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The Growth of Derivatives of Multipliers of Fractional Cauchy Transforms

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In this paper we prove a number of sharp results on the permissible growth of derivatives of multipliers of fractional Cauchy transforms. For example, we prove that if $f \in M_1$ then there exists a positive constant C such that

$$\int_{-\pi}^{\pi} |f'(re^{i\theta})|\, d\theta \le C\|f\|_{M_1} \frac{1}{1-r} \frac{1}{1+\log(1/(1-r))}$$

for $r \in [0,1)$. This result is proved to be sharp.

1. INTRODUCTION

Let $\Delta = \{z: |z| < 1\}$ and let $\Gamma = \{z: |z| = 1\}$. Let M denote the set of complex-valued Borel measures on Γ. For each $\alpha > 0$ let F_α denote the family of functions f having the property that there exists a measure $\mu \in M$ such that

$$f(z) = \int_\Gamma \frac{1}{(1-\bar{x}z)^\alpha}\, d\mu(x) \tag{1}$$

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for $|z| < 1$. In (1) and throughout this paper each logarithm means the principal branch. $\mathcal{F}_\alpha$ is a Banach space with respect to the norm defined by

$$\|f\|_{\mathcal{F}_\alpha} = \inf\{\|\mu\|\}, \tag{2}$$

where μ varies over all measures in $\mathcal{M}$ for which (1) holds and where $\|\mu\|$ denotes the total variation norm of μ. For $\alpha = 0$, let $\mathcal{F}_0$ denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = f(0) + \int_\Gamma \log \frac{1}{(1 - \bar{x}z)} \, d\mu(x) \tag{3}$$

for $|z| < 1$. $\mathcal{F}_0$ is a Banach space with respect to the norm defined by $\|f\|_{\mathcal{F}_0} = \inf\{\|\mu\|\} + |f(0)|$ when μ varies over all measures in $\mathcal{M}$ for which (3) holds. A function f is called a multiplier of $\mathcal{F}_\alpha$ provided $fg \in \mathcal{F}_\alpha$ for every $g \in \mathcal{F}_\alpha$. Let M_α denote the set of multipliers of $\mathcal{F}_\alpha$. $\mathcal{M}_\alpha$ is a Banach space with respect to the norm defined by

$$\|f\|_{M_\alpha} = \sup\left\{\|fg\|_{\mathcal{F}_\alpha} : g \in \mathcal{F}_\alpha, \|g\|_{\mathcal{F}_\alpha} \leq 1\right\}. \tag{4}$$

Some properties of M_α were derived in [7].

Suppose f is analytic on Δ. Define $m(r) = \max_{|z|=r} |f'(z)|$ and $I(r) = \int_{-\pi}^{\pi} |f'(re^{i\theta})| \, d\theta$ for $r \in [0, 1)$. In this paper we determine the sharp growth of $m(r)$ and $I(r)$ as f varies over $M_\alpha (\alpha > 0)$.

Throughout the paper we use $C_1, C_2, \ldots,$ etc., to denote certain absolute constants. Also, we use $A, B, C, \ldots,$ etc., to denote constants depending on various parameters or assumptions. The meaning of $A, B, C, \ldots,$ etc., may change within an argument or even within a line.

2. GROWTH OF $M(r)$ FOR $f \in M_\alpha$ $\alpha > 0$

Since $M_\alpha \subset H^\infty$ [7] for $\alpha \geq 0$ we have $f'(z) = 0(1/(1 - |z|))$ whenever $f \in M_\alpha$ $(\alpha \geq 0)$. More precisely if $f \in M_\alpha$ then there exists a constant C such that $|f'(z)| \leq C\|f\|_{H^\infty}(1 - r)^{-1}$ for $z \in \Delta$, $r = |z|$. Our first theorem shows that this result is sharp and cannot be improved in M_α for $\alpha > 0$. The construction also answers positively a question raised by T. H. MacGregor in [9]. He proved that for each A such that $0 < A < 1$, there is a function f analytic in Δ satisfying $|f(z)| \leq 1$ $(z \in \Delta)$ such that $\overline{\lim}_{r \to 1^-} (1 - r^2)M(r) \geq A$ and he asked whether $A = 1$ is possible.

