A complete many-valued logic with product-conjunction

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1 Introduction

Many-valued logics have become a subject of increased interest as logics of vagueness, i.e. fuzzy logics; intermediate truth values are understood as degrees of truth of fuzzy propositions (see [20]). Many-valued semantics underlying fuzzy set theory have advocated the use of some binary operations in the unit interval $[0,1]$ to generalize the classical boolean truth functions on $\{0,1\}$. In particular, the so-called *t-norms* and their duals *t-conorms* ([17, 18, 1, 13]) have become popular to model conjunction and disjunction-like operations with fuzzy sets. A (continous) t-norm is a commutative and associative binary operation in $[0,1]$, non-decreasing in both variables and having 1 and 0 as neutral and absorbent elements respectively.

In this paper we investigate some logics whose set of truth values is the real interval $[0,1]$ and we concentrate our attention to logics having a conjunction whose truth function $t(x,y)$ is a t-norm, and having a corresponding residuated implication (or, as Pavelka [14] observes, the conjunction and the implication form an adjoint couple); i.e., if $i(x,y)$ is the truth function of the implication then

$$z \leq i(x,y) \text{ iff } t(x,z) \leq y.$$

There are three main examples: (1) *Łukasiewicz's logic* [11] with the conjunction $x\&y = \max(0, x+y-1)$ and implication $x \rightarrow_L y = \min(1, 1-x+y)$, (2) *Gödel's logic* [4] with the conjunction $x \wedge y = \min(x,y)$ and the implication $x \rightarrow_G y = 1$ for $x \leq y$ and $x \rightarrow_G y = y$ otherwise, and finally (3) *product logic* with the conjunction $x \odot y = x.y$ and implication $x \rightarrow_P y = 1$ for $x \leq y$ and $x \rightarrow_P y = y/x$ otherwise. We also have truth constants 0, 1 (absolute falsity and absolute truth). Each of the above implications $\rightarrow$ defines its negation as

$\neg x = x \rightarrow 0$; for Łukasiewicz's logic we get $\neg x = 1 - x$, for Gödel's logic and product logic we get Gödel's negation $\neg 0 = 1$, $\neg x = 0$ for $x > 0$.

Note that each (continuous) t-norm is a "mixture" of the three above t-norms. Namely, Ling has proved [10] that any archimedean t-norm, i.e. a t-norm having 0 and 1 as the only idempotent elements, is isomorphic either to the t-norm *product* $t(x, y) = x.y$ or to the *Łukasiewicz*'s t-norm $t(x, y) = max(0, x + y - 1)$, and that any other continous t-norm is either the t-norm *minimum* $t(x, y) = min(x, y)$, or is an ordinal sum of the t-norm *min* and archimedean t-norms (see e.g. [10] or [13] for the exact formulation.)

In Łukasiewicz's logic, implication and 0 can be taken as primitives and other connectives (including Łukasiewicz's conjunction and the minimum conjunction as well as their de Morgan duals - disjunctions $\min(1, x + y)$, $max(x, y)$) are definable. The Łukasiewicz's propositional logic has an elegant axiomatization by four schemes, which was shown complete with respect to 1-tautologies (formulas having identically the value 1) by Rose and Rosser [15] (see [5] for a full proof and very detailed information on Łukasiewicz's and Gödel's logic). As was shown by Scarpelini [16], the corresponding predicate calculus is not recursively axiomatizable, i.e. the set of its 1-tautologies is not recursively enumerable. On the other hand, both Łukasiewicz's propositional and predicate logic has a "graded" version, originally formulated and investigated by Pavelka [14] and Novák [12]; for very simplified versions see [7], [8].

Gödel's logic has, besides the connectives above, also the disjunction with maximum as truth function as a primitive connective; Gödel's propositional logic is completely axiomatized by the axioms of intuitionistic propositional logic plus the linearity axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$; see [5] for a completeness proof. The corresponding Gödel's predicate logic has a finitary (recursive) axiomatization and hence the set of its 1-tautologies is recursively enumerable (see[19]; surprisingly, Takeuti and Titani do not refer to Gödel and their axiomatization allows simplifications following from the completeness of the mentioned axiom system of propositional logic). An analogon of Pavelka's graded logic is not possible since Gödel's implication is not continuous (see [14]).

