det Df(ru)| \\ &\leq (2c(B)r + 2\alpha)\frac{(1+r)^{\alpha-c(B)-1}}{(1-r)^{\alpha+c(B)+1}} \end{aligned} \tag{5.4}$$

for $r \in (0, 1)$. This implies that there exists the finite upper limit δ_2 in the second equality in (5.2). It follows that for every $\varepsilon > 0$, there exists $r_0 \in (0, 1)$ such that

$$\left|\frac{d}{dr}\det Df(ru_0)\right| \leq (\delta_2 + \varepsilon)(2c(B)r + 2\alpha)\frac{(1+r)^{\alpha-c(B)-1}}{(1-r)^{\alpha+c(B)+1}}$$

for $r \in (r_0, 1)$. Therefore, we have

$$\begin{aligned} |\det Df(ru_0)| - |\det Df(r_0u_0)| &\leq \int_{r_0}^{r}\left|\frac{d}{d\rho}\det Df(\rho u_0)\right|d\rho \\ &\leq (\delta_2 + \varepsilon)\int_{r_0}^{r}(2c(B)\rho + 2\alpha)\frac{(1+\rho)^{\alpha-c(B)-1}}{(1-\rho)^{\alpha+c(B)+1}}d\rho \\ &= (\delta_2 + \varepsilon)\left[\frac{(1+r)^{\alpha-c(B)}}{(1-r)^{\alpha+c(B)}} - \frac{(1+r_0)^{\alpha-c(B)}}{(1-r_0)^{\alpha+c(B)}}\right] \end{aligned} \tag{5.5}$$

for $r \in (r_0, 1)$. Multiplying this inequality by $(1-r)^{\alpha+c(B)}/(1+r)^{\alpha-c(B)}$ and letting $r \to 1^-$, we have $\delta_0 \leq \delta_2 + \varepsilon$. Since ε is arbitrary, we have $\delta_0 \leq \delta_2$.

From inequality (5.4), it also follows that

$$\left|\frac{d}{dr}\det Df(ru_0)\right|\frac{(1-r)^{\alpha+c(B)+1}}{(2c(B)r + 2\alpha)(1+r)^{\alpha-c(B)-1}} \leq |\det Df(ru_0)|\frac{(1-r)^{\alpha+c(B)}}{(1+r)^{\alpha-c(B)}}$$

for $r \in (0, 1)$. Letting $r \to 1^-$, we have $\delta_2 \leq \delta_0$. Thus, $\delta_2 = \delta_0$.

Since

$$\max_{\|z\|=r} \frac{|D(\det Df)(z)(z)|}{\|z\|} = \max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right|,$$

$\left| \frac{d}{dr} \det Df(ru) \right|$ attains its maximum on $\|u\| = 1$ at a maximal tripotent u by Proposition 2.3. Therefore, from inequality (5.4), we also have

$$\begin{aligned} \max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right| &\le \frac{2c(B)r + 2\alpha}{1-r^2} M(r, \det Df) \\ &\le (2c(B)r + 2\alpha) \frac{(1+r)^{\alpha - c(B) - 1}}{(1-r)^{\alpha + c(B) + 1}} \end{aligned} \tag{5.6}$$

for $r \in (0, 1)$. This implies that there exists the finite upper limit

$$\delta_3 = \limsup_{r \to 1^-} \max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right| \frac{(1-r)^{\alpha + c(B) + 1}}{2(c(B)r + \alpha)(1+r)^{\alpha - c(B) - 1}}.$$

It follows that for every $\varepsilon > 0$, there exists $r_1 \in (0, 1)$ such that

$$\max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right| \le (\delta_3 + \varepsilon)(2c(B)r + 2\alpha) \frac{(1+r)^{\alpha - c(B) - 1}}{(1-r)^{\alpha + c(B) + 1}}$$

for $r \in (r_1, 1)$. Therefore, we have

$$\begin{aligned} |\det Df(ru_0)| - |\det Df(r_1 u_0)| &\le \int_{r_1}^{r} \left| \frac{d}{d\rho} \det Df(\rho u_0) \right| d\rho \\ &\le \int_{r_1}^{r} \max_{\|u\|=1} \left| \frac{d}{d\rho} \det Df(\rho u) \right| d\rho \\ &\le (\delta_3 + \varepsilon) \int_{r_1}^{r} (2c(B)\rho + 2\alpha) \frac{(1+\rho)^{\alpha - c(B) - 1}}{(1-\rho)^{\alpha + c(B) + 1}} d\rho \\ &= (\delta_3 + \varepsilon) \left[\frac{(1+r)^{\alpha - c(B)}}{(1-r)^{\alpha + c(B)}} - \frac{(1+r_1)^{\alpha - c(B)}}{(1-r_1)^{\alpha + c(B)}} \right] \end{aligned} \tag{5.7}$$