LEMMA 1. *If $w, z \in \Delta$ then*

$$1 - \left| \frac{w - z}{1 - \bar{w}z} \right| \leq 2 \frac{1 - |w|}{1 - |z|}. \tag{5}$$

Additionally if $1 - |w| \leq \frac{1}{4}(1 - |z|)$ *then*

$$\left|\frac{1 - \overline{w}z}{w - z}\right| - 1 \leq 4\frac{1 - |w|}{1 - |z|}. \tag{6}$$

Proof. We note that (5) is well known and we omit its proof. To verify (6) we observe that the additional assumption and (5) imply $|(w - z)/(1 - \overline{w}z)| \geq 1/2$. This fact together with (5) easily gives (6).

Let B denote the set of functions $g \in H^\infty$ with $\|g\|_{H^\infty} \leq 1$.

LEMMA 2. *There exists a function $f \in B$ such that*

$$\overline{\lim_{r \to 1^-}}\ (1 - r^2)M(r) = 1. \tag{7}$$

Proof. We will construct such an f in the form

$$f(z) = \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z - z_k}{1 - \bar{z}_k z}, \tag{8}$$

where $|z_k| < 1$, $\sum_{k=1}^{\infty}(1 - |z_k|) < +\infty$ and additionally

$$\lim_{n \to \infty} \left(1 - |z_n|^2\right)|f'(z_n)| = 1. \tag{9}$$

Since $(1 - |z_n|^2)|f'(z_n)| \leq (1 - r_n^2)M(r_n) \leq 1 (r_n = |z_n|)$, (9) implies (7). Let $\{z_k\}$ be any sequence satisfying $z_k \in \Delta$ $(k = 1, 2, \ldots)$ and $\sum_{k=1}^{\infty}(1 - |z_k|) < +\infty$. Recall that $\lim_{|z| \to 1^-} |(z - z_0)/(1 - \bar{z}_0 z)| = 1$, $z_0 \in \Delta$. Using this fact it is easy to prove that there exists a subsequence $\{z_{m_n}\}$ of $\{z_n\}$ such that

$$1 - \frac{1}{n} \leq \prod_{k=1}^{n-1} \left|\frac{z_k - z_{m_n}}{1 - \bar{z}_k z_{m_n}}\right| \leq 1 \tag{10}$$

for $n = 1, 2, \ldots$. We now denote this subsequence by $\{z_n\}$. By choosing another subsequence and denoting it by $\{z_n\}$ we can also have $1 - |z_{n+1}| \leq (1/n)(1 - |z_n|)$ for $n = 1, 2, \ldots$. Hence for each $k \geq 1$ we infer that

$$\frac{1 - |z_{n+k}|}{1 - |z_n|} \leq \frac{1}{n^k}. \tag{11}$$

It follows from (10) that

$$\lim_{n \to \infty} \prod_{k=1}^{n-1} \left|\frac{z_k - z_n}{1 - \bar{z}_k z_n}\right| = 1. \tag{12}$$

Using the elementary inequality $1 + x \leq e^x$, (6), and (11) we obtain

$$
\begin{aligned}
1 &\geq \prod_{k>n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| = \left(\prod_{k>n} \left[1 + \left(\left| \frac{1 - \bar{z}_k z_n}{z_k - z_n} \right| - 1 \right) \right]^{-1} \right) \\
&\geq \left[\exp\left(\sum_{k>n} \left(\left| \frac{1 - \bar{z}_k z_n}{z_k - z_n} \right| - 1 \right) \right) \right]^{-1} \\
&\geq \exp\left(-4 \sum_{k>n} \frac{1 - |z_k|}{1 - |z_n|} \right) \geq \exp\left(-4 \sum_{k=1}^{\infty} \frac{1}{n^k} \right) \\
&= \exp\left(\frac{-4}{n-1} \right). \qquad (13)
\end{aligned}
$$

It follows from (13) that

$$\lim_{n\to\infty} \prod_{k>n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| = 1. \qquad (14)$$

We obtain from (12) and (14)

$$\lim_{n\to\infty} \prod_{k\neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| = 1. \qquad (15)$$

Any easy computation gives

$$|f'(z_n)| = \frac{1}{1 - |z_n|^2} \prod_{k\neq n} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right|. \qquad (16)$$

Clearly (15) and (16) imply (9).

