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Linear invariance of locally biholomorphic mappings in the unit ball of a JB*-triple

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ABSTRACT

We study the notion of linear invariance on the unit ball of a JB*-triple X, and we obtain some connection between the norm-order of a linear invariant family and the starlikeness of order $1/2$. Also, we give some result concerning the radius of univalence of some linear invariant families. Finally, if the dimension of X is finite and if the norm-order of a linear invariant family is finite, then we prove the normality of the linear invariant family and we also obtain upper bounds on the distortion and the growth of mappings in a linear invariant family with specified norm-order. In particular, our results are valid for the classical Cartan domains and the unit polydisc.

1. Introduction

Linear invariance, introduced by Pommerenke [30,31] has been a powerful tool in extending many ideas of univalent function theory to the study of locally univalent functions on the unit disc. Recently Barnard, FitzGerald and Gong [1], Pfaltzgraff [26], Pfaltzgraff and Suffridge [27–29], Hamada and Kohr [12,13,15,16] and Graham, Hamada, Kohr and Suffridge [5] studied the linear invariant families in several complex variables. Barnard, FitzGerald and Gong [1] studied a linear invariant family on the Euclidean unit ball in $\mathbb{C}^2$ and Pfaltzgraff [26], Pfaltzgraff and Suffridge [27,28], Godula, Liczberski and Starkov [7], Liczberski and Starkov [24], studied linear invariant families on the Euclidean unit ball in $\mathbb{C}^n$ by using the (trace) order of a linear invariant family.

On the other hand, several interesting results, concerning the norm-order of a linear invariant family and some connections with starlikeness, convexity and other geometric properties of holomorphic mappings in $\mathbb{C}^n$, were recently obtained by Pfaltzgraff and Suffridge [29]. Also they showed a number of growth, covering and distortion results for mappings that belong to a linear invariant family on the Euclidean unit ball in $\mathbb{C}^n$. Hamada and Kohr generalized the results in [29] to the unit ball in a complex Hilbert space in [12] and to the unit polydisc in [13]. For linear invariant families in several complex variables, see also the books [3,6] and the references therein.

In this paper we continue the study of linear invariance on the unit ball B of a JB*-triple. All four types of classical Cartan domains and their infinite dimensional analogues are the open unit balls of JB*-triples, and the same holds for any finite or infinite product of these domains ([18], see also [11,20]). Thus the unit balls of JB*-triples are natural generalizations of the unit disc in $\mathbb{C}$ and we have a setting in which a large number of bounded symmetric homogeneous domains may be studied simultaneously. We obtain some connection between the norm-order of a linear invariant family and the starlikeness of order $1/2$. Also, we give some result concerning the radius of univalence of some linear invariant families. Finally, when

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the dimension of X is finite, and the norm-order of a linear invariant family is finite, we will prove the normality of the linear invariant family and we also obtain upper bounds on the distortion and the growth of mappings in a linear invariant family with specified norm-order.

2. Preliminaries

Let B be the unit ball in a complex Banach space X. Let Y be a complex Banach space. A holomorphic mapping $f : B \to Y$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. A holomorphic mapping $f : B \to Y$ is said to be biholomorphic if $f(B)$ is a domain in Y, f^{-1} exists and holomorphic on $f(B)$. A biholomorphic mapping $f : B \to Y$ is said to be convex if $f(B)$ is a convex domain. Let X^* be the dual space of X. For each $x \in X \setminus \{0\}$, we define

$$T(x) = \left\{x^* \in X^*:\ \left\|x^*\right\| = 1,\ x^*(x) = \|x\|\right\}.$$

By the Hahn–Banach theorem, $T(x)$ is nonempty. Let $f : B \to X$ be a locally biholomorphic mapping. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. We say that f is a starlike mapping of order α if

$$\left|\frac{1}{\|x\|}x^*\left(\left[Df(x)\right]^{-1}f(x)\right) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$$

for $x \in B\backslash\{0\}$, $x^* \in T(x)$.

Let $L(X, Y)$ denote the set of continuous linear operators from X into Y. Let I_X be the identity in $L(X, X)$.

Let $\mathcal{LS}(B)$ denote the family of locally biholomorphic mappings from B to X, normalized by $f(0) = 0$ and $Df(0) = I_X$.

We recall that a JB*-triple is a complex Banach space X together with a continuous mapping (called Jordan triple product)

$$X \times X \times X \to X, \qquad (x, y, z) \mapsto \{x, y, z\}$$

such that for all elements in X the following conditions (J_1)–(J_4) hold, where for every $x, y \in X$, the operator $x \Box y$ on X is defined by $z \mapsto \{x, y, z\}$:

- (J_1) $\{x, y, z\}$ is symmetric bilinear in the outer variable x, z and conjugate linear in the inner variable y,
- (J_2) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$ (Jordan triple identity),
- (J_3) $x \Box x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,
- (J_4) $\|\{x, x, x\}\| = \|x\|^3$.

It is known [21, p. 523] that in this definition condition (J_4) can be replaced by $\|x \Box x\| = \|x\|^2$ and that

$$\|x \Box y\| \leqslant \|x\| \cdot \|y\|$$

holds for all $x, y \in X$. Then, we have

$$\left\|\{x, y, z\}\right\| \leqslant \|x\| \cdot \|y\| \cdot \|z\|, \quad \text{for all } x, y, z. \tag{2.1}$$

Example 2.1. Let S be a locally compact topological space and let $C_0(S)$ be the Banach space of all continuous complex valued functions f on S vanishing at infinity with $\|f\| = \sup|f(S)|$. Then $C_0(S)$ is a JB*-triple with $\{f, g, h\} = f\bar{g}h$.

A linear subspace $I \subset X$ is called a *subtriple* if $\{I, I, I\} \subset I$.

For every $a \in X$, let $Q_a : X \to X$ be the conjugate linear operator defined by $Q_a(x) = \{a, x, a\}$. This operator is called the quadratic representation and it satisfies the fundamental formula

$$Q_{Q_a(b)} = Q_a Q_b Q_a$$

for all $a, b \in X$. For every $x, y \in X$, the Bergman operator $B(x, y) \in L(X, X)$ is defined by

$$B(x, y) = I_X - 2x \Box y + Q_x Q_y.$$

From (2.1), we have

$$\left\|B(x, y)\right\| \leqslant \left(1 + \|x\| \cdot \|y\|\right)^2, \quad x, y \in X. \tag{2.2}$$

In case $\|x \Box y\| < 1$, the spectrum of $B(x, y)$ lies in $\{z \in \mathbb{C}:\ |z - 1| < 1\}$. In particular, the fractional power $B(x, y)^r \in GL(X)$ exists for every $r \in \mathbb{R}$ in a natural way (cf. [21, p. 517]).

Let B be the unit ball of a JB*-triple X. Then, for each $a \in B$, the Möbius transformation g_a defined by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x \Box a)^{-1}x, \tag{2.3}$$

is a biholomorphic mapping of B onto itself with $g_a(0) = a$, $g_a(-a) = 0$ and $g_{-a} = g_a^{-1}$.