The third logic, based on the product conjunction, seems not to be studied in a similar extent as the two logics above. In this paper we show that the propositional logic based on product and the corresponding implication is completely axiomatized by a finite set of axiom schemes. It is important that this logic contains both minimum and maximum as defined connectives. (Note that minimum is definable in each t-norm based logic, cf. [3].) In proving the completeness we apply the same method as Gottwald for Łukasiewicz's logic [5], i.e. we algebraize the logic (introducing a notion of product algebras) and show, among other things, that each product algebra is a subdirect product of linearly ordered product algebras. At the end we present some open problems.

2 Product Logic: Basic Definitions

Formulas are built from propositional variables and truth constants 0, 1 using connectives $\odot, \rightarrow$ (both binary). The *semantics* is as above, i.e. the truth functions are defined as follows, for any $x, y \in [0,1]$,

$$\begin{aligned} x \odot y &= x \cdot y \\ x \rightarrow y &= \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases} \end{aligned}$$

for any $x, y \in [0,1]$. We take the freedom of denoting the truth function of a connective by the same symbol as the connective itself. From these primitive connectives, one can define the following four derived connectives:

$$\begin{aligned} \neg\varphi &\quad \text{is} \quad \varphi \rightarrow 0, \\ \varphi \wedge \psi &\quad \text{is} \quad \varphi \odot (\varphi \rightarrow \psi) \\ \varphi \vee \psi &\quad \text{is} \quad ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \varphi \leftrightarrow \psi &\quad \text{is} \quad (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi) \end{aligned}$$

The corresponding truth functions for these connectives are:

$$\begin{aligned} \neg 0 &= 1; \\ \neg x &= 0, \text{ for } x > 0, \\ x \wedge y &= min(x,y), \\ x \vee y &= max(x,y) \\ x \leftrightarrow y &= min(x \rightarrow y, y \rightarrow x) = (x \rightarrow y).(y \rightarrow x) \end{aligned}$$

(Trivial checking.)

Lemma 1 *The following formulas are 1-tautologies:*

for $\rightarrow$:

(A1)	$\varphi \rightarrow (\psi \rightarrow \varphi)$	*(adding assumptions)*
(A2)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$	*(transitivity)*
(A3)	$\varphi \rightarrow 1,\ 0 \rightarrow \varphi$	*(extremals)*

for $\rightarrow, \odot$:

(A4)	$(\varphi \odot \psi) \rightarrow (\psi \odot \varphi)$	*(commutativity)*
(A5)	$(\varphi \odot (\psi \odot \chi)) \rightarrow ((\varphi \odot \psi) \odot \chi)$	*(associativity)*
	$((\varphi \odot \psi) \odot \chi) \rightarrow (\varphi \odot (\psi \odot \chi))$	*(associativity)*
(A6)	$((\varphi \odot \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$	*(residuation)*
	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \odot \psi) \rightarrow \chi)$	*(residuation)*
(A7)	$(\varphi \rightarrow \psi) \rightarrow ((\varphi \odot \chi) \rightarrow (\psi \odot \chi))$	*(monotonicity)*
(A8)	$\neg\neg\varphi \rightarrow ((\varphi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\varphi \rightarrow \psi))$	*(cancellation)*

for $\wedge, \vee, \rightarrow$:

(A9)	$(\chi \rightarrow \varphi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow (\varphi \wedge \psi)))$	*(implied conjunction)*
(A10)	$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)))$	*(implying disjunction)*
(A11)	$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$	*(pre-linearity)*
(A12)	$(\varphi \wedge \neg\varphi) \rightarrow 0$	*(contradiction)*

Definition 1 (product logic ΠL) *We take the above 1-tautologies for axioms and allow modus ponens as the only deduction rule. (Various optimizations and redundancies are possible.) This logic will be called ΠL (product logic).*

Soundness of ΠL with respect to the above semantics is clear: it is very easy to check that any truth-evaluation respecting the above truth-functions evaluates each axiom to 1, and that modus ponens preserves 1-tautologies. The rest of the paper is devoted to prove that ΠL is also complete for 1-tautologies. The following two next lemmas show some interesting theorems of ΠL.