THEOREM 1. *There exists a function $f \in B$ such that $f \in M_\alpha$ for all $\alpha > 0$ and*

$$\overline{\lim_{r\to 1^-}} \, (1 - r) M(r) = 1. \qquad (17)$$

Proof. Since $M_\alpha \subset M_1$ for $0 < \alpha \leq 1$ [7] we need only consider the case $\alpha \in (0, 1)$. It is clear from the construction of the Blaschke product in Lemma 2, that the Blaschke product satisfying (17) can be constructed as a subproduct of an arbitrary Blaschke product. On the other hand if the condition

$$\sup_\theta \sum_{n=1}^{\infty} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^\alpha < +\infty \qquad (0 < \alpha \leq 1) \qquad (18)$$

is satisfied by a sequence $\{z_n\}$, then it holds for my subsequence of $\{z_n\}$. Now (18) implies $f(z) = \prod_{k=1}^{n}(\bar{z}_k/z_k)((z - z_k)/(1 - \bar{z}_k z)) \in M_\alpha$ [3]. So to prove the theorem it is enough to prove that there is an infinite Blaschke product satisfying (18). To this end let $z_n = (1 - 4^{-n})e^{i2^{-n}}$. It is easy to prove that if γ denotes an arc on Γ with length t satisfying $0 < t < \pi$ then the length of the chord connecting the endpoints of γ, which we denote by $l(t)$ satisfies

$$l(t) \geq \frac{2}{\pi} t. \tag{19}$$

If $-\pi \leq \theta \leq 0$, then using (19) we may estimate

$$|e^{i\theta} - z_n| \geq |1 - z_n| \geq \frac{2}{\pi} 2^{-n}. \tag{20}$$

For these θ's we find that

$$\sum_{n=1}^{\infty} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^{\alpha} \leq \sum_{n=1}^{\infty} \left(\frac{4^{-n}}{(2/\pi) 2^{-n}} \right)^{\alpha} = \left(\frac{\pi}{2} \right)^{\alpha} \frac{1}{2^{\alpha} - 1}. \tag{21}$$

When $0 < \theta < \pi$, let $N = \min\{n \colon 2^{-n} \leq \theta\}$. Then if $n \geq N + 1$ we have

$$|e^{i\theta} - z_n| \geq \frac{2}{\pi} \frac{1}{2^{N+1}}. \tag{22}$$

It follows from (22) that

$$\begin{aligned} \sum_{n \geq N+1} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^{\alpha} &\leq \sum_{n \geq N+1} \left(\frac{4^{-n}}{\frac{2}{\pi} 2^{-N-1}} \right)^{\alpha} = \left(\frac{\pi}{2} \right)^{\alpha} \sum_{n=1}^{\infty} \left(\frac{4^{N-n}}{2^{-N-1}} \right)^{\alpha} \\ &\leq \left(\frac{\pi}{2} \right)^{\alpha} \sum_{n=1}^{\infty} \left(\frac{4^{-N-n}}{4^{-N-1}} \right)^{\alpha} \\ &= \left(\frac{\pi}{2} \right)^{\alpha} \sum_{n=0}^{\infty} 4^{-\alpha n} = \left(\frac{\pi}{2} \right)^{\alpha} \frac{4^{\alpha}}{4^{\alpha} - 1}. \end{aligned} \tag{23}$$

If on the other hand $n \leq N - 2$ then

$$|e^{i\theta} - z_n| \geq \frac{\pi}{2} \frac{1}{2^{n+1}}. \tag{24}$$

Hence, as in (21), we have

$$\sum_{n \le N-2} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^{\alpha} \le \left(\frac{\pi}{2} \right)^{\alpha} \sum_{n=1}^{N-2} \frac{1}{(2^{n+1})^{\alpha}} < \left(\frac{\pi}{2} \right)^{\alpha} \frac{1}{2^{\alpha} - 1}. \quad (25)$$