Proposition 2.2. *Let g_a be as above. Then for any $a \in B$, g_a extends biholomorphically to a neighborhood of $\overline{B}$ and we have*

$$\left[Dg_a(0)\right]^{-1} D^2 g_a(0)(x, y) = -2\{x, a, y\}, \tag{2.4}$$

$$\left\| Dg_a(0) \right\| \leqslant 1, \tag{2.5}$$

$$\left\| \left[Dg_a(0)\right]^{-1} \right\| = \frac{1}{1 - \|a\|^2}, \tag{2.6}$$

$$Dg_{\zeta a}(0) = Dg_a(0), \quad |\zeta| = 1, \tag{2.7}$$

$$g_a(a) = \frac{2}{1 + \|a\|^2} a, \tag{2.8}$$

$$g_a(x) = x + a - \{x, a, x\} + O\left(\|a\|^2\right), \tag{2.9}$$

$$\left[Dg_a(0)\right]^{-1} = I_X + O\left(\|a\|^2\right). \tag{2.10}$$

Moreover, we have

$$\frac{1}{1 - \|g_{-z}(w)\|^2} \leqslant \frac{(1 + \|w\| \cdot \|z\|)^2}{(1 - \|w\|^2)(1 - \|z\|^2)}, \quad z, w \in B. \tag{2.11}$$

Proof. Since $\|x \square a\| \leqslant \|x\| \cdot \|a\|$, g_a and $g_a^{-1} = g_{-a}$ extend holomorphically to $\|x\| < 1/\|a\|$. Then, g_a extends biholomorphically to a neighborhood of $\overline{B}$. Since

$$g_a(x) = a + B(a, a)^{1/2}\left[x - (x \square a)x\right] + O\left(\|x\|^3\right) = a + B(a, a)^{1/2}\left[x - \{x, a, x\}\right] + O\left(\|x\|^3\right),$$

we have

$$Dg_a(x)(y) = B(a, a)^{1/2}\left[y - \{y, a, x\} - \{x, a, y\}\right] + O\left(\|x\|^2\right)$$

and

$$D^2 g_a(0)(y, z) = B(a, a)^{1/2}\left[-\{y, a, z\} - \{z, a, y\}\right] = -2B(a, a)^{1/2}\{y, a, z\}.$$

Since $Dg_a(0) = B(a, a)^{1/2}$, we obtain (2.4). By [22, Corollary 3.6], we obtain (2.5) and (2.6). Since

$$B(\zeta a, \zeta a) = B(a, a), \quad |\zeta| = 1,$$

we obtain (2.7). Since the JB*-subtriple of X generated by a, denoted by X_a, is isometrically isomorphic to $\mathcal{C}_0(S)$ for some locally compact subset $S \subset \mathbb{R}$ [21], it is easy to see that in X_a and hence in X, we have

$$g_a(a) = \frac{2}{1 + \|a\|^2} a.$$

Thus, we obtain (2.8). Since $B(a, a)^{1/2} = I_X + O(\|a\|^2)$, we have (2.10) and

$$g_a(x) = a + B(a, a)^{1/2}\left[x - \{x, a, x\}\right] + O\left(\|a\|^2\right) = a + x - \{x, a, x\} + O\left(\|a\|^2\right).$$

Since

$$\frac{1}{1 - \|g_{-z}(w)\|^2} = \left\| B(w, w)^{-1/2} B(w, z) B(z, z)^{-1/2} \right\|, \quad z, w \in B \tag{2.12}$$

by [25, Proposition 3.1], we obtain (2.11) from (2.2) and (2.6). □

$x \in X$ is called *regular* if $x \square x \in GL(X)$ and $x \in X$ is called a *triponent* if $\{x, x, x\} = x$. A point $u \in \overline{B}$ is said to be a *real (resp. complex) extreme point of* $\overline{B}$ if the only $x \in X$ satisfying $\|u + \lambda x\| \leqslant 1$ for all real (resp. complex) numbers λ with $|\lambda| \leqslant 1$ is $x = 0$. We call $u \in \overline{B}$ *holomorphically extreme in* $\overline{B}$ if for every open neighborhood U of $0 \in \mathbb{C}$ and every holomorphic mapping $f : U \to X$ the conditions $f(0) = u$ and $f(U) \subset \overline{B}$ imply that $f'(0) = 0$. $u \in \partial B$ is called a *simple boundary point of* B if $u + ty \in \partial B$, $y \in X$, $t \in \mathbb{C}$, $|t| < 1$ always implies $y = 0$. The following result is obtained in Kaup and Upmeier [23, Proposition 3.5].

Proposition 2.3. *Let B be the unit ball of a JB*-triple X and $u \in X$. Then the following conditions are equivalent*:

(i) *u is a regular triponent in X;*
(ii) *u is holomorphically extreme in $\overline{B}$;*

(iii) *u is a complex extreme point of $\overline{B}$;*
(iv) *u is a simple boundary point of B.*

Let $\mathcal{E}$ be the set of all complex extreme points of $\overline{B}$. As a corollary of the above proposition, we obtain the following maximum principle for holomorphic mappings on the unit ball of a JB*-triple. When B is the unit ball of a J*-algebra, see Harris [18, Theorem 9]. By the Krein–Milman theorem (see e.g. [8, Chapter 4]), it is known that if $\overline{B}$ is a compact subset of X, then $\mathcal{E}$ is nonempty.

Proposition 2.4. *Let B be the unit ball of a JB*-triple X and let $\mathcal{E}$ denote the set of all complex extreme points of $\overline{B}$. If $\mathcal{E} \neq \emptyset$, then:*

(i) *Let $g_a \in \mathrm{Aut}(B)$ given in (2.3). Then $g_a(\mathcal{E}) = \mathcal{E}$ for any $a \in B$;*
(ii) *Let Y be a complex Banach space. Let $f : B \to Y$ be a holomorphic mapping with a continuous and bounded extension to $B \cup \mathcal{E}$. Then*

$$\|f(x)\| \leqslant \sup\{\|f(u)\|: u \in \mathcal{E}\}, \quad x \in B.$$

Moreover, f is completely determined by its value on $\mathcal{E}$.

Proof. (i) Since $g_a^{-1} = g_{-a}$, it suffices to show that $g_a(\mathcal{E}) \subset \mathcal{E}$ for any $a \in B$. Let $v = g_a(u)$, where $u \in \mathcal{E}$. Assume that $v + \lambda x \in \overline{B}$ for $|\lambda| \leqslant 1$. Let

$$h(\lambda) = g_a^{-1}(v + \lambda x), \quad \lambda \in U.$$

Then h is holomorphic on U by Proposition 2.2, $h(0) = g_a^{-1}(v) = u$ and $h(U) \subset \overline{B}$. Since u is a holomorphic extreme point by Proposition 2.3, we must have $h'(0) = 0$. This implies that $Dg_a^{-1}(v)(x) = 0$. Since g_a^{-1} extends biholomorphically to a neighborhood of $\overline{B}$, we obtain $x = 0$. Thus, $v \in \mathcal{E}$.

(ii) By the mean value property for vector valued holomorphic functions, we obtain

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f\big(g_x\big(e^{i\theta}u\big)\big)\,d\theta,$$

where $u \in \mathcal{E}$. Since $g_x(e^{i\theta}u) \in \mathcal{E}$ for $\theta \in [0, 2\pi]$ by (i), we obtain (ii). □

3. Linear invariance in *X*

We define the notion of linear invariant families and the norm-order in the unit ball B of a complex Banach space X.

Definition 3.1. Let B be the unit ball of a complex Banach space X. Then a family $\mathcal{F}$ is called a *linear-invariant family* if:

(i) $\mathcal{F} \subset \mathcal{LS}(B)$,

and

(ii) $\Lambda_\phi(f) \in \mathcal{F}$, for all $f \in \mathcal{F}$ and $\phi \in \mathrm{Aut}\,B$,

where $\mathrm{Aut}\,B$ denotes the set of biholomorphic automorphisms of B, and $\Lambda_\phi(f)$ is the *Koebe transform*

$$\Lambda_\phi(f)(x) = \big[D\phi(0)\big]^{-1}\big[Df\big(\phi(0)\big)\big]^{-1}\big(f\big(\phi(x)\big) - f\big(\phi(0)\big)\big), \tag{3.1}$$

for all $x \in B$.

Note that the Koebe transform has the group property $\Lambda_\psi \circ \Lambda_\phi = \Lambda_{\phi\circ\psi}$.

If $\mathcal{F}$ is a linear invariant family, we define two types of *norm-order* of $\mathcal{F}$ (cf. [29]), given by

$$\|\mathrm{ord}\|_{X,1}\mathcal{F} = \sup_{f\in\mathcal{F}}\sup_{\|y\|=1}\left\{\frac{1}{2}\big\|D^2f(0)(y,\cdot)\big\|\right\}$$

and

$$\|\mathrm{ord}\|_{X,2}\mathcal{F} = \sup_{f\in\mathcal{F}}\sup_{\|y\|=1}\left\{\frac{1}{2}\big\|D^2f(0)(y,y)\big\|\right\}.$$

It is clear that $\|\operatorname{ord}\|_{X,1}\mathcal{F} \geqslant \|\operatorname{ord}\|_{X,2}\mathcal{F}$. Since

$$D^2 f(0)(y,z) = \frac{1}{2}\bigl\{D^2 f(0)(y+z, y+z) - D^2 f(0)(y,y) - D^2 f(0)(z,z)\bigr\},$$

we obtain $\|\operatorname{ord}\|_{X,1}\mathcal{F} \leqslant 3\|\operatorname{ord}\|_{X,2}\mathcal{F}$. Moreover, if X is a Hilbert space, then $\|\operatorname{ord}\|_{X,1}\mathcal{F} = \|\operatorname{ord}\|_{X,2}\mathcal{F}$ by Hörmander [19, Theorem 4].

