ON A CONJECTURE OF CLUNIE AND SHEIL-SMALL

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In this note I prove the following result which was conjectured by Clunie and Sheil-Small.

THEOREM. *Let f be a normalized convex function on $\Delta = \{z : |z| < 1\}$ and $w \notin f(\Delta)$. Let*

$$F(z) = \frac{wf(z)}{w-f(z)} = z + c_2 z^2 + c_3 z^3 + \ldots$$

Then there exists a constant A, independent of f and w, such that $|c_n| \leqslant A$.

I should like to thank Dr. Sheil-Small for mentioning this problem to me. It remains to find the best possible value of A: the present method depends on integral mean estimates and is unlikely to give this.

I also give an integral formula which exactly represents a dense subclass of the normalized convex functions.

LEMMA. *Let f be a normalized convex function. Then for $|z| = r < 1$, we have*

$$\left|z^2 \frac{f'(z)}{f^2(z)}\right| \geqslant \left(\frac{r}{\arcsin r}\right)^2 > \frac{4}{\pi^2}.$$

Moreover this is sharp for each fixed r: the extremal function satisfies $f'(z) = (1-2rz+z^2)^{-1}$.

We remark that in the larger class of functions starlike of order $\frac{1}{2}$, the sharp lower bound for the left hand side is $\sqrt{(1-r^2)}$: the extremal function is $z(1-2rz+z^2)^{-1/2}$. For normalized univalent functions the sharp lower bound is $1-r^2$: this follows from the Goluzin inequality.

Proof of the lemma. Within the class K of normalized convex functions, the functions such that

$$f'(z) = \prod_{j=0}^{n} (1-\zeta_j z)^{-\beta_j} \tag{1}$$

where $\beta_j > 0, \sum \beta_j = 2, |\zeta_j| = 1$, are dense in the usual topology. We may confine our attention to this subclass, moreover we may assume that $z = r$. Let $\zeta_j = \exp(i\theta_j)$ for each j so that

$$|f'(r)| = \prod_{j=0}^{n} (1-2r\cos\theta_j + r^2)^{-\beta_j/2}$$

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Next,

$$|f(r)| \leqslant \int_0^r |f'(\rho)|d\rho$$
$$\leqslant \prod_{j=0}^{n}\left(\int_0^r \frac{d\rho}{1-2\rho\cos\theta_j+\rho^2}\right)^{\beta_j/2} \tag{2}$$

by Hölder's inequality: here we need the fact that $\sum(\beta_j/2) = 1$. Thus

$$|f(r)|^2 \leqslant \prod_{j=0}^{n}\left(\frac{1}{\sin\theta_j}\arcsin\left(\frac{r\sin\theta_j}{\sqrt{(1-2r\cos\theta_j+r^2)}}\right)\right)^{\beta_j}$$

(with the obvious interpretation if $\sin\theta_j = 0$) and so

$$\left|\frac{f^2(r)}{r^2 f'(r)}\right| \leqslant \prod_{j=0}^{n}\left\{\frac{\sqrt{(1-2r\cos\theta_j+r^2)}}{r\sin\theta_j}\arcsin\left(\frac{r\sin\theta_j}{\sqrt{(1-2r\cos\theta_j+r^2)}}\right)\right\}^{\beta_j}.$$

Since $x^{-1}\arcsin x$ is increasing on (0, 1) the maximum of the right-hand side is attained when $\cos\theta_j = r$ for every j. This gives the result stated. Finally, there is equality in (2) if and only if $f'(\rho) > 0$ for $0 \leqslant \rho \leqslant r$, and this isolates the extremal function given above.

Remark. Let us denote by K' the functions defined by (1), with $f(0) = 0$. Then we have

$$\frac{f(z)}{z} = \frac{1}{\Gamma(\beta_0)\Gamma(\beta_1)\ldots\Gamma(\beta_n)}\int_S \frac{x_0^{\beta_0-1}x_1^{\beta_1-1}\ldots x_n^{\beta_n-1}dx_1\ldots dx_n}{1-(x_0\zeta_0+x_1\zeta_1+\ldots+x_n\zeta_n)z}$$

where the integral is over the sloping face of the $(n+1)$-dimensional simplex, viz

$$x_0 \geqslant 0,\quad x_1 \geqslant 0,\ldots,x_n \geqslant 0,\quad x_0+x_1+\ldots+x_n = 1.$$

This formula does not seem to be in the literature: it has the advantage over the familiar integral representation formula for functions with real part $> \frac{1}{2}$ that it only represents convex functions. By convexity, $x_0\zeta_0+x_1\zeta_1+\ldots+x_n\zeta_n$ is in the closed unit disc: we can if we wish use Poisson's formula to get a measure on the circle. Sheil-Small has remarked that we also get an immediate formula for $K' * K'$, the class of convolutions of elements of K'.

The formula depends on the beta-function formula:

$$\int_S x_0^{\alpha_0-1}x_1^{\alpha_1-1}\ldots x_n^{\alpha_n-1}dx_1\ldots dx_n = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_0+\alpha_1+\ldots+\alpha_n)}, \tag{3}$$

which holds when the α's are positive. From (1), we have

$$f'(z) = \prod_{j=0}^{n} \left\{ \frac{1}{\Gamma(\beta_j)} \sum_{m=0}^{\infty} \frac{\Gamma(m+\beta_j)}{m!} \zeta_j^m z^m \right\}$$

and we apply (3) to $\prod \Gamma(m_j+\beta_j)$, noting that $\sum \beta_j = 2$. This represents $f'(z)$ as an integral over S, and we collect powers of z, put together $(x_0\zeta_0+x_1\zeta_1+\ldots+x_n\zeta_n)^m$ by the multinomial theorem, and integrate with respect to z.

Proof of the theorem. The function $F(z)$ is univalent, moreover we may check that

$$\frac{z^2 F'(z)}{F^2(z)} = \frac{z^2 f'(z)}{f^2(z)}$$

so that by the lemma we have

$$\left| z^2 \frac{F'(z)}{F^2(z)} \right| > \frac{4}{\pi^2}.$$

It follows from this and the Koebe distortion theorem that

$$|F(z)| < \frac{\pi^2 r}{4} \left| z \frac{F'(z)}{F(z)} \right| < \frac{(\pi^2/2)r}{1-r}.$$

A result of Littlewood and Paley ([1], p. 186; [2], Theorem 5.3) states that if g is univalent on Δ and there exist fixed M_1 and $\alpha > \frac{1}{2}$ such that

$$M(r, g) \leqslant \frac{M_1 r}{(1-r)^\alpha} \, (r < 1),$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} |g'(re^{i\theta})| d\theta \leqslant \frac{M_2}{(1-r)^\alpha}, \qquad M_2 = M_2(M_1, \alpha).$$

We apply this with $M_1 = \pi^2/2, \alpha = 1$. Therefore

$$\frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta})| d\theta \leqslant \frac{M_2}{1-r}$$

and it follows that $|c_n| \leqslant M_2 e$.

References

1. G. M. Goluzin, *Geometric theory of functions of a complex variable.* Translations of Mathematical Monographs (26), A. M. S. (Providence R. I., 1969).
2. Ch. Pommerenke, *Univalent functions* (Vandenhoeck and Ruprecht, Göttingen 1975).

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