for $r \in (r_1, 1)$. Multiplying this inequality by $(1-r)^{\alpha + c(B)}/(1+r)^{\alpha - c(B)}$ and letting $r \to 1^-$, we have $\delta_0 \le \delta_3 + \varepsilon$. Since ε is arbitrary, we have $\delta_0 \le \delta_3$.

From inequality (5.6), it also follows that

$$\max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right| \frac{(1-r)^{\alpha + c(B) + 1}}{(2c(B)r + 2\alpha)(1+r)^{\alpha - c(B) - 1}} \le M(r, \det Df) \frac{(1-r)^{\alpha + c(B)}}{(1+r)^{\alpha - c(B)}}$$

for $r \in (0, 1)$. Letting $r \to 1^-$, we have $\delta_3 \le \delta_0$. Thus, $\delta_3 = \delta_0$.

Let

$$p(r) = \int_{r_1}^{r} \max_{\|u\|=1} \left| \frac{d}{d\rho} \det Df(\rho u) \right| d\rho, \quad r \in (r_1, 1). \tag{5.8}$$

Since $\delta_3 = \delta_0$, we obtain from (5.7)

$$\begin{aligned} |\det Df(ru_0)| - |\det Df(r_1 u_0)| &\le p(r) \\ &\le (\delta_0 + \varepsilon) \left[\frac{(1+r)^{\alpha - c(B)}}{(1-r)^{\alpha + c(B)}} - \frac{(1+r_1)^{\alpha - c(B)}}{(1-r_1)^{\alpha + c(B)}} \right] \end{aligned}$$

for $r \in (r_1, 1)$. Multiplying this inequality by $(1-r)^{\alpha + c(B)}/(1+r)^{\alpha - c(B)}$ and letting $r \to 1^-$, we have

$$\delta_0 \le \liminf_{r \to 1^-} p(r) \frac{(1-r)^{\alpha + c(B)}}{(1+r)^{\alpha - c(B)}} \le \limsup_{r \to 1^-} p(r) \frac{(1-r)^{\alpha + c(B)}}{(1+r)^{\alpha - c(B)}} \le \delta_0 + \varepsilon.$$

Thus,

$$\lim_{r \to 1^-} p(r) \frac{(1-r)^{\alpha + c(B)}}{(1+r)^{\alpha - c(B)}} = \delta_0. \tag{5.9}$$

By applying the maximum principle for holomorphic functions to

$$D(\det Df)(\zeta w)(w), \quad \|w\| = 1, \zeta \in U,$$

we obtain that

$$\max_{\|z\|\le r} \frac{|D(\det Df)(z)(z)|}{\|z\|} = \max_{\|z\|=r} \frac{|D(\det Df)(z)(z)|}{\|z\|} = \max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right|$$

for $r \in (0, 1)$. Therefore, $\max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right|$ is a non-decreasing continuous function of $r \in [0, 1)$. This implies that the function p, defined in (5.8), is differentiable, $p'(r) = \max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right|$ for $r \in [0, 1)$ and p' does not decrease. By Hardy [36, Theorem 112] (see also [16, Lemma 1]) and (5.9), we have

$$\lim_{r\to 1^-} \max_{\|u\|=1} \left| \frac{d}{dr} \det Df(ru) \right| \frac{(1-r)^{\alpha+c(B)+1}}{2(c(B)r+\alpha)(1+r)^{\alpha-c(B)-1}}$$
$$= \lim_{r\to 1^-} p'(r)(1-r)^{\alpha+c(B)+1} \frac{1}{2(c(B)r+\alpha)(1+r)^{\alpha-c(B)-1}} = \delta_0.$$

This completes the proof. □

Remark 5.2. The maximal tripotent $u_0 = u_0(f)$ in Theorem 5.1 is called the direction of the maximal growth of the mapping $f \in \mathcal{U}_\alpha(B)$.

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