We now give some examples of linear invariant families in the unit ball B of a complex Banach space X.

Example 3.2. $S(B)$, the set of all biholomorphic mappings in $\mathcal{LS}(B)$. If X is a complex Hilbert space of dimension n, where $n > 1$, the linear invariant family $S(B)$ does not have finite norm-order (see [29], cf. [1]).

Example 3.3. $\mathcal{U}_\alpha(B)$, the union of all linear invariant families contained in $\mathcal{LS}(B)$ with norm-order not greater than α. This is a generalization of the universal linear invariant families $\mathcal{U}_\alpha = \mathcal{U}_\alpha(\Delta)$ considered in [30].

Example 3.4. If $\mathcal{G}$ is a nonempty subset of $\mathcal{LS}(B)$, then the linear invariant family generated by $\mathcal{G}$ is the family

$$\Lambda[\mathcal{G}] = \bigl\{\Lambda_\phi(g)\colon\ g \in \mathcal{G},\ \phi \in \operatorname{Aut} B\bigr\}.$$

The linear invariance is a consequence of the group property of the Koebe transform. Obviously, $\Lambda[\mathcal{G}] = \mathcal{G}$ if and only if $\mathcal{G}$ is a linear-invariant family. In the case of the unit Euclidean ball and the unit polydisc in $\mathbb{C}^n$, this example provided a useful technique for generating many interesting mappings (see [26–28]). For example, we can use a single mapping f from $\mathcal{LS}(B)$ to generate the linear invariant family $\Lambda[\{f\}]$. The family $\Lambda[\{i\}]$, generated by the identity mapping $i(x) = x$, consists of all the Koebe transforms of $i(x)$.

Example 3.5. $\mathcal{K}(B)$, the set of convex mapping in $\mathcal{LS}(B)$.

As in the proof of [29, Theorem 5.1], we obtain the following result. We will see later that $\|\operatorname{ord}\|_{X,2}\mathcal{K}(B) = 1$, if B is the unit ball of a JB*-triple. We remark that if $X = \ell^1$ is the complex Banach space of summable complex sequences, then $\|\operatorname{ord}\|_{X,2}\mathcal{K}(B) = 0$, since the only mapping $f \in \mathcal{K}(B)$ is the identity mapping [32, Corollary 1].

Proposition 3.6. *Let B be the unit ball of a complex Banach space X and let $\mathcal{K}(B)$ be the set of normalized convex mappings on B. Then $\|\operatorname{ord}\|_{X,2}\mathcal{K}(B) \leqslant 1$.*

When B is the unit ball of a JB*-triple X, we have the following first-order approximation formula for the Koebe transform of f.

Lemma 3.7. *Let $g_a \in \operatorname{Aut}(B)$ given in (2.3). If $f \in \mathcal{LS}(B)$, then*

$$\begin{aligned}&\bigl[Dg_a(0)\bigr]^{-1}\bigl[Df\bigl(g_a(0)\bigr)\bigr]^{-1}\bigl(f\bigl(g_a(x)\bigr) - f\bigl(g_a(0)\bigr)\bigr)\\ &\quad = f(x) + Df(x)\bigl(a - \{x,a,x\}\bigr) - a - D^2 f(0)\bigl(a, f(x)\bigr) + O\bigl(\|a\|^2\bigr), \quad a \to 0.\end{aligned}$$

Proof. Since

$$f(x) = x + \frac{1}{2}D^2 f(0)(x,x) + \frac{1}{6}D^3 f(0)(x,x,x) + \cdots,$$

we have

$$f\bigl(g_a(0)\bigr) = a + O\bigl(\|a\|^2\bigr), \tag{3.2}$$

and

$$\bigl[Df\bigl(g_a(0)\bigr)\bigr]^{-1} = I_X - D^2 f(0)(a,\cdot) + O\bigl(\|a\|^2\bigr). \tag{3.3}$$

Since

$$f(x+y) = f(x) + Df(x)y + O\bigl(\|y\|^2\bigr),$$

we obtain from (2.9) that

$$f\bigl(g_a(x)\bigr) = f\bigl(x + a - \{x,a,x\} + O\bigl(\|a\|^2\bigr)\bigr) = f(x) + Df(x)\bigl(a - \{x,a,x\}\bigr) + O\bigl(\|a\|^2\bigr). \tag{3.4}$$

From (2.10) and (3.3), we have

$$\big[Dg_a(0)\big]^{-1}\big[Df\big(g_a(0)\big)\big]^{-1} = I_X - D^2f(0)(a,\cdot) + O\big(\|a\|^2\big). \tag{3.5}$$

Then from (3.2), (3.4) and (3.5), we have

$$\begin{aligned}&\big[Dg_a(0)\big]^{-1}\big[Df\big(g_a(0)\big)\big]^{-1}\big(f\big(g_a(x)\big) - f\big(g_a(0)\big)\big)\\ &\quad= \big(I_X - D^2f(0)(a,\cdot)\big)\big[f(x) + Df(x)\big(a - \{x,a,x\}\big) - a\big] + O\big(\|a\|^2\big)\\ &\quad= f(x) + Df(x)\big(a - \{x,a,x\}\big) - a - D^2f(0)\big(a, f(x)\big) + O\big(\|a\|^2\big).\end{aligned}$$

This completes the proof. □

The following useful result is a natural extension to JB*-triples of [30, Lemma 1.2] (cf. [12,13,27]).

Lemma 3.8. *Let $\mathcal{F}$ be a linear-invariant family on the unit ball B of a JB*-triple X with $\|\operatorname{ord}\|_{X,1}\mathcal{F} = \alpha$ and $\|\operatorname{ord}\|_{X,2}\mathcal{F} = \beta$. Then*

$$\alpha = \sup_{f\in\mathcal{F}}\sup\left\{\left\|\frac{1}{2}\Phi(f,x,y,z) - \{y,x,z\}\right\|:\ \|x\|<1,\ \|y\| = \|z\| = 1\right\}, \tag{3.6}$$

and

$$\beta = \sup_{f\in\mathcal{F}}\sup\left\{\left\|\frac{1}{2}\Phi(f,x,y,y) - \{y,x,y\}\right\|:\ \|x\|<1,\ \|y\| = 1\right\}, \tag{3.7}$$

where

$$\Phi(f,x,y,z) = \big[Dg_x(0)\big]^{-1}\big[Df(x)\big]^{-1}D^2f(x)\big(Dg_x(0)y, Dg_x(0)z\big).$$

Proof. It is clear that

$$\sup_{f\in\mathcal{F}}\sup\left\{\left\|\frac{1}{2}\Phi(f,x,y,z) - \{y,x,z\}\right\|:\ \|x\|<1,\ \|y\| = \|z\| = 1\right\} \geqslant \alpha.$$

On the other hand, let $f \in \mathcal{F}$ and $\phi = g_x$ where $x \in B$. It is clear that $F \in \mathcal{F}$, where $F(w) = \Lambda_\phi(f)(w)$, $w \in B$. Therefore, we have

$$\left\|\frac{1}{2}D^2F(0)(y,z)\right\| \leqslant \alpha, \quad y,z\in X,\ \|y\| = \|z\| = 1. \tag{3.8}$$