We infer from (23) and (25) that for each $\theta \in (0, \pi)$

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^{\alpha} &\le \sum_{n \le N-2} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^{\alpha} + \sum_{N-1 \le n \le N} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n|} \right)^{\alpha} \\ &\quad + \sum_{n \ge N+1} \left(\frac{1 - |z_n|}{|e^{i\theta} - z_n} \right)^{\alpha} \\ &\le \left(\frac{\pi}{2} \right)^{\alpha} \frac{1}{2^{\alpha} - 1} + 2 + \left(\frac{\pi}{2} \right)^{\alpha} \frac{4^{\alpha}}{4^{\alpha} - 1}. \end{aligned} \quad (26)$$

It follows from (21) and (26) that (18) holds.

Remark. Theorem 1 shows that for $f \in M_{\alpha} (\alpha > 0)$ the estimate $f'(z) = 0(1/(1 - |z|))$ is sharp in a strong sense. We next prove that when $f \in M_0$ this estimate on $f'(z)$ can be improved. This result is also sharp.

THEOREM 2. *Suppose* $f \in M_0$. *Then*

$$|f'(z)| \le 4\|f\|_{M_0} \frac{1}{(1 - r)\log(1/(1 - r))}, \quad (27)$$

where $r = |z| > 0$ *and* (27) *is sharp*.

Proof. For each $x, |x| = 1$ since $f \in M_0$ we have

$$\left\| f(z)\log\frac{1}{1 - \bar{x}z} \right\|_{F_0} \le \|f\|_{M_0}. \quad (28)$$

It follows [6] from (28) that

$$\left\| \left(f(z)\log\frac{1}{1 - \bar{x}z} \right)' \right\|_{F_1} \le \|f\|_{M_0}. \quad (29)$$

But $(f(z)\log(1/(1 - \bar{x}z)))' = f'(z)\log(1/(1 - \bar{x}z)) + \bar{x}f(z)(1/(1 - \bar{x}z))$.
Hence

$$\left\| f'(z)\log\frac{1}{1 - \bar{x}z} \right\|_{F_1} \le \left\| \left(f(z)\log\frac{1}{1 - \bar{x}z} \right)' \right\|_{F_1} + \left\| \bar{x} \cdot f(z) \frac{1}{1 - \bar{x}z} \right\|_{F_1}. \quad (30)$$

It follows from (29), (30), and the fact that $M_0 \subset M_1$ [4] that

$$\left\| f'(z) \log \frac{1}{1 - \bar{x}z} \right\|_{\mathcal{F}_1} \leq \|f\|_{M_0} + \|f\|_{M_1} \leq 2\|f\|_{M_0} \tag{31}$$

since $\|f\|_{M_1} \leq \|f\|_{M_0}$ [4].

For any $g \in \mathcal{F}_1$ we have $|g(z)| \leq 2\|g\|_{\mathcal{F}_1}/(1 - |z|)$ $(z \in \Delta)$. Applying this to $f'(z)\log(1/(1 - \bar{x}z))$ and using (31) above we find that

$$\left| f'(z) \log \frac{1}{1 - \bar{x}z} \right| \leq \frac{4\|f\|_{M_0}}{1 - |z|}. \tag{32}$$

We take $x = \bar{z}/|z|$ in (32) and obtain (27) after dividing by $\log(1/(1 - |z|))$.

To prove that (27) is sharp, suppose there is a positive function $\epsilon(r)$ satisfying $\lim_{r \to 1^-} \epsilon(r) = 0$ and such that

$$|f'(z)| \leq 4\|f\|_{M_0} \left(\frac{1}{\log(1/(1 - r))} \right) \frac{1}{(1 - r)} \epsilon(r) \tag{33}$$

for each $z \in \Delta$ and each $f \in M_0$. Let $g_\rho(z) = \log(1/(1 - \rho z))$ for $\rho \in (0, 1)$ and note $g_\rho \in M_0$ [6]. In [5] it was proved that there is a constant C independent of $r = |z|$ such that

$$\|g_\rho\|_{M_0} \leq C \log \frac{1}{1 - \rho} \tag{34}$$

for $\rho > 1/2$. Also $g'_\rho(z) = \rho/(1 - pz)$. If (33) were true for all $f \in M_0$ then for $f = g_\rho$ we would have

$$\frac{\rho}{|1 - \rho z|} \leq 4C \log \frac{1}{1 - \rho} \frac{1}{(\log(1/(1 - |z|)))} \frac{1}{(1 - |z|)} \epsilon(r). \tag{35}$$