Lemma 2 *The following formulas are provable in ΠL:*

(1) $\varphi \odot \psi \to \varphi$, $\psi \odot \varphi \to \varphi$; $\varphi \to \varphi$,
$\varphi \odot 0 \to 0$, $\varphi \to (1 \odot \varphi)$, *(extremals)*
$\varphi \to (\psi \to (\varphi \odot \psi))$

(2) $(\varphi \leftrightarrow \psi) \to (\varphi \to \psi)$, $(\varphi \leftrightarrow \psi) \to (\psi \to \varphi)$,
$(\varphi \to \psi) \to ((\psi \to \varphi) \to (\varphi \leftrightarrow \psi))$
$(\varphi \leftrightarrow \psi) \to ((\varphi \to \chi) \leftrightarrow (\psi \to \chi))$
$(\varphi \leftrightarrow \psi) \to ((\chi \to \varphi) \leftrightarrow (\chi \to \psi))$
$(\varphi \leftrightarrow \psi) \to ((\varphi \odot \chi) \leftrightarrow (\psi \odot \chi))$

(3) $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$, $(\varphi \odot \psi) \to (\varphi \wedge \psi)$

(4) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$

(5) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

(6) $(\varphi \wedge (\psi \wedge \chi)) \leftrightarrow (\varphi \wedge \psi) \wedge \chi)$

(7) *commutativity and associativity of $\vee$ - analogous*

(8) $\varphi \leftrightarrow (\varphi \wedge \varphi)$, $(\varphi \vee \varphi) \leftrightarrow \varphi$ *(idempotence)*

(9) $\varphi \leftrightarrow ((\varphi \wedge \psi) \vee \varphi)$, $\varphi \leftrightarrow ((\varphi \vee \psi) \wedge \varphi)$

(10) $(\varphi \to \psi) \to (\varphi \leftrightarrow (\varphi \wedge \psi))$
$(\varphi \to \psi) \to (\varphi \leftrightarrow (\varphi \wedge \psi))$
$(\varphi \to \psi) \to (\varphi \leftrightarrow (\psi \odot (\psi \to \varphi))$
$(\varphi \to \psi) \to (\psi \leftrightarrow (\varphi \vee \psi)$

(11) $(\varphi \to \psi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi))$

(12) $(\varphi \to \psi) \to ((\varphi \vee \chi) \to (\psi \vee \chi))$

(13) $((\varphi \odot \psi) \to 0) \to ((\varphi \wedge \psi) \to 0)$
$(\varphi \odot \varphi \to 0) \to (\varphi \to 0)$

(14) $\neg 0 \leftrightarrow 1$; $\neg 1 \leftrightarrow 0$; $\neg\varphi \vee \neg\neg\varphi$.

(15) $\varphi \odot (\psi \vee \chi) \leftrightarrow (\varphi \odot \psi) \vee (\varphi \odot \chi)$

(16) *the same for $\odot, \wedge$*

(17) $(\varphi \vee \psi) \odot (\varphi \vee \psi) \to ((\varphi \odot \varphi) \vee (\psi \odot \psi))$

(18) $((\varphi \wedge \psi) \to \chi) \leftrightarrow ((\varphi \to \chi) \vee (\psi \to \chi))$;
$((\varphi \vee \psi) \to \chi) \leftrightarrow ((\varphi \to \chi) \wedge (\psi \to \chi))$

(19) $(\varphi \to (\psi \wedge \chi)) \leftrightarrow ((\varphi \to \psi) \wedge (\varphi \to \chi))$;
$(\varphi \to (\psi \vee \chi)) \leftrightarrow ((\varphi \to \psi) \vee (\varphi \to \chi))$

Proof:

(1) $\vdash \varphi \to (\psi \to \varphi)$ by (A1), thus $\vdash (\varphi \odot \psi) \to \varphi$ by (A6); furthermore, $\vdash (\psi \odot \varphi) \to \varphi$ by (A4). From this, $\vdash \psi \to (\varphi \to \varphi)$ by (A6), thus $\vdash (\varphi \to \varphi)$ (substitute an axiom for ψ). $\vdash \varphi \odot 0 \to 0$ follows from $\vdash (\varphi \odot \psi) \to \psi$. The proof of $\vdash \varphi \to (1 \odot \varphi)$ is as follows. By definition of $\wedge$, (A7) and (A3) we have, $\vdash ((\phi \to 1) \to 1) \to ((\phi \odot (\phi \to 1)) \to (\phi \odot 1))$. But by (A12) we have $\vdash (\phi \to 1) \to 1$. Therefore, by modus ponens, we prove $\vdash (\phi \wedge 1) \to (\phi \odot 1)$. Now, by (A2), (A4) and (A6) we have $\vdash (\phi \to (\phi \wedge 1)) \to (\phi \to (\phi \odot 1))$, but on the other hand, by (A9) we have $\vdash (\phi \to \phi) \to ((\phi \to 1) \to (\phi \to (\phi \wedge 1)))$. Thus, since $(\phi \to \phi)$ and $(\phi \to 1)$ are provable, by repeated application of modus ponens we have $\vdash (\phi \to (\phi \wedge 1))$, and finally by modus ponens with the previous implication and (A4) we have $\vdash \phi \to (1 \odot \phi)$. Finally, $\vdash \varphi \to (\psi \to (\varphi \odot \psi))$ follows from $(\varphi \odot \psi) \to (\varphi \odot \psi)$ by (A6).