If we differentiate twice the mapping $F = \Lambda_\phi(f)$, given by (3.1), we obtain that

$$DF(w) = \big[D\phi(0)\big]^{-1}\big[Df\big(\phi(0)\big)\big]^{-1}Df\big(\phi(w)\big)D\phi(w), \quad w\in B,$$

and

$$\begin{aligned}D^2F(w)(y,z) &= \big[D\phi(0)\big]^{-1}\big[Df\big(\phi(0)\big)\big]^{-1}\big\{D^2f\big(\phi(w)\big)\big(D\phi(w)y, D\phi(w)z\big) + Df\big(\phi(w)\big)D^2\phi(w)(y,z)\big\},\\ &\quad y,z\in X.\end{aligned}$$

Evaluating at $w = 0$, we obtain that

$$D^2F(0)(y,z) = \Phi(f,x,y,z) + \big[Dg_x(0)\big]^{-1}D^2g_x(0)(y,z).$$

Hence, from (2.4) and this equality, we have

$$D^2F(0)(y,z) = \Phi(f,x,y,z) - 2\{y,x,z\}.$$

Finally, from (3.8) and the last relation, one concludes that

$$\left\|\frac{1}{2}\Phi(f,x,y,z) - \{y,x,z\}\right\| \leqslant \alpha,$$

for all $x \in B$ and $y,z \in X$, $\|y\| = \|z\| = 1$. Thus, we obtain (3.6). Putting $z = y$ in the above argument, we obtain (3.7). This completes the proof. □

Note that in the case of one complex variable, the relations (3.6) and (3.7) are equivalent to

$$\alpha=\beta=\sup_{f\in\mathcal{F}}\sup_{|b|<1}\left|\frac{1}{2}\left(1-|b|^2\right)\frac{f''(b)}{f'(b)}-\bar{b}\right|$$

(compare with [30, Lemma 1.2]).

Pfaltzgraff and Suffridge [29, Theorem 3.1] proved recently that if $\mathcal{M}$ is a linear invariant family on the Euclidean unit ball of $\mathbb{C}^n$, then $\|\operatorname{ord}\|\mathcal{M}\geqslant 1$. Hamada and Kohr obtained the extension of this result to the unit ball of a complex Hilbert space in [12, Theorem 3.2] and to the unit polydisc in [13, Theorem 3.2]. In the following we obtain the extension of this result to the unit ball of a JB*-triple.

Theorem 3.9. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a JB*-triple X. Then $\|\operatorname{ord}\|_{X,2}\mathcal{F}\geqslant 1$.*

Proof. We will use an argument similar to that in the proof of [29, Theorem 3.1]. Let $\beta=\|\operatorname{ord}\|_{X,2}\mathcal{F}$ and let $x\in B\setminus\{0\}$ be fixed. Putting $y=\frac{x}{\|x\|}$, $x\in B\setminus\{0\}$, in (3.7), we obtain that

$$\beta\geqslant\left\|\frac{1}{2\|x\|^2}\Phi(f,x,x,x)-\frac{1}{\|x\|^2}\{x,x,x\}\right\|,$$

where

$$\Phi(f,x,x,x)=\left[Dg_x(0)\right]^{-1}\left[Df(x)\right]^{-1}D^2f(x)\left(Dg_x(0)x,Dg_x(0)x\right).$$

Therefore, we have

$$\beta\geqslant\left|\frac{1}{2\|x\|^2}z^*\left(\Phi(f,x,x,x)-2\{x,x,x\}\right)\right|, \tag{3.9}$$

where $z^*\in T(\{x,x,x\})$. Further, let

$$h(\zeta)=\frac{\zeta}{2\|x\|^2}z^*\left(\Psi(f,\zeta,x)\right),\quad |\zeta|<\frac{1}{\|x\|},$$

where

$$\Psi(f,\zeta,x)=\left[Dg_x(0)\right]^{-1}\left[Df(\zeta x)\right]^{-1}D^2f(\zeta x)\left(Dg_x(0)x,Dg_x(0)x\right).$$

Then h is a holomorphic function on $|\zeta|<1/\|x\|$ and by (2.7)

$$\Phi(f,\zeta x,\zeta x,\zeta x)=\zeta^2\Psi(f,\zeta,x),\quad |\zeta|=1.$$

Since $h(0)=0$, for every r with $r<1/\|x\|$, there exists a value of ζ with $|\zeta|=r$ such that $\operatorname{Re}h(\zeta)\leqslant 0$.

We now replace x by ζx and z^* by $\frac{\bar{\zeta}}{|\zeta|}z^*$ in (3.9), where $|\zeta|=1$ so that $\operatorname{Re}h(\zeta)\leqslant 0$. Then we deduce that

$$\beta\geqslant\left|h(\zeta)-\frac{\|x\|^3}{\|x\|^2}\right|\geqslant-\Re h(\zeta)+\|x\|\geqslant\|x\|,$$

because we have used the fact that $\operatorname{Re}h(\zeta)\leqslant 0$. Hence, $\beta\geqslant\|x\|$ for all $x\in B$. Therefore $\|\operatorname{ord}\|_{X,2}\mathcal{F}\geqslant 1$. This completes the proof. □

As a corollary of Proposition 3.6 and Theorem 3.9, we obtain the following result (cf. [29, Theorem 5.1]).

Corollary 3.10. *Let B be the unit ball of a JB*-triple X and let $\mathcal{K}(B)$ be the set of normalized convex mappings on B. Then $\|\operatorname{ord}\|_{X,2}\mathcal{K}(B)=1$.*

Next, we give a result on a lower bound for starlikeness. Hamada and Kohr [14] (cf. [17]) proved the following sufficient condition for starlikeness on the unit ball of a complex Banach space.

Proposition 3.11. *Let f be a locally biholomorphic mapping on the unit ball B of a complex Banach space with $f(0)=0$. If*

$$\left\|\left[Df(x)\right]^{-1}D^2f(x)(x,\cdot)\right\|\leqslant 1,\quad x\in B,$$

then f is a starlike mapping of order $1/2$ on B.

Using the above sufficient condition, we will prove the following theorem (cf. [29, Theorems 5.5 and 5.7]).

Theorem 3.12. *Let $\mathcal{F}$ be a linear-invariant family on the unit ball B of a JB*-triple X with $\|\operatorname{ord}\|_{X,1}\mathcal{F}=\alpha<\infty$. If $f\in\mathcal{F}$, then f is a starlike mapping of order $1/2$ on B_{r_s}, where $r_s\in(0,1)$ is the unique solution of the equation*

$$\frac{2r^2+2\alpha r}{(1-r^2)^2}=1.$$

Proof. From Lemma 3.8,

$$\left\|\left[Df(x)\right]^{-1}D^2f(x)\left(Dg_x(0)y,Dg_x(0)z\right)\right\|\leqslant 2\left\|Dg_x(0)\{y,x,z\}\right\|+2\alpha\left\|Dg_x(0)\right\|\cdot\|y\|\cdot\|z\|.$$

Also, we have $\|Dg_x(0)\|\leqslant 1$ and $\|[Dg_x(0)]^{-1}\|\leqslant 1/(1-\|x\|^2)$ from (2.5) and (2.6). Therefore, putting $y=[Dg_x(0)]^{-1}x$ and $z=[Dg_x(0)]^{-1}w$ with $\|w\|=1$ and using (2.1), we obtain that

$$\begin{aligned}\left\|\left[Df(x)\right]^{-1}D^2f(x)(x,w)\right\|&\leqslant 2\left\|Dg_x(0)\right\|\cdot\left\|\left[Dg_x(0)\right]^{-1}x\right\|\cdot\|x\|\cdot\left\|\left[Dg_x(0)\right]^{-1}w\right\|\\&\quad+2\alpha\left\|Dg_x(0)\right\|\cdot\left\|\left[Dg_x(0)\right]^{-1}x\right\|\cdot\left\|\left[Dg_x(0)\right]^{-1}w\right\|\\&\leqslant\frac{2r^2+2\alpha r}{(1-r^2)^2},\end{aligned}$$

where $r=\|x\|$. From Proposition 3.11, f is a starlike mapping of order $1/2$ on B_{r_s}. This completes the proof. □

Before to give the following result, we have to introduce some notations, as follows. This result relates the radius of univalence of a linear invariant family with the radius of nonvanishing of this family.