In particular for $z = \rho$ we infer from (35) that $\rho/(1 + \rho) \leq 4C\epsilon(\rho)$ which is a contradiction.

3. GROWTH OF $I(r)$ FOR $f \in M_\alpha$ $\alpha > 0$

In this section we prove sharp results on the growth of the integral means of f' when $f \in M_\alpha$ $\alpha > 0$. Recall that the radial variation of these functions [7, 10] are uniformly bounded in all directions. In keeping with the principle of moderation described in [8, p. 200] for $\alpha > 0$, $\alpha \neq 1$ the integral means of f' for $f \in M_\alpha$ are no more restricted in terms of growth as $r \to 1^-$ than those of any function in $\mathcal{F}_\alpha$. However, for $f \in M_1$ the growth of $I(r)$ is more restricted than for arbitrary functions in $\mathcal{F}_1$. The following technical lemma is needed in the proof of Theorem 3 below.

LEMMA 3. *Let ϕ and ψ be non-negative measurable functions on Γ. Then*

$$\sup_\tau \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it})\psi(e^{it}e^{i\tau})\,dt \geq \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\,dt \frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it})\,dt. \tag{36}$$

Proof. Note $1 = (1/2\pi)\int_0^{2\pi} d\tau$ and so by Tonnelli's Theorem we have

$$\begin{aligned}
&\sup_\tau \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\psi(e^{it}e^{i\tau})\,dt \\
&\quad\geq \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\psi(e^{it}e^{i\tau})\,dt \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}d\tau\frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\psi(e^{it}e^{i\tau})\,dt \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\left(\frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it}e^{i\tau})\,d\tau\right)dt \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\left(\frac{1}{2\pi}\int_0^{2\pi}\psi(e^{i\tau})\,d\tau\right)dt \\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{it})\,dt\frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it})\,dt.
\end{aligned} \tag{37}$$

So (36) holds.

We let $k_1(z) = 1/(1-z)$ for $z \in \Delta$.

LEMMA 4. *There are positive constants A_1 and A_2 such that for $0 \leq r < 1$ and any $f \in \mathcal{F}_1$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi}|f'(re^{it})|\,dt \leq A_1\frac{\|f\|_{\mathcal{F}_1}}{1-r} \tag{38}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}|k_1(re^{it})|\,dt \geq A_2\left(\log\frac{1}{1-r}+1\right). \tag{39}$$

Proof. The argument is standard using (1) and Lemma 3 in [3].

THEOREM 3. *There is a constant C such that for each $r \in [0,1)$ and each $g \in M_1$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi}|g'(re^{it})|\,dt \le \frac{C\|g\|_{M_1}}{(1-r)\big(\log(1/(1-r))+1\big)}. \tag{40}$$

Proof. Since $g \in M_1$ we have

$$\|g(z)k_1(e^{i\tau}z)\|_{\mathcal{F}_1} \le \|g\|_{M_1} \tag{41}$$

for all τ and by (38) with $f(z) = g(z)k_1(e^{i\tau}z)$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}|g'(re^{it})k_1(e^{i\tau}re^{it}) + g(re^{it})e^{i\tau}k_1'(e^{i\tau}re^{it})|\,dt \le \frac{A_1\|g\|_{M_1}}{1-r}. \tag{42}$$

Using $\|g\|_{H^\infty} \le \|g\|_{M_1}$ and Lemma 3 in [3] we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}|g(re^{it})e^{i\tau}k_1'(e^{i\tau}re^{it})|\,dt\\ &\qquad \le \|g\|_{H^\infty}\frac{1}{2\pi}\int_0^{2\pi}|k_1'(re^{it})|\,dt \le \frac{A_3\|g\|_{M_1}}{1-r}.\end{aligned} \tag{43}$$

We infer from (42) and (43) that

$$\frac{1}{2\pi}\int_0^{2\pi}|g'(re^{it})k_1(e^{i\tau}re^{it})|\,dt \le \frac{A_4\|g\|_{M_1}}{1-r}, \tag{44}$$

where A_4 is independent of g.