(2) It follows from the definition of $\leftrightarrow$ using (1). (Exercise: prove that $\varphi \leftrightarrow \psi$ can be equivalently defined as $(\varphi \to \psi) \wedge (\psi \to \varphi)$.)

(3) $\vdash (\varphi \wedge \psi) \to \varphi$ is evident from the definition of $\wedge$; we show $\vdash (\varphi \wedge \psi) \to \psi$, i.e. $\vdash (\varphi \odot (\varphi \to \psi)) \to \psi$, i.e. $\vdash ((\varphi \to \psi) \odot \varphi) \to \psi$, i.e. $\vdash (\varphi \to \psi) \to (\varphi \to \psi)$, which is evident from (1). Finally, $\vdash \varphi \odot \psi \to \varphi \wedge \psi$ follows from (1) and (A9).

(4) We show $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$ (see (3)) and $\vdash \varphi \to ((\psi \to \varphi) \to \varphi)$ (from (A1)); thus $\vdash \varphi \to (\varphi \vee \psi)$ by (A9) and the definition of $\vee$. $\vdash \varphi \to (\psi \vee \varphi)$ follows from $\vdash (\varphi \vee \psi) \to (\psi \vee \varphi)$, which follows from the definition of $\vee$ and from (5).

(5) Use $\vdash ((\varphi \wedge \psi) \to \psi) \to (((\varphi \wedge \psi) \to \varphi) \to ((\varphi \wedge \psi) \to (\psi \wedge \varphi)))$ (i.e. (A9)) and modus ponens.

(6) Show $\vdash (\varphi \wedge (\psi \wedge \chi)) \to \varphi$, $\vdash (\varphi \wedge (\psi \wedge \chi)) \to \psi$, $\vdash (\varphi \wedge (\psi \wedge \chi)) \to \chi$ (using (3)); thus $\vdash (\varphi \wedge (\psi \wedge \chi)) \to (\varphi \wedge \psi)$ and $\vdash (\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)$ by double use of (A9).

(7) It is analogous to (6).

(8) Use $\vdash (\varphi \to \varphi) \to ((\varphi \to \varphi) \to (\varphi \to (\varphi \wedge \varphi)))$ (i.e. (A9)) and (1). For $\vee$ analogously using (A10).

(9) $\vdash \varphi \to ((\varphi \wedge \psi) \vee \varphi)$ by (4); conversely, use $\vdash \varphi \to \varphi$, $\vdash (\varphi \wedge \psi) \to \varphi$ and (A10).

(10) It suffices to show $\vdash (\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$; but by (A9) we have $\vdash (\varphi \to \varphi) \to ((\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi)))$. The second formula follows from the first using definition and commutativity of $\wedge$. The proof of the last formula is analogous.

(11) We have $\vdash (\varphi \to \psi) \to ((\varphi \wedge \chi) \to \psi)$, $\vdash (\varphi \wedge \chi) \to \chi$, further $\vdash ((\varphi \wedge \chi) \to \psi) \to (((\varphi \wedge \chi) \to \chi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi)))$, thus $\vdash (\varphi \to \psi) \to (((\varphi \wedge \chi) \to \chi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi)))$, thus $\vdash ((\varphi \wedge \chi) \to \chi) \to ((\varphi \to \psi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi)))$, hence (11) is provable.

(12) dually from (11).