Let

$$r_0=r_0(\mathcal{F})=\sup\left\{r>0:\ f(x)\neq 0,\ 0<\|x\|<r,\ f\in\mathcal{F}\right\}$$

and let $r_1=r_1(\mathcal{F})$ denote the radius of univalence of the linear invariant family $\mathcal{F}$, i.e.

$$r_1=\sup\{r>0:\ f\text{ is univalent on }B_r,\ f\in\mathcal{F}\}.$$

Then, we obtain the following result. This result is a generalization of [30, Lemma 2.4], [29, Theorem 5.11], [12, Theorem 3.4] and [13, Theorem 3.5] to the unit ball of a JB*-triple. We remark that if $\|\operatorname{ord}\|_{X,1}\mathcal{F}=\alpha<\infty$, then $r_0>0$ from Theorem 3.12.

Theorem 3.13. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a JB*-triple X. Assume that $r_0(\mathcal{F})>0$. Then*

$$r_1=\frac{r_0}{1+\sqrt{1-r_0^2}}.$$

Proof. Let $f\in\mathcal{F}$ and $r\leqslant\frac{r_0}{1+\sqrt{1-r_0^2}}$. Also, let $y,z\in B_r$ with $y\neq z$. Let

$$F(w;x)=\left[Dg_x(0)\right]^{-1}\left[Df\left(g_x(0)\right)\right]^{-1}\left(f\left(g_x(w)\right)-f\left(g_x(0)\right)\right),\quad w,x\in B,\tag{3.10}$$

where g_x is the biholomorphic automorphism of B, given in (2.3). Clearly, $F(\cdot;x)\in\mathcal{F}$, for all $x\in B$, and if we set $x=y$ and $w=g_y^{-1}(z)$ in (3.10), we obtain that

$$F\left(g_y^{-1}(z);y\right)=\left[Dg_y(0)\right]^{-1}\left[Df(y)\right]^{-1}\left(f(z)-f(y)\right).\tag{3.11}$$

From (2.11), we obtain

$$1-\left\|g_{-y}(z)\right\|^2\geqslant\frac{(1-\|y\|^2)(1-\|z\|^2)}{(1+\|y\|\cdot\|z\|)^2}>\frac{(1-r^2)^2}{(1+r^2)^2}.$$

Therefore, we have

$$\left\|g_y^{-1}(z)\right\|=\left\|g_{-y}(z)\right\|<\frac{2r}{1+r^2}\leqslant r_0.$$

Since $g_y^{-1}(z)\neq 0$ for $y\neq z$, we have $F(g_y^{-1}(z);y)\neq 0$. Then, we conclude from (3.11) that $f(y)\neq f(z)$, that means f is univalent on B_r. Therefore, $r_1\geqslant\frac{r_0}{1+\sqrt{1-r_0^2}}$. Also, since $r_0>0$, we deduce that $r_1>0$.

In the second part of this proof, we will show that $r_1 \leqslant \frac{r_0}{1+\sqrt{1-r_0^2}}$. To this end, let $x \in B$ with $0 < \|x\| < \frac{2r_1}{1+r_1^2}$. Then there exists $a \in B$ such that $x = g_a(a)$ and $0 < \|a\| < r_1$ by (2.8). After short computations, we obtain the following relations

$$F(a;a) = \left[Dg_a(0)\right]^{-1}\left[Df(a)\right]^{-1}\left(f(x) - f(a)\right)$$

and

$$F(-a;a) = -\left[Dg_a(0)\right]^{-1}\left[Df(a)\right]^{-1} f(a),$$

where F is defined by (3.10). Therefore, we have

$$f(x) = Df(a)Dg_a(0)\left(F(a;a) - F(-a;a)\right).$$

Since $0 < \|a\| < r_1$, $F(a;a) \neq F(-a;a)$. Hence, $f(x) \neq 0$. This implies that $r_0 \geqslant \frac{2r_1}{1+r_1^2}$. This is equivalent to $r_1 \leqslant \frac{r_0}{1+\sqrt{1-r_0^2}}$. This completes the proof. □

Corollary 3.14. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a JB*-triple X. Assume that $r_0(\mathcal{F}) = 1$. Then $\mathcal{F}$ is a family of normalized univalent mappings on B.*

4. Finite dimensional case

In this section, we consider about the linear invariant families on the unit ball B of a finite dimensional JB*-triple X.

When the dimension of X is finite, we obtain the normality of a linear invariant family with finite norm-order. This is a generalization of [29, Theorem 3.2] to the unit ball of a finite dimensional JB*-triple.

Theorem 4.1. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a JB*-triple X. If $\|\operatorname{ord}\|_{X,2}\mathcal{F} = \beta < \infty$, then $\mathcal{F}$ is a locally uniformly bounded family. In particular, in the case of a finite dimensional JB*-triple X, $\mathcal{F}$ is a normal family.*

Proof. Since $\|\operatorname{ord}\|_{X,1}\mathcal{F} \leqslant 3\beta < \infty$, we obtain from Theorem 3.12 that, if $f \in \mathcal{F}$, then the mapping $\tilde{f}$ given by

$$\tilde{f}(x) = \frac{1}{r_s} f(r_s x)$$

is a normalized starlike mapping of order 1/2 on B. By [17, Theorem 3.1] (see also [4, Theorem 2.2], [10, Corollary 14], [9, Theorem 3.1]), we obtain that

$$\left\|\tilde{f}(x)\right\| \leqslant \frac{\|x\|}{1 - \|x\|}, \quad x \in B.$$

Therefore, in view of Cauchy's integral formula, we obtain that there exist $\delta_1 > 0$ and $M_1 > 0$ which is independent of $f \in \mathcal{F}$, such that $\|f\|$ and $\|Df\|$ are bounded by M_1 on B_{δ_1}. Now apply this result to the Koebe transform $F(y;x) = \Lambda_{g_x}(f)(y)$, where $\|x\| \leqslant \delta_1$. If we set $y = x$, then we obtain that

$$\left\|f\left(g_x(x)\right) - f(x)\right\| = \left\|Df(x)Dg_x(0)F(x;x)\right\|$$

and

$$\left\|Df\left(g_x(x)\right)\right\| = \left\|Df(x)Dg_x(0)DF(x;x)\left[Dg_x(x)\right]^{-1}\right\|.$$

Let $\delta_2 = 2\delta_1/(1+\delta_1^2) > \delta_1$. Since for any $y \in B_{\delta_2}$, there exists $x \in B_{\delta_1}$ such that $y = g_x(x)$ by (2.8), we get a uniform bound M_2 for $\|f\|$ and $\|Df\|$ on the ball of radius δ_2. We continue the above process and we obtain a sequence δ_j of the radius of the ball centered at zero on which we have a uniform bound M_j for $\|f\|$ and $\|Df\|$ over all $f \in \mathcal{F}$, where

$$\delta_{j+1} = \frac{2\delta_j}{1+\delta_j^2}.$$

The sequence δ_j is increasing and bounded above by 1. Therefore, it has a limit. If the limit is r, then we have $r = 2r/(1+r^2)$. Therefore, $r = 1$. This implies that for any $R \in (0,1)$, there exists a number j_0 such that $R < \delta_{j_0}$. Thus, f is uniformly bounded on B_R. This completes the proof. □

Next, we will prove the distortion theorem. This theorem is a generalization of [29, Theorem 4.1] and [13, Theorem 4.2] to the unit ball of a finite dimensional JB*-triple. For the distortion theorems of normalized convex mappings, see [2] and the references therein.

Theorem 4.2. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a finite dimensional JB*-triple X. If $\|\operatorname{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then*

$$\|Df(x)\| \leqslant \frac{(1+\|x\|)^{\alpha-1}}{(1-\|x\|)^{\alpha+1}}, \quad x \in B$$

for all $f \in \mathcal{F}$.

As a corollary, we obtain the following growth theorem as in the proof of [29, Theorem 4.2]. The following theorem is a generalization of [29, Theorem 4.2] and [13, Theorem 4.3] to the unit ball of a finite dimensional JB*-triple.