Taking the sup over τ and applying Lemma 3 with $\phi(e^{it}) = |g'(re^{it})|$ and $\psi(e^{it}) = |k_1(re^{it})|$ we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}|g'(re^{it})|\,dt \le \frac{2A_1\|g\|_{M_1}}{(1-r)(1/2\pi)\int_0^{2\pi}|k_1(re^{it})|\,dt}. \tag{45}$$

Clearly (39) and (45) imply (40). We next prove that Theorem 3 is sharp.

THEOREM 4. *Let $\epsilon(r)$ be a positive non-increasing function on $(0,1)$ such that $\lim_{r\to 1^-}\epsilon(r) = 0$. Then there is a function $f \in M_1$ for which*

$$\varlimsup_{r\to 1^-}\frac{(1-r)\log(1/(1-r))}{\epsilon(r)}\min_{|z|=r}|f'(z)| = +\infty \tag{46}$$

and

$$\varlimsup_{r\to 1^-}\frac{(\log(1/(1-r)))(1-r)}{\epsilon(r)}\left(\int_0^{2\pi}|f'(re^{i\theta})|^p\,d\theta\right)^{1/p} = +\infty \tag{47}$$

for each $p > 0$.

Proof. Clearly (46) implies (47). To prove (46) it suffices to show that such a function f exists which has the following property: there is a sequence $\{r_k\}$ such that $0 \leq r_k < 1$, $\lim_{k\to\infty} r_k = 1$, and for every θ

$$(1 - r_k)\log \frac{1}{1 - r_k}|r_k f'(r_k e^{i\theta})| \geq \frac{1}{6}\epsilon(r_k) \tag{48}$$

for all sufficiently large values of k. It is clear (48) implies (46) by first applying (48) with ϵ replaced by $\sqrt{\epsilon}$. Let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{\log \lambda_n} \frac{1}{n^2} z^{\lambda_n}, \tag{49}$$

where $\{\lambda_n\}$ is a strictly increasing sequence of positive integers, suitably selected in terms of ϵ in a manner described below. We have $\sum_{n=1}^{\infty} \log \lambda_n (1/\log \lambda_n)(1/n^2) < +\infty$ ($\lambda_n \geq 2$ for all n) and so $f \in M_1$ [10].

The sequence $\{\lambda_n\}$ can be defined so that the following three inequalities hold for all positive integers k:

$$\frac{\lambda_{k+1}}{\log \lambda_{k+1}} \geq \left(\frac{\lambda_k}{\log \lambda_k}\right)^2 \tag{50}$$

$$\epsilon\left(1 - \frac{1}{\lambda_k}\right) \leq \frac{1}{k^2} \tag{51}$$

$$\frac{\lambda_{k+1}}{\log \lambda_{k+1}} \geq \frac{(k+1)^3}{n^2} \frac{\lambda_n}{\log \lambda_n}, \qquad n = 1, 2, \ldots, k. \tag{52}$$

Each of the conditions (50), (51), and (52) can be inductively satisfied so that at any stage in the induction the next term in the sequence is permitted to take on all arbitrarily large values ($x/\log x \to +\infty$ as $x \to +\infty$); thus there is a sequence satisfying all three conditions.

The proof that (50), (51), and (52) imply (48) is similar to the proof of Theorem 1 in [1], and we omit it.

We finish the paper by giving sharp results on the growth of integral means of derivatives of functions in M_α for $\alpha > 0$, $\alpha \neq 1$.