(13) We want to prove $((\varphi \odot \psi) \to 0) \to ((\varphi \wedge \psi) \to 0)$, i.e. $(\varphi \to (\psi \to 0)) \to ((\varphi \wedge \psi) \to 0)$, i.e. $(\varphi \to \neg\psi) \to ((\varphi \wedge \psi) \to 0)$, thus finally $((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to 0$. The following chains of implications are provable.
$((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to [(\varphi \to \neg\psi) \odot \varphi] \to \neg\psi$,
$((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to [(\varphi \to \neg\psi) \odot \psi] \to \psi$, thus
$((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to [\psi \wedge \neg\psi] \to 0$.

(14) We have $\vdash \neg 0 \to 1$ by (A3); conversely, $\neg 0$ is $0 \to 0$, hence $\vdash \neg 0$, thus $\vdash 1 \to \neg 0$. Furthermore $\vdash 0 \to \neg 1$ by (A3); conversely, $\vdash \neg 1 \to (1 \to 0) \to [1 \odot (1 \to 0)] \to [1 \wedge (1 \to 0)] \to 0$ by (1), (3) and (A12). We prove $\neg\varphi \vee \neg\neg\varphi$; during the proof we prove some other important formulas with the negation, namely

(a) $(\varphi \to \neg\varphi) \to \neg\varphi$,
(b) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$,
(c) $\varphi \to \neg\neg\varphi$,
(d) $\neg\neg\neg\varphi \to \neg\varphi$.

(a) follows from $\vdash ((\varphi \odot \varphi) \to 0) \to (\varphi \to 0)$, i.e. $\vdash (\varphi \to (\varphi \to 0)) \to (\varphi \to 0)$, which is $\vdash (\varphi \to \neg\varphi) \to \neg\varphi$. (b) follows from transitivity: $\vdash (\varphi \to \psi) \to ((\psi \to 0) \to (\varphi \to 0))$. (c) follows from $\vdash (1 \to \varphi) \to (\neg\varphi \to 0)$. (d) follows from (c) and (b). Now observe the following provable chains of implications:

$$(\neg\varphi \to \neg\neg\varphi) \to \neg\neg\varphi \to (\neg\varphi \vee \neg\neg\varphi)$$

$$(\neg\neg\varphi \to \neg\varphi) \to (\neg\neg\varphi \to \neg\neg\neg\varphi) \to \neg\neg\neg\varphi \to \neg\varphi \to (\neg\varphi \vee \neg\neg\varphi)$$

Thus $\vdash \neg\varphi \vee \neg\neg\varphi$ using (A10) and (A11).

(15) $\vdash ((\varphi \odot \chi) \vee (\psi \odot \chi)) \to ((\varphi \vee \psi) \odot \chi)$ using (A7), (A10) and (4); we prove the converse implication. This means to prove

$$\vdash ((\varphi \vee \psi) \odot \chi) \to [((\varphi \odot \chi) \to (\psi \odot \chi)) \to (\psi \odot \chi)]$$

and the same formula with φ, ψ in [...] exchanged (by (A9) and the definition of $\vee$). After obvious transformations we have to prove

$$\vdash ((\varphi \odot \chi) \to (\psi \odot \chi)) \to [((\varphi \vee \psi) \odot \chi) \to (\psi \odot \chi)].$$

Denote this formula by (*). We use a proof by cases $\neg\chi, \neg\neg\chi$.
$\vdash \neg\chi \rightarrow (\chi \rightarrow 0) \rightarrow [((\varphi \vee \psi) \odot \chi) \rightarrow 0] \rightarrow [((\varphi \vee \psi) \odot \chi) \rightarrow (\psi \odot \chi)] \rightarrow (*)$,
thus $\vdash \neg\chi \rightarrow (*)$. On the other hand,
$\vdash \neg\neg\chi \rightarrow [((\varphi \odot \chi) \rightarrow (\psi \odot \chi)) \rightarrow (\varphi \rightarrow \psi)] \rightarrow [((\varphi \odot \chi) \rightarrow (\psi \odot \chi)) \rightarrow ((\varphi \vee \psi) \leftrightarrow \psi)] \rightarrow (*)$;
thus $\vdash (\neg\chi \vee \neg\neg\chi) \rightarrow (*)$ and hence $\vdash (*)$.