Theorem 4.3. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a finite dimensional JB*-triple X. If $\|\operatorname{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then*

$$\|f(x)\| \leqslant \frac{1}{2\alpha}\left\{\left(\frac{1+\|x\|}{1-\|x\|}\right)^{\alpha} - 1\right\}$$

for all $f \in \mathcal{F}$ and $x \in B$.

We give a few remarks and lemmas in preparation for the proof of Theorem 4.2. Now, assume that $\mathcal{F}$ is a linear invariant family which satisfies the assumptions of Theorem 4.2. Since $\mathcal{F}$ is a normal family from Theorem 4.1, $\operatorname{cl}(\mathcal{F})$ is also a linear invariant family such that $\|\operatorname{ord}\|_{X,1}\operatorname{cl}(\mathcal{F}) = \alpha$. Thus, we may assume that $\mathcal{F}$ is compact in Theorems 4.2 and 4.3. For a point $x_0 = r_0u_0$, where u_0 is a complex extreme point of $\overline{B}$ and $0 < r_0 < 1$, we consider the extremal problem $\sup\{\|Dh(x_0)\|\colon h \in \mathcal{F}\}$, and let $F \in \mathcal{F}$ be an extremal mapping as follows:

$$\|DF(x_0)\| = \sup\{\|Dh(x_0)\|\colon h \in \mathcal{F}\}. \tag{4.1}$$

Then the definition of the operator norm insures the existence of a point y_0 with $\|y_0\| = 1$ such that $\|DF(x_0)\| = \|DF(x_0)y_0\|$. Clearly the extremal mapping also satisfies $\|DF(x_0)y_0\| \geqslant \|Dh(x_0)y\|$ for all $h \in \mathcal{F}$, and all points $y \in \overline{B}$.

Lemma 4.4. *Let F be an extremal mapping defined by (4.1). Then it also has the extremal property*

$$\|DF(x_0)\| = \max\{\|DF(e^{it}x_0)\|\colon 0 \leqslant t \leqslant 2\pi\}.$$

Proof. If this were false, then there would be a point $x_1 = e^{it}x_0$ with $0 < t < 2\pi$ such that $\|DF(x_1)\| > \|DF(x_0)\|$. Let $F_t(x) = e^{-it}F(e^{it}x)$ that must belong to the linear invariant family $\mathcal{F}$. Then $DF_t(x) = DF(e^{it}x)$ which yields the contradiction $\|DF_t(x_0)\| > \|DF(x_0)\|$. □

We can prove the following *rotation lemma* by an argument similar to that in the proof of Pfaltzgraff and Suffridge [29, Lemma 4.4].

Lemma 4.5. *Let F be an extremal mapping defined by (4.1). Then we obtain that $(DF(x_0)y_0)^*(D^2F(x_0)(x_0, y_0))$ is real.*

Proof. For t near 0,

$$DF(e^{it}x_0)y_0 = DF(x_0 + (e^{it}-1)x_0)y_0 = DF(x_0)y_0 + itD^2F(x_0)(x_0, y_0) + O(t^2).$$

Therefore,

$$0 \geqslant \operatorname{Re}(DF(x_0)y_0)^*(DF(e^{it}x_0)y_0 - DF(x_0)y_0) = \operatorname{Re}(DF(x_0)y_0)^*(itD^2F(x_0)(x_0, y_0)) + O(t^2).$$

Upon dividing by $|t|$ and considering both $t \to 0+$ and $t \to 0-$, we obtain that

$$\operatorname{Im}(DF(x_0)y_0)^*(D^2F(x_0)(x_0, y_0)) = 0.$$

This completes the proof. □

For r near r_0 we have the following formula.

Lemma 4.6. *Let F be an extremal mapping defined by (4.1). Then we obtain that*

$$\begin{aligned}\frac{\operatorname{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0 - DF(x_0)y_0)}{r - r_0} &= \frac{1}{1-r_0^2}\{\operatorname{Re}(DF(x_0)y_0)^*(D^2F(0)(u_0, DF(x_0)(y_0)))\} \\ &\quad + \frac{r_0}{1-r_0^2}\operatorname{Re}(DF(x_0)y_0)^*(DF(x_0)[\{y_0, u_0, u_0\} + \{u_0, u_0, y_0\}]) \\ &\quad + O(r - r_0), \quad r \to r_0.\end{aligned}$$

Proof. Let $z=\alpha x_0/\|x_0\|^2$, where $\alpha\in\mathbb{C}\setminus\{0\}$. For small $\|z\|$, the Koebe transform $F(x;z)$ can be viewed as a perturbation of the extremal $F(x)$. By Lemma 3.7, we have

$$\begin{aligned} F(x;z) &= Dg_z(0)^{-1}DF\big(g_z(0)\big)^{-1}\big[F\big(g_z(x)\big)-F\big(g_z(0)\big)\big] \\ &= F(x)+DF(x)\big(z-\{x,z,x\}\big)-z-D^2F(0)\big(z,F(x)\big)+O\big(\|z\|^2\big). \end{aligned}$$

Then we have

$$\begin{aligned} DF(x;z)(\cdot) &= DF(x)(\cdot)+D^2F(x)\big(z-\{x,z,x\},\cdot\big) \\ &\quad - DF(x)\big\{(\cdot),z,x\big\}-DF(x)\big\{x,z,(\cdot)\big\}-D^2F(0)\big(z,DF(x)(\cdot)\big)+O\big(\|z\|^2\big). \end{aligned}$$

Evaluating at x_0 and y_0, we find that

$$\begin{aligned} &\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(DF(x_0;z)(y_0)-DF(x_0)(y_0)\big) \\ &\quad=\operatorname{Re}\big\{\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)(z,y_0)\big)\big\}-\operatorname{Re}\big\{\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)\big(\{x_0,z,x_0\},y_0\big)\big)\big\} \\ &\qquad-\operatorname{Re}\big\{\big(DF(x_0)y_0\big)^*\big(DF(x_0)\big[\{y_0,z,x_0\}+\{x_0,z,y_0\}\big]\big)\big\}-\operatorname{Re}\big\{\big(DF(x_0)y_0\big)^*\big(D^2F(0)\big(z,DF(x_0)(y_0)\big)\big)\big\} \\ &\qquad+O\big(\|z\|^2\big). \end{aligned} \tag{4.2}$$

Since $\{u_0,u_0,u_0\}=u_0$ by Proposition 2.3, we have $\{x_0,z,x_0\}=\bar\alpha x_0$. Therefore, we obtain

$$\begin{aligned} \operatorname{Re}\big\{\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)\big(\{x_0,z,x_0\},y_0\big)\big)\big\} &= \operatorname{Re}\big\{\bar\alpha\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)(x_0,y_0)\big)\big\} \\ &= \operatorname{Re}\big\{\alpha\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)(x_0,y_0)\big)\big\}, \end{aligned}$$

where the last equality follows from Lemma 4.5. We will divide (4.2) by $\|z\|$ and deduce a necessary condition as it tends to zero. Since

$$\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(DF(x_0;z)(y_0)-DF(x_0)(y_0)\big)\leqslant 0,$$

we obtain

$$\begin{aligned} 0 &\leqslant -\big(1-r_0^2\big)\operatorname{Re}\left\{\big(DF(x_0)y_0\big)^*\left(D^2F(x_0)\left(\frac{z}{\|z\|},y_0\right)\right)\right\} \\ &\quad+\operatorname{Re}\left\{\big(DF(x_0)y_0\big)^*\left(DF(x_0)\left[\left\{y_0,\frac{z}{\|z\|},x_0\right\}+\left\{x_0,\frac{z}{\|z\|},y_0\right\}\right]\right)\right\} \\ &\quad+\operatorname{Re}\left\{\big(DF(x_0)y_0\big)^*\left(D^2F(0)\left(\frac{z}{\|z\|},DF(x_0)(y_0)\right)\right)\right\}+O\big(\|z\|\big). \end{aligned}$$

We let $\|z\|\to 0$, noting that z can be replaced by $e^{it}z$ with arbitrary real t. Then we conclude that

$$\begin{aligned} 0 &= -\big(1-r_0^2\big)\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)(u_0,y_0)\big) \\ &\quad+\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(DF(x_0)\big[\{y_0,u_0,x_0\}+\{x_0,u_0,y_0\}\big]\big) \\ &\quad+\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(0)\big(u_0,DF(x_0)(y_0)\big)\big), \end{aligned}$$

where $u_0=x_0/r_0$, $r_0=\|x_0\|$. Then we have

$$\begin{aligned} \big(1-r_0^2\big)\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)(u_0,y_0)\big) &= r_0\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(DF(x_0)\big[\{y_0,u_0,u_0\}+\{u_0,u_0,y_0\}\big]\big) \\ &\quad+\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(0)\big(u_0,DF(x_0)(y_0)\big)\big). \end{aligned}$$

By the usual expansion computation for r near r_0 and Lemma 4.5, we have the formula

$$\frac{\operatorname{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0-DF(x_0)y_0)}{r-r_0}=\operatorname{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(x_0)(u_0,y_0)\big)+O(r-r_0).$$

Then the result follows by applying the preceding step. □

Proof of Theorem 4.2. Since the dimension of X is finite, $\mathcal{E}\neq\emptyset$. It suffices to prove that for fixed $u_0\in\mathcal{E}$,

$$\big\|Df(ru_0)\big\|\leqslant\frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}} \tag{4.3}$$

for all $r\in(0,1)$.

