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A COEFFICIENT ESTIMATE FOR MULTIVALENT FUNCTIONS

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ABSTRACT. By making use of extreme point theory we obtain bounds on the coefficients of a class of functions, multivalent in the unit disk, closely related to the bounds conjectured by Goodman.

I. **Introduction.** Let $S(p,q)$, p and q integers, $1 \leqslant q \leqslant p$, be the class of functions $f(z)$ analytic in $\Delta = \{z\colon |z| < 1\}$ with power series expansion $f(z) = \sum_{n=q+1}^{\infty} a_n z^n$, $z \in \Delta$, and for which there exists a $\rho = \rho(f)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{1.1}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) d\theta = 2p\pi \quad \text{for } z = re^{i\theta},\ \rho < r < 1. \tag{1.2}$$

Functions in $S(p,q)$ are p-valent and are referred to as multivalent starlike functions [2]. We let $S_1(p,q)$ be the subclass of $S(p,q)$ of functions which are analytic on $|z| = 1$ and satisfy (1.1) and (1.2) on $|z| = 1$.

We define the class $K(p,q)$ [5] to be the class of functions $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$, $z \in \Delta$, for which there exists $g(z)$ in some $S(p,t)$, an α, $0 \leqslant \alpha \leqslant 2\pi$, and a ρ, $0 < \rho < 1$, such that

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for } \rho < |z| < 1. \tag{1.3}$$

Functions in $K(p,q)$ are called multivalent close-to-convex functions. We let $K_1(p,q)$ be the subclass of functions $f(z)$ in $K(p,q)$ which are analytic on $|z| = 1$ and for which there exists $g(z)$ in some $S_1(p,t)$ and an α, $0 \leqslant \alpha \leqslant 2\pi$, such that (1.3) holds on $|z| = 1$. It is known [5] that if $f(z)$ is in $K(p,q)$, $zf'(z)$ has exactly p zeros in Δ. We thus divide $K(p,q)$ into subclasses according to the location of the nonzero zeros of $zf'(z)$. Let α_i, $i = 1,2,\ldots,p-q$, be arbitrary complex numbers satisfying $0 < |\alpha_i| < 1$, $i = 1,2,\ldots,p-q$, and define $K(p,q,\alpha_1,\alpha_2,\ldots,\alpha_{p-q})$ to be the class of functions $f(z)$ such that $f(z)$ is in $K(p,q)$ and $zf'(z) = 0$ for $z = \alpha_i$, $i = 1,2,\ldots,p-q$. Furthermore we let $K_1(p,q,\alpha_1,\ldots,\alpha_{p-q})$ be the subclass of functions in

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$K(p,q,\alpha_1,\alpha_2,\ldots,\alpha_{p-q})$ which are also in $K_1(p,q)$.

A class related to $K(p,q,\alpha_1,\ldots,\alpha_{p-q})$ is the class $\hat{K}(p,q,\alpha_1,\ldots,\alpha_{p-q})$ defined by the following: $f(z)$ is in $\hat{K}(p,q,\alpha_1,\ldots,\alpha_{p-q})$ if $f(z)$ is analytic in Δ, $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ for z in Δ and $zf'(z)$ has the representation

$$zf'(z) = \frac{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i)}\, p(z)h(z) \tag{1.4}$$

where $h(z)$ is in $S(p,p)$ and $p(z)$ satisfies $p(0) = q$ and there exists $\delta > 0$ such that $\operatorname{Re} e^{i\delta}p(z) > 0$ for z in Δ.

It has been conjectured by Goodman [1] that for a function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, which is analytic and p-valent in Δ, the coefficients satisfy the inequalities

$$|a_n| \leqslant \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!\,(p-k)!\,(n-p-1)!\,(n^2-k^2)}\, |a_k|$$

for $n \geqslant p+1$. These inequalities are known to be true for $K(p,p)$ and $K(p,p-1)$ [6] but not for $K(p,q)$ if $q < p-1$. The situation is similar for the classes $S(p,q)$ except that the inequalities are known to be true for functions in $S(p,q)$, $1 \leqslant q \leqslant p$, whose power series have real coefficients [3]. In §III of this paper we show that by making use of linear methods, inequalities closely related to those conjectured by Goodman [1] can be proven for the class $K(p,q,\alpha_1,\ldots,\alpha_{p-q})$.

In [4], by making use of extreme point theory, we obtained results concerning $K(p,p)$. In particular, we obtained the sharp bounds on the coefficients of any function majorized by a function in $K(p,p)$. We also found the sharp bounds on the coefficients of any function majorized by a function in $S(p,p)$. We remark that this same result can be proven for the class $K(p,q)$ but again as in [4] we do not give the simple proof.