THEOREM 5. *If $f \in M_\alpha$ then there exists a constant C depending only on α such that, for each $p \geq 1$*

$$\left(\int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta\right)^{1/p} \leq \frac{C\|f\|_{M_\alpha}}{(1 - r)} \tag{53}$$

whenever $\alpha > 1$. *Additionally if* $0 < \alpha < 1$ *then*

$$\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leq \frac{C\|f\|_{M_\alpha}}{(1-r)^\alpha} \tag{54}$$

and

$$\left(\int_0^{2\pi} |f'(re^{i\theta})|^p\, d\theta\right)^{1/p} = o\left[\frac{1}{(1-r)^\alpha}\right] \tag{55}$$

as $r \to 1^-$ *for each* $p \in (0,1)$.

Proof. The proofs of (53) and (54) are routine using Lemma 3 in [3] and the facts that $\|f\|_{H^\infty} \leq \|f\|_{M_\alpha}$, $M_\alpha \subset \mathcal{F}_\alpha$, and the μ in (1) can be picked so that $\|\mu\| \leq 2\|f\|_{M_\alpha}$. Finally, (55) follows directly from Theorem 3 in [2, p. 237] with $\alpha + 1$ playing the role of α.

Remark. When $p > 1$ we can prove that the left-hand side of (55) is $0(1/(1-r)^{\frac{\alpha}{p}+1-\frac{1}{p}}$. We cannot prove it is sharp in M_α but it is sharp in $\mathcal{F}_\alpha \cap H^\infty$.

We first show that (53) is sharp.

THEOREM 6. *Suppose* $\alpha > 1$ *and* $\epsilon(r)$ *is a positive non-increasing function on* $(0,1)$ *such that* $\lim_{r\to 1^-} \epsilon(r) = 0$. *Then there is a function* $f \in M_\alpha$ *such that*

$$\overline{\lim_{r\to 1^-}}\ \frac{1-r}{\epsilon(r)} \min_{(z)=r} |f'(z)| = +\infty \tag{56}$$

and

$$\overline{\lim_{r\to 1^-}}\ \frac{1-r}{\epsilon(r)} \left(\int_0^{2\pi} |f'(re^{i\theta})|^p\, d\theta\right)^{1/p} = +\infty \tag{57}$$

for each $p > 0$.

Proof. Clearly (57) follows from (56). In [1] it was proved that there is a sequence (increasing) of positive integers $\{\lambda_n\}$ such that for $f(z) = \sum_{n=1}^\infty (1/n^2) z^{\lambda_n}$, (56) holds. Since $\sum_{n=1}^\infty (1/n^2) < +\infty$, $f \in M_\alpha$ for each $\alpha > 1$ [3].

We next prove that (54) and (55) are sharp.

THEOREM 7. *Let* $\epsilon(r)$ *be a positive non-increasing function on* $(0,1)$ *such that* $\lim_{r\to 1^-} \epsilon(r) = 0$ *and* $\alpha \in (0,1)$. *Then there is a function* $g \in M_\alpha$ *for which*

$$\overline{\lim_{r\to 1^-}}\ \left\{\frac{(1-r)^\alpha}{\epsilon(r)} \min_{(z)=r} |g'(z)|\right\} = +\infty \tag{58}$$

and

$$\overline{\lim_{r \to 1^-}} \frac{(1-r)^{\alpha}}{\epsilon(r)} \left(\int_0^{2\pi} |g'(re^{i\theta})|^p \, d\theta \right)^{1/p} = +\infty \tag{59}$$

for each $p > 0$.

Proof. Let $\alpha + 1$ play the role of α in Theorem 7 [2, p. 248]. The function f constructed there [2, p. 250] has the form $f(z) = \sum_{n=1}^{\infty} (1/n^2) A_{\lambda_n} z^{\lambda_n}$ where $\{\lambda_n\}$ is an increasing sequence of positive integers and $A_n = 0\ (n^{\alpha - 1})$. Define $g(z) = \int_0^z f(\tau)\, d\tau$ and suppose $g(z) = \sum_{n=1}^{\infty} a_n z^n$. Then it is easily verified that $\sum_{n=1}^{\infty} n^{1-\alpha} |a_n| < +\infty$ and so $f \in M_{\alpha}$ [3]. Now (58) follows directly from Theorem 7 [2, p. 248]. Clearly (59) follows from (58).

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