Throughout the proof, F will be an extremal mapping for the maximum problem associated with the points $x_0 = r_0 u_0$ and y_0, i.e., $\|DF(x_0)y_0\| = \|DF(x_0)\| \geqslant \|Dh(x_0)\|$ for all $h \in \mathcal{F}$. We now assume that (4.3) is false. Hence there exist $r_0 \in (0,1)$, a point y_0 with $\|y_0\| = 1$ and $F \in \mathcal{F}$ such that

$$\sup_{h\in\mathcal{F}}\|Dh(x_0)\| = \|DF(x_0)\| = \|DF(x_0)y_0\| > \frac{(1+r_0)^{\alpha-1}}{(1-r_0)^{\alpha+1}}.$$

Since the function

$$\psi(t,s) = \frac{(1+t)^{s-1}}{(1-t)^{s+1}} = \left(\frac{1+t}{1-t}\right)^s \frac{1}{1-t^2}, \quad 0 \leqslant t < 1,$$

is an increasing function of s such that $\lim_{s\to\infty}\psi(t,s) = \infty$ for $0 < t < 1$, there exists a number $\beta_0 > \alpha$ such that $\|DF(x_0)y_0\| = \psi(r_0, \beta_0)$. Consequently, for small $\varepsilon > 0$ and $r_0 - \varepsilon < r < r_0 + \varepsilon$ there is a C^1 smooth function $\beta(r)$ such that $\beta(r_0) = \beta_0$ and

$$\mathrm{Re}\big(DF(x_0)y_0\big)^*\big(DF(ru_0)y_0\big) = \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} = \psi\big(r, \beta(r)\big).$$

Thus

$$\frac{\mathrm{Re}(DF(x_0)y_0)^*(DF(ru_0)y_0 - DF(x_0)y_0)}{r-r_0} = \frac{\psi(r,\beta(r)) - \psi(r_0,\beta_0)}{r-r_0}.$$

From Lemma 4.6 and this equality, we have

$$\begin{aligned}\frac{\psi(r,\beta(r)) - \psi(r_0,\beta_0)}{r-r_0} &= \frac{1}{1-r_0^2}\mathrm{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(0)\big(u_0, DF(x_0)(y_0)\big)\big) \\ &\quad + \frac{r_0}{1-r_0^2}\mathrm{Re}\big(DF(x_0)y_0\big)^*\big(DF(x_0)\big[\{y_0,u_0,u_0\} + \{u_0,u_0,y_0\}\big]\big) \\ &\quad + O(r-r_0), \quad r \to r_0.\end{aligned}$$

As $r \to r_0$, the limit of the left-hand side of this formula is

$$\big(\psi_r + \beta'\psi_\beta\big)_{r=r_0} = \psi(r_0,\beta_0)\left\{\frac{2(\beta_0+r_0)}{1-r_0^2} + \beta'(r_0)\log\left(\frac{1+r_0}{1-r_0}\right)\right\}.$$

Noting that $\psi(r_0,\beta_0) = \|DF(x_0)\|$, we deduce that

$$\begin{aligned}\|DF(x_0)\|\left\{\frac{2(\beta_0+r_0)}{1-r_0^2} + \beta'(r_0)\log\left(\frac{1+r_0}{1-r_0}\right)\right\} &= \frac{1}{1-r_0^2}\mathrm{Re}\big(DF(x_0)y_0\big)^*\big(D^2F(0)\big(u_0, DF(x_0)(y_0)\big)\big) \\ &\quad + \frac{r_0}{1-r_0^2}\mathrm{Re}\big(DF(x_0)y_0\big)^*\big(DF(x_0)\big[\{y_0,u_0,u_0\} + \{u_0,u_0,y_0\}\big]\big) \\ &\leqslant \|DF(x_0)\|\frac{2(r_0+\alpha)}{1-r_0^2}.\end{aligned}$$

Our assumption that $\beta_0 > \alpha$ now implies that $\beta'(r_0) < 0$. Hence there exists an r with $0 < r < r_0$ such that $\beta(r) > \beta(r_0) = \beta_0 > \alpha$ and that

$$\mathrm{Re}\big(DF(x_0)y_0\big)^*\big(DF(ru_0)y_0\big) = \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}.$$

Let F_1 be an extremal such that

$$\|DF_1(ru_0)\| = \sup\big\{\|Dh(ru_0)\|\colon h \in \mathcal{F}\big\}.$$

We then have

$$\|DF_1(ru_0)\| \geqslant \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}.$$

Let

$$r_\infty = \inf\left\{0 < r < r_0\colon \sup_{h\in\mathcal{F}}\|Dh(ru_0)\| > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}\right\}.$$

Then there are a sequence $\{y_\nu\}$ with $\|y_\nu\| = 1$, a sequence of numbers $\{r_\nu\}$ decreasing to r_∞ and a sequence $\{F_\nu\} \subset \mathcal{F}$ such that

$$\big\|DF_\nu(x_\nu)\big\| = \big\|DF_\nu(x_\nu)y_\nu\big\| > \frac{(1+r_\nu)^{\beta_0-1}}{(1-r_\nu)^{\beta_0+1}}, \tag{4.4}$$

where $x_\nu = r_\nu u_0$. We may assume that y_ν tends to y with $\|y\| = 1$. Since $\mathcal{F}$ is compact, there exists a subsequence of $\{F_\nu\}$ that converges locally uniformly to a mapping $f \in \mathcal{F}$. If $r_\infty > 0$, then

$$\big\|Df(r_\infty u_0)y\big\| \geqslant \frac{(1+r_\infty)^{\beta_0-1}}{(1-r_\infty)^{\beta_0+1}}.$$

Hence we can repeat the argument just given and deduce that there exists an r with $0 < r < r_\infty$ such that $\beta(r) > \beta_0 > \alpha$ and

$$\sup_{h\in\mathcal{F}}\big\|Dh(ru_0)\big\| = \frac{(1+r)^{\beta(r)-1}}{(1-r)^{\beta(r)+1}} > \frac{(1+r)^{\beta_0-1}}{(1-r)^{\beta_0+1}}.$$

This contradicts with the definition of r_∞. So, $r_\infty = 0$.

From (4.4) we have

$$\big\|DF_\nu(x_\nu)y_\nu\big\| > 1 + 2\beta_0 r_\nu + O\big(r_\nu^2\big).$$

Equivalently one has the inequality

$$\begin{aligned}\big\|D^2F_\nu(0)(r_\nu u_0, y_\nu) + O\big(r_\nu^2\big)\big\| &= \big\|DF_\nu(x_\nu)y_\nu - y_\nu\big\| \\ &> 1 + 2\beta_0 r_\nu - \|y_\nu\| + O\big(r_\nu^2\big) \\ &\geqslant 2\beta_0 r_\nu + O\big(r_\nu^2\big).\end{aligned}$$

Thus dividing by r_ν and letting $\nu \to \infty$ through the subsequence we obtain

$$\big\|D^2f(0)(u_0, y)\big\| \geqslant 2\beta_0 > 2\alpha = 2\|\operatorname{ord}\|_{X,1}\mathcal{F}.$$

This contradiction completes the proof. □

Using Theorem 4.3 and arguments similar to those in the proof of [2, Lemma 2.7], we obtain the following result.