(16) $\vdash (\varphi \wedge \psi) \odot \chi \rightarrow ((\varphi \odot \chi) \wedge (\psi \odot \chi))$ by (3) and (A9); we prove the converse, i.e. $\vdash ((\varphi \odot \chi) \wedge (\psi \odot \chi)) \rightarrow (\varphi \wedge \psi) \odot \chi$, i.e.
$\vdash [(\varphi \odot \chi) \odot ((\varphi \odot \chi) \rightarrow \psi \odot \chi)] \rightarrow ((\varphi \wedge \psi) \odot \chi)$, or
$\vdash ((\varphi \odot \chi) \rightarrow (\psi \odot \chi)) \rightarrow [(\varphi \odot \chi) \rightarrow ((\varphi \wedge \psi) \odot \chi)]$.
This is proved by cases $\neg\chi$, $\neg\neg\chi$ as in (15).

(17) We write φ^2 for $\varphi \odot \varphi$ etc.; we have to prove $(\varphi \vee \psi)^2 \leftrightarrow (\varphi^2 \vee \psi^2)$. Now, by (15), $\vdash (\varphi \vee \psi)^2 \leftrightarrow (\varphi^2 \vee \psi^2 \vee \varphi \odot \psi)$, thus it suffices to prove $\vdash (\varphi \odot \psi) \rightarrow (\varphi^2 \vee \psi^2)$ to get $(\varphi^2 \vee \psi^2 \vee \varphi \odot \psi) \leftrightarrow \varphi^2 \vee \psi^2$.
Prove by cases $(\varphi \rightarrow \psi)$, $(\psi \rightarrow \varphi)$:
$\vdash (\varphi \rightarrow \psi) \rightarrow ((\varphi \odot \psi) \rightarrow \psi^2) \rightarrow ((\varphi \odot \psi) \rightarrow (\varphi^2 \vee \psi^2)$ and dually;
thus $\vdash [(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)] \rightarrow [(\varphi \vee \psi)^2 \leftrightarrow (\varphi^2 \vee \psi^2)]$, which gives the result.

(18) We only prove the first formula. It suffices to show $\vdash ((\varphi \wedge \psi) \rightarrow \chi) \rightarrow ((\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi))$, since in the other direction is straightforward by (A10). We prove it by cases $\varphi \rightarrow \psi$, $\psi \rightarrow \varphi$. The following implications are provable:

$(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\varphi \wedge \psi))$

$(\varphi \rightarrow (\varphi \wedge \psi)) \rightarrow (((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,

$(\varphi \rightarrow \psi) \rightarrow (((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$, and thus

$(\varphi \rightarrow \psi) \rightarrow (((\varphi \wedge \psi) \rightarrow \chi) \rightarrow ((\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)))$
Analogously, one proves $(\psi \rightarrow \varphi) \rightarrow (((\varphi \wedge \psi) \rightarrow \chi) \rightarrow ((\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)))$.

(19) The proofs are very similar to (18). In one direction it is easy from (A9) and (A10) respectively. The converse implications are proved by cases $\chi \rightarrow \psi$, $\psi \rightarrow \chi$.

This completes the proof. □

Lemma 3 *For each n, ΠL proves*

$$(\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n,$$

where α^n is $\alpha \odot \ldots \odot \alpha, n$ times.

Proof: First show $\vdash \varphi \to (\varphi \to (\varphi \odot \varphi))$ (using residuation, adding assumption); thus, $\vdash [(\varphi \to \psi) \vee (\psi \to \varphi)]^2$. By distributivity (cf. Lemma 2 (17)) we get $(\varphi \to \psi)^2 \vee (\psi \to \varphi)^2$. Iterate and use $\vdash ((\varphi \to \psi)^{n+1} \vee (\psi \to \varphi)^{n+1}) \to ((\varphi \to \psi)^n \vee (\psi \to \varphi)^n)$ (easy from $\vdash \varphi^{n+1} \to \varphi^n$ and Lemma 2 (12), used doubly). □

3 Product Algebras

To prove completeness for our product logic ΠL, we next define what we call *product algebras* and show that they play an analogous role for ΠL as *MV-Algebras* do for the infinitely many-valued Łukasiewicz's logic (see for instance [5]). Namely, one can prove that the quotient algebra of classes of equivalent ΠL-formulas is a product algebra. Moreover, the unit interval equipped with the truth functions of product logic is an special linearly ordered product algebra because every valid identity there can be shown to be also valid in every linearly ordered product algebra. Then completeness for 1-tautologies comes from the fact that each product algebra is a subdirect product of linearly ordered product algebras.