II. Classes of functions related to $K(p,q)$.

THEOREM 1. $K(p,q,\alpha_1,\ldots,\alpha_{p-q}) \subset \hat{K}(p,q,\alpha_1,\ldots,\alpha_{p-q})$.

PROOF. Suppose first that $f(z)$ is in $K_1(p,q,\alpha_1,\ldots,\alpha_{p-q})$; then [5] there exists $h(z)$ in $S_1(p,p)$ and a β such that $\operatorname{Re}[e^{i\beta}zf'(z)/h(z)] > 0$ on $|z| = 1$. As in [5] we easily see that

$$g(z) = \frac{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i)}\, h(z) \quad \text{is in } S_1(p,q).$$

Let $\delta = \beta - \arg\prod_{i=1}^{p-q}(-\alpha_i)$; then $\operatorname{Re} e^{i\delta}[zf'(z)/g(z)] > 0$ for $|z| = 1$. But since $zf'(z)/g(z)$ is analytic for $|z| \leqslant 1$ we have $\operatorname{Re}(e^{i\delta}zf'(z)/g(z)) > 0$ for $|z| \leqslant 1$. Letting $p(z) = zf'(z)/g(z)$ we obtain (1.1) for $zf'(z)$.

Next, if $f(z)$ is analytic only for $|z| < 1$, there exists a ρ, $0 < \rho < 1$, such that $g_r(z) = r^{-q}f(rz)$ is in $K_1(p,q)$ for $\rho < r < 1$. Since $g_r'(z) = 0$ for $z = \alpha_i/r$, $i = 1,2,\ldots,p-q$, we have

$$zg_r'(z) = \frac{\prod_{i=1}^{p-q}(z-\alpha_i/r)(1-\bar{\alpha}_i z/r)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i/r)}\, p_r(z)h_r(z) \tag{1.5}$$

where $h_r(z)$ is in $S_1(p,q)$, $p_r(0) = q$ and there exists δ_r such that $\operatorname{Re}(e^{i\delta_r}p_r(z)) > 0$. Using the fact that the families of functions $e^{i\delta_r}p_r(z)$ and $h_r(z)$ belong to normal and compact families, we easily obtain (1.1) from (1.5) upon passing to the limit.

We define the class $\widehat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ to be the class of functions $f(z)$, analytic in Δ, such that $zf'(z)$ has the representation

$$(1.6) \qquad zf'(z) = \frac{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\overline{\alpha}_i z)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i)}\, p(z) z^p \prod_{s=1}^{2p}(1-e^{-i\theta_s}z)^{-1}$$

where α_i, $0 < |\alpha_i| < 1$, are fixed for $i = 1,2, \ldots, p-q$; $p(0) = q$ and there exists δ so that $\operatorname{Re}(e^{i\delta}p(z)) > 0$ for z in Δ and $0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_{2p} < 2\pi$.

The class $\widehat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is defined to be the class of functions $f(z)$ such that $zf'(z)$ has the representation (1.6) except that no requirement that the θ_i all be different is made. The following lemma is then obvious.

LEMMA 1. *$\widehat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is dense in $\widehat{\widehat{C}}(p,q,\alpha_1, \ldots, \alpha_{p-q})$.*

We let $D(p,q)$ be the class of functions $f(z)$, analytic in Δ, with $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ for z in Δ such that $zf'(z)$ is analytic on $|z| = 1$ with the exception of $2p$ simple poles on $|z| = 1$ and such that there exists a $\beta = \beta(f)$ so that $\operatorname{Im} e^{i\beta}(e^{-i\beta/q}zf'(e^{-i\beta/q}z))$ changes sign exactly $2p$ times on $|z| = 1$ [i.e., at the poles of $zf'(z)$].

THEOREM 2. *Every function in $\widehat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is the uniform limit in compacta of functions in $D(p,q)$.*

PROOF. Let $f(z)$ be in $\widehat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ and suppose that in the representation (1.6) the function $p(z)$ is analytic on $|z| = 1$ and $\operatorname{Re} e^{i\delta}p(z) > 0$ for some δ on $|z| = 1$. We can write

$$(1.7) \qquad e^{i\delta}p(z) = \frac{z^{p-q}e^{i\delta}\prod_{i=1}^{p-q}(-\alpha_i)}{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\overline{\alpha}_i z)}\, \frac{\prod_{s=1}^{2p}(1-e^{-i\theta_s}z)}{z^p}\, zf'(z).$$