Theorem 4.7. *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a finite dimensional JB*-triple X. If $\|\operatorname{ord}\|_{X,1}\mathcal{F} = \alpha < \infty$, then for $x, y \in B$, we have*

$$\big\|f(x) - f(y)\big\| \leqslant \frac{1}{2\alpha}\big(\exp\big(2\alpha C_B(x,y)\big) - 1\big)\min\big\{\big\|Df(x)Dg_x(0)\big\|, \big\|Df(y)Dg_y(0)\big\|\big\}.$$

Proof. Fix $x, y \in B$ and define $F : B \to X$ by

$$F(z) = \big[D\phi(0)\big]^{-1}\big[Df(y)\big]^{-1}\big(f\big(\phi(z)\big) - f(y)\big) \quad (z \in B),$$

where $\phi = g_y$. Then $F \in \mathcal{F}$. In view of Theorem 4.3, we have

$$\big\|F(z)\big\| \leqslant \frac{1}{2\alpha}\left\{\left(\frac{1+\|z\|}{1-\|z\|}\right)^{\alpha} - 1\right\} = \frac{1}{2\alpha}\big\{\exp\big(2\alpha C_B(z,0)\big) - 1\big\}$$

for all $z \in B$. It follows that, for $z = g_y^{-1}(x)$,

$$\begin{aligned}\big\|f(x) - f(y)\big\| &= \big\|Df(y)Dg_y(0)F(z)\big\| \\ &\leqslant \big\|Df(y)Dg_y(0)\big\| \cdot \big\|F(z)\big\| \\ &\leqslant \big\|Df(y)Dg_y(0)\big\| \cdot \frac{1}{2\alpha}\big\{\exp\big(2\alpha C_B(z,0)\big) - 1\big\} \\ &= \big\|Df(y)Dg_y(0)\big\| \cdot \frac{1}{2\alpha}\big\{\exp\big(2\alpha C_B(x,y)\big) - 1\big\}.\end{aligned}$$

Changing the roles of x and y, one deduces the desired result. □

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References

[1] R.W. Barnard, C.H. FitzGerald, S. Gong, A distortion theorem for biholomorphic mappings in $\mathbb{C}^2$, Trans. Amer. Math. Soc. 344 (1994) 907–924.
[2] C.-H. Chu, H. Hamada, T. Honda, G. Kohr, Distortion theorems for convex mappings on homogeneous balls, J. Math. Anal. Appl. 369 (2010) 437–442.
[3] S. Gong, Convex and Starlike Mappings in Several Complex Variables, Math. Appl. (China Ser.), vol. 435, Kluwer Acad. Publ./Science Press, Dordrecht/Beijing, 1998.
[4] I. Graham, H. Hamada, G. Kohr, Parametric representation of univalent mappings in several complex variables, Canad. J. Math. 54 (2002) 324–351.
[5] I. Graham, H. Hamada, G. Kohr, T.J. Suffridge, Extension operators for locally univalent mappings, Michigan Math. J. 50 (2002) 37–55.
[6] I. Graham, G. Kohr, Geometric Function Theory in One and Higher Dimensions, Monogr. Textb. Pure Appl. Math., vol. 255, Marcel Dekker, Inc., New York, 2003.
[7] J. Godula, P. Liczberski, V. Starkov, Order of linearly invariant family of mappings in $\mathbb{C}^n$, Complex Var. Theory Appl. 42 (2000) 89–96.
[8] D.J. Hallenbeck, T.H. MacGregor, Linear Problems and Convexity Techniques in Geometric Function Theory, Pitman, Boston, 1984.
[9] H. Hamada, T. Honda, Sharp growth theorems and coefficient bounds for starlike mappings in several complex variables, Chin. Ann. Math. Ser. B 29 (2008) 353–368.
[10] H. Hamada, T. Honda, G. Kohr, Growth theorems and coefficient bounds for univalent holomorphic mappings which have parametric representation, J. Math. Anal. Appl. 317 (2006) 302–319.
[11] H. Hamada, T. Honda, G. Kohr, Bohr's theorem for holomorphic mappings with values in homogeneous balls, Israel J. Math. 173 (2009) 177–187.
[12] H. Hamada, G. Kohr, Linear invariance of locally biholomorphic mappings in Hilbert spaces, Complex Var. Theory Appl. 47 (2002) 277–289.
[13] H. Hamada, G. Kohr, Linear invariant families on the unit polydisc, Mathematica (Cluj) 44 (67) (2002) 153–170.
[14] H. Hamada, G. Kohr, Simple criterions for strongly starlikeness and starlikeness of certain order, Math. Nachr. 254–255 (2003) 165–171.
[15] H. Hamada, G. Kohr, Order of linear invariant families on the ball and polydisc of $\mathbb{C}^n$, Rev. Roumaine Math. Pures Appl. 48 (2003) 143–151.
[16] H. Hamada, G. Kohr, Roper–Suffridge extension operator and the lower bound for the distortion, J. Math. Anal. Appl. 300 (2004) 454–463.
[17] H. Hamada, G. Kohr, P. Liczberski, Starlike mappings of order α on the unit ball in complex Banach spaces, Glas. Mat. Ser. III 36 (56) (2001) 39–48.
[18] L.A. Harris, Bounded symmetric homogeneous domains in infinite dimensional spaces, in: T.L. Hayden, T.J. Suffridge (Eds.), Proceedings on Infinite Dimensional Holomorphy, Internat. Conf., Univ. Kentucky, Lexington, KY, 1973, in: Lecture Notes in Math., vol. 364, Springer, Berlin, 1974, pp. 13–40.
[19] L. Hörmander, On a theorem of Grace, Math. Scand. 2 (1954) 55–64.
[20] L.K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains, Transl. Math. Monogr., vol. 6, American Mathematical Society, Providence, RI, 1963.
[21] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in Banach spaces, Math. Z. 183 (1983) 503–529.
[22] W. Kaup, Hermitian Jordan triple systems and the automorphisms of bounded symmetric domains, in: Non-associative Algebra and Its Applications, Oviedo, 1993, in: Math. Appl., vol. 303, Kluwer Acad. Publ., Dordrecht, 1994, pp. 204–214.
[23] W. Kaup, H. Upmeier, Jordan algebras and symmetric Siegel domains in complex Banach spaces, Math. Z. 157 (1977) 179–200.
[24] P. Liczberski, V. Starkov, Regularity theorems for linearly invariant families of holomorphic mappings in $\mathbb{C}^n$, Ann. Univ. Mariae Curie-Sklodowska Sect. A 54 (2000) 61–73.
[25] P. Mellon, Holomorphic invariance on bounded symmetric domains, J. Reine Angew. Math. 523 (2000) 199–223.
[26] J.A. Pfaltzgraff, Distortion of locally biholomorphic maps of the n-ball, Complex Var. Theory Appl. 33 (1997) 239–253.
[27] J.A. Pfaltzgraff, T.J. Suffridge, Linear invariance, order and convex maps in $\mathbb{C}^n$, Complex Var. Theory Appl. 40 (1999) 35–50.
[28] J.A. Pfaltzgraff, T.J. Suffridge, An extension theorem and linear invariant families generated by starlike maps, Ann. Univ. Mariae Curie-Sklodowska Sect. A 53 (1999) 193–207.
[29] J.A. Pfaltzgraff, T.J. Suffridge, Norm order and geometric properties of holomorphic mappings in $\mathbb{C}^n$, J. Anal. Math. 82 (2000) 285–313.
[30] C. Pommerenke, Linear-invariante familien analytischer funktionen I, Math. Ann. 155 (1964) 108–154.
[31] C. Pommerenke, Linear-invariante familien analytischer funktionen II, Math. Ann. 156 (1964) 226–262.
[32] T.J. Suffridge, Starlike and convex maps in Banach spaces, Pacific J. Math. 46 (1973) 575–589.