Definition 2 *A* product algebra *is an algebra* $\mathcal{A} = \langle A, \odot, \to, 0, 1\rangle$ *such that, defining*

$$\begin{aligned} x \wedge y &= x \odot (x \to y) \\ x \vee y &= ((x \to y) \to y) \wedge ((y \to x) \to x) \\ \neg x &= x \to 0 \end{aligned}$$

the following conditions are satisfied:

- $\mathcal{A} = \langle A, \odot, \to, \wedge, \vee, 0, 1\rangle$ *is a* residuated lattice *(see [14]), i.e.*
  - **(i)** $\langle A, \wedge, \vee, 0, 1\rangle$ *is a lattice* ($x \leq y \equiv x \wedge y = x \equiv x \vee y = y$)*, 1 and 0 are the top and bottom elements respectively,*
  - **(ii)** $\langle \odot, \to \rangle$ *is an* adjoint couple *on A,that is:*
    1. $\odot$ *and* $\to$ *are binary operations on A,*
    2. $\odot$ *is non-decreasing in both variables,*
    3. $\to$ *is non-increasing in the first variable and non-decreasing in the second one, and*
    4. *the adjointness condition* $x \leq (y \to z)$ *iff* $(x \odot y) \leq z$ *holds.*
  - **(iii)** $\langle A, \odot, 1\rangle$ *is a commutative monoid, that is,* $\odot$ *is associative and commutative, and* $1 \odot x = x$
- $((x \to y) \vee (y \to x)) = 1$ *(pre-linearity)*
- $\neg\neg z \odot (x \odot z \to y \odot z) \leq (x \to y)$ *(pre-cancellation)*
- $x \wedge \neg x = 0$ *(bottom)*

- $x \odot (y \vee z) \leq (x \odot y) \vee (x \odot z)$, *(distributive laws)*
  $x \odot (y \wedge z) \geq (x \odot y) \wedge (x \odot z)$

Lemma 4 *Any product algebra satisfies the following properties:*
(i) $x \leq y$ *iff* $(x \to y) = 1$
(ii) $x \leq y$ *implies* $x = y \odot (y \to x)$
(iii) $\neg 0 = 1$, $\neg 1 = 0$
Furthermore, if the algebra is linearly ordered it also sastisfie:
(iv) $\neg x = 0$, *for* $x > 0$
(v) if $z \neq 0$ *then* $x \odot z = y \odot z$ *iff* $x = y$
(vi) if $z \neq 0$ *then* $x \odot z < y \odot z$ *iff* $x < y$ *(*$x < y$ *means* $x \leq y$ *and* $x \neq y$*).*

Proof: (i) It is true in each residuated lattice.
(ii) It follows from the definition of $\wedge$ from $\odot, \to$.
(iii) It easily follows from (i) and (bottom).
(iv) Here $\wedge$ is minimum and $x \wedge \neg x = 0$ and thus, if $x > 0$ then $\neg x = 0$.
(v) If $z \neq 0$ then $\neg\neg z = 1$, thus if $x \odot z \leq y \odot z$ then $(x \odot z \to y \odot z) = 1$, thus $x \to y = 1$, $x \leq y$. This gives also (vi). □

Remark. Observe that the above definition of product algebras, even if cumbersome, can be written as a sequence of (universally quantified) Horn clauses. Therefore the class of product algebras is a quasivariety and thus it is closed under direct products and subalgebras (see e.g. [6]).

Notice also that the only finite linearly ordered product algebra is the two elements boolean algebra $\{0, 1\}$. This is an easy consequence of the fact that in any finite chain the above cancellation property forbids the identification of meet and product. Moreover, any boolean algebra is a product algebra, taking product as meet, since it is a subdirect product of copies of $\{0, 1\}$.

For our purpoes, the most interesting and notorious examples of product algebras are given in the following lemma.

Lemma 5
(i) The unit interval with truth functions is a linearly ordered product algebra.
(ii) The algebra of classes of equivalent formulas in ΠL *is a product algebra (not linearly ordered.)*

Next lemma shows that every provable formula in ΠL is interpreted in any product algebra as its top element.

Lemma 6 *For each formula* φ*, replace each propositional variable* p_i *by an object variable* x_i*; you get a term* τ_φ *of the language of product algebras. If* φ *is provable then the identity* $\tau_\varphi = 1$ *holds in each product algebra.*

Proof: Verify that $\tau_\varphi = 1$ is a true identity of product algebras for each axiom (A1) - (A12) and observe that in each product algebra if $x = 1$ and $x \to y = 1$, i.e. $1 \le x \to y$, thus $x = 1 \odot x \le y$, then $y = 1$; thus modus ponens preserves 1-tautologicity of our product algebras.