It is easily seen that

$$\arg\left[z^{p-q}e^{i\delta}\prod_{i=1}^{p-q}(-\alpha_i) \Big/ \prod_{i=1}^{p-q}(z-\alpha_i)(1-\overline{\alpha}_i z)\right]$$

is constant on $|z| = 1$ and that $\arg \prod_{s=1}^{2p}(1-e^{-i\theta_s}z)/z^p$ is constant for z on the arc between $e^{i\theta_j}$ and $e^{i\theta_{j+1}}$ and that the argument changes by π as z goes from the arc between $e^{i\theta_j}$ to $e^{i\theta_{j+1}}$ to the arc between $e^{i\theta_{j+1}}$ to $e^{i\theta_{j+2}}$. It follows then that there exists a line through the origin such that $zf'(z)$ lies on one side of the line for z on the arc between $e^{i\theta_j}$ and $e^{i\theta_{j+1}}$ and lies on the other side of the line for z on the arc from $e^{i\theta_{j+1}}$ to $e^{i\theta_{j+2}}$. Thus there exists a $\beta = \beta(f)$ such that $\operatorname{Im} e^{i\beta}(e^{-i\beta/q}zf'(e^{-i\beta/q}z))$ changes sign exactly $2p$ times on $|z| = 1$.

If in the representation (1.6), $p(z)$ is not analytic on $|z| = 1$, then for each $r < 1$ we let $g_r(z)$ be given by

$$(1.8) \qquad zg_r'(z) = \frac{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\overline{\alpha}_i z)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i)}\, p(rz) z^p \prod_{s=1}^{2p}(1-e^{-i\theta_s}z)^{-1}.$$

By the first part of the proof, $g_r(z)$ is in $D(p,q)$. By taking a sequence r_n converging to 1 we have $g_{r_n}(z)$ converges uniformly to $f(z)$ in compacta in Δ.

III. Coefficient problem.

THEOREM 3. *Let* $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$, *z in* Δ, *be in* $D(p,q)$; *then for* $n \geqslant p + 1$,

$$|a_n| \leqslant \sum_{k=q}^{p} \frac{2k(n+p)!}{(p+k)!\,(p-k)!\,(n-p-1)!\,(n^2-k^2)} |a_k| \tag{2.1}$$

where $a_q = 1$.

PROOF. We may obviously assume that $\beta = 0$. With this assumption the proof follows the proof of Theorem 1 in [7] with $zf'(z)$ identified with the function in the statement of the theorem. We notice first that Lemma 1 of [7] goes through with $f(z)$ in the statement of that lemma identified with $zf'(z)$ where $f(z)$ is in $D(p,q)$. It is important to note that in the special case $p = q = 1$, the quantities θ_i and θ_j in the proof of the lemma are such that $zf'(z)$ has its two simple poles on $|z| = 1$ at $e^{i\theta_i}$ and $e^{i\theta_j}$ and thus the two simple poles of $e^{i\mu}zf'(e^{i\mu}z)$ are at $e^{i\nu}$ and $e^{-i\nu}$. Thus $(z + z^{-1} - 2\cos\nu)$ $e^{i\mu}zf'(e^{i\mu}z)$ is analytic on $|z| = 1$. The proof of (1.12) on p. 411 in [7] can now be carried out. Let $g(z)$ be defined as on p. 411 where we identify $\phi(z)$ with $zf'(z)$; then, as noted above, $g(z)$ is analytic on $|z| = 1$. This is sufficient to carry out the proof. The proof starting on p. 413 can then be followed exactly giving us inequalities on the coefficients of $zf'(z)$ which in turn give us (2.1).

REMARK 1. There are two minor misprints on p. 415 of [7]. The summation in (4.19) should be $\sum_{s=1}^{p-2}$ instead of $\sum_{s=1}^{p-1}$ and in (4.21) the numerators in the square brackets should both be 1.

REMARK 2. Equality cannot be attained in (2.1). An examination of the proof of (1.12) on p. 411 in [7] yields the fact that if equality occurs, then necessarily ν is a multiple of π [i.e. we need $|(\sin k\nu)/\sin\nu| = k$ for $k = 1, 2, \ldots, n$]. This would imply that $zf'(z)$ has a multiple pole on $|z| = 1$, which cannot be the case for a function in $D(p,q)$.

COROLLARY 1. *If*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n,$$

z in Δ, *is in* $\hat{\hat{C}}(p,q,\alpha_1, \ldots, \alpha_{p-q})$, *then the inequalities* (2.1) *are satisfied.*

PROOF. This follows immediately upon combining Lemma 1 and Theorems 2 and 3.

We let $\mathcal{H}B$ denote the closed convex hull of any set B of functions analytic in Δ. The closed convex hulls and extreme points of a variety of classes of multivalent functions were determined in [4].

THEOREM 4. $\mathcal{H}\hat{\hat{C}}(p,q,\alpha_1, \ldots, \alpha_{p-q}) = \mathcal{H}\hat{K}(p,q,\alpha_1, \ldots, \alpha_{p-q})$.

PROOF. We need only prove

$$\mathcal{H}\hat{\hat{C}}'(p,q,\alpha_1, \ldots, \alpha_{p-q}) = \mathcal{H}\hat{K}'(p,q,\alpha_1, \ldots, \alpha_{p-q})$$

where B' denotes the class of derivatives of functions in any class B.

Let X be the unit circle and $\mathcal{P}$ the set of probability measures on X. If $zf'(z)$ is given by (1.4) then by the Herglotz representation for functions of positive real part we have

$$p(z) = \int_X \frac{x + e^{-i\delta}z}{x - z}\, d\mu(x) \tag{2.2}$$

and from [4] we have

$$h(z) = \int_X \frac{z^p}{(1 - xz)^{2p}}\, dv(x) \tag{2.3}$$

where μ and v are in $\mathcal{P}$.

Combining (1.4), (2.2), and (2.3) it can be seen that

$$f'(z) = q\int_\Gamma \frac{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \overline{\alpha}_i z)}{\prod_{i=1}^{p-q}(-\alpha_i)} \frac{z^{q-1}(1 - xz)}{(1 - yz)^{2p+1}}\, d\mu(x,y) \tag{2.4}$$

where $\Gamma = X \times X$ and μ is a probability measure on Γ. Letting $g'(z)$ be the kernel function in (2.4) we see that $g'(z)$ has the representation (1.4) with $h(z) = z^p/(1 - yz)^{2p}$ and $p(z) = (1 - xz)/(1 - yz)$. It follows that (2.4) with μ varying over all probability measures on Γ, gives $\mathcal{H}\hat{K}'(p,q,\alpha_1, \ldots, \alpha_{p-q})$.

The representation formula for $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is like (1.4) except that $h(z)$ is replaced by a function of the form $z^p\prod_{s=1}^{2p}(1 - e^{-i\theta_s}z)^{-1}$ which is in $S(p,p)$. It is thus seen that (2.4) also holds for $f(z)$ in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$. We note that if $g'(z)$ denotes the kernel function in (2.4) then $g(z)$ is in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$. Thus (2.4) with μ varying over the probability measures on Γ also gives $\mathcal{H}\hat{C}'(p,q,\alpha_1, \ldots, \alpha_{p-q})$, thereby proving the theorem.

THEOREM 5. *If $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ is in $\mathcal{H}\hat{K}(p,q,\alpha_1, \ldots, \alpha_{p-q})$, then for $n \geqslant p + 1$,*

$$|a_n| \leqslant \sum_{k=q}^{p} \frac{2k(n + p)!}{(p + k)!\,(p - k)!\,(n - p - 1)!\,(n^2 - k^2)} b_k \tag{2.5}$$

where $a_q = 1$ and $b_k = \sup|a_k|$, $k = 1,2,, \ldots, p$, the sup *being taken over all functions in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$.*

PROOF. We first note that the b_k, $k = 1,2, \ldots, p$, exist. This follows from the fact that if $f(z)$ is in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$, then the coefficients of $f'(z)$ are majorized by the coefficients of

$$\frac{q\prod_{i=1}^{p-q}(z + |\alpha_i|)(1 + |\alpha_i|z)}{\prod_{i=1}^{p-q}|\alpha_i|} \frac{z^{q-1}(1 + z)}{(1 - z)^{2p+1}}.$$

[Note that the above function is not in general the derivative of a function in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$.]

By Corollary 1, inequalities (2.1) and, hence, (2.5) are satisfied in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$. It is then easily seen that (2.5) holds in

$$\mathcal{H}\hat{C}(p,q,\alpha_1,\alpha_2, \ldots, \alpha_{p-q}) = \mathcal{H}\hat{K}(p,q,\alpha_1, \ldots, \alpha_{p-q}).$$

We remark that the result above clearly holds for the class $K(p,q, \alpha_1, \ldots, \alpha_{p-q})$. It is also clear that the linear methods used in this paper will not give the exact Goodman conjecture [1] but only the closely related inequality proven in Theorem 5.

REFERENCES

1. A. W. Goodman, *On some determinants related to p-valent functions*, Trans. Amer. Math. Soc. **63**(1948), 175–192. MR**9**, 421.

2. ——, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. **68**(1950), 204–223. MR**11**, 508.

3. A. W. Goodman and M. S. Robertson, *A class of multivalent functions*, Trans. Amer. Math. Soc. **70**(1951), 127–136. MR**12**, 691.

4. D. J. Hallenbeck and A. E. Livingston, *Applications of extreme point theory to classes of multivalent functions*, Trans. Amer. Math. Soc. (to appear).

5. A. E. Livingston, *p-valent close-to-convex functions*, Trans. Amer. Math. Soc. **115**(1965), 161–179. MR**33** #7520.

6. ——, *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc. **21**(1969), 545–552. MR**39** #4378.

7. M. S. Robertson, *A coefficient problem for functions regular in an annulus*, Canad. J. Math. **4**(1952), 407–423. MR**14**, 460.

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