We only verify (A6), (A9), (A10). Now $(x \odot y) \to z \le x \to (y \to z)$ iff $x \odot ((x \odot y \to z) \le y \to z$ iff $x \odot y \odot ((x \odot y) \to z) \le z$, and the last inequality is true $(a \odot (a \to b) \le b)$.Conversely, to show $x \to (y \to z) \le (x \odot y) \to z$ show

$$x \odot y \odot (x \to (y \to z)) = y \odot x \odot (x \to (y \to z)) \le y \odot (y \to z) \le z.$$

this proves (A6).

We verify (A10), i.e. $(x \to z) \odot (y \to z) \le (x \vee y) \to z$. Compute: $(x \vee y) \odot ((x \to z) \odot (y \to z)) = (x \odot (x \to z) \odot (y \to z)) \vee (y \odot (y \to z) \odot (x \to z)) \le z \odot (y \to z) \vee z \odot (x \to z) \le z \vee z = z$. (distributivity used).

Finally, we prove (A9), i.e. $(z \to x) \odot (z \to y) \le z \to (x \wedge y)$.
$z \odot (z \to x) \odot (z \to y) \le x \odot (z \to y) \le x$, analogously $z \odot (z \to x) \odot (z \to y) \le y$, thus $z \odot (z \to x) \odot (z \to y) \le x \wedge y$. Note that the distributivity for $\odot, 1$ appears to be redundant. □

Next step is to prove that each product algebra is a subdirect product of linearly ordered product algebras.

Definition 3 *Let $\mathcal{A}$ be a product algebra. A* filter *is a set $F \subseteq A$ such that, for each $a, b \in A$:*

$$a \in F \text{ and } b \in F \text{ implies } a \odot b \in F$$
$$a \in F \text{ and } a \le b \text{ implies } b \in F.$$

Furthermore, F is an ultrafilter *iff for each $c, d \in A$,*

$$(c \to d) \in F \text{ or } (d \to c) \in F.$$

Lemma 7 *Let $\mathcal{A}$ be a product algebra and let F be a filter. Define the corresponding equivalence*

$$a \sim_F b \text{ iff } (a \to b) \in F \text{ and } (b \to a) \in F,$$

i.e. $a \sim_F b$ iff $(a \leftrightarrow b) \in F$. Then:
(1) $\sim_F$ is a congruence, and therefore the corresponding quotient algebra $\mathcal{A}/{\sim_F}$ is a product algebra too.
(2) $\mathcal{A}/{\sim_F}$ is linearly ordered iff F is an ultrafilter.

Proof: As usual. □

Theorem 1 *Let $\mathcal{A}$ be a product algebra and $a \in A$, $a \ne 1$. Then there is an ultrafilter F on A not containing a.*

Proof: Start with $F_0 = \{1\}$ $(a \notin F_0)$ and succesively process all pairs $(c \to d), (d \to c)$; if F is a filter not containing $(c \to d), (d \to c)$ then create F_1, F_2 by adjoining these elements, i.e. $F_1 = \{u \mid (\exists v \in F)(\exists n \text{ natural })(v \odot (c \to d)^n \leq u)\}$, and similarly for F_2 and $(d \to c)$. We show next that either $a \notin F_1$ or $a \notin F_2$. If $a \in F_1$ and $a \in F_2$ then, for some $v \in F$ and n natural, $v \odot (c \to d)^n \leq a$ and $v \odot (d \to c)^n \leq a$, thus $a \geq (v \odot (c \to d)^n) \vee (v \odot (d \to c)^n) = v \odot ((c \to d)^n \vee ((d \to c)^n) = v \odot 1 = v$, thus $a \in F$, contradiction. □

Corollary 1 *Each product algebra is a subdirect product of linearly ordered product algebras.*

Proof: From last theorem, it is easy to check that, for any algebra $\mathcal{A}$, $\cap\{F \mid F$ is an ultrafilter of $\mathcal{A}\} = \{1\}$. Therefore, the intersection of the corresponding congruences $\sim_F$ is just the minimum congruence, that is, the identity. The corollary then comes from Lemma 7 (2) by applying the standard result about subdirect products of algebras (see e.g. [2]) saying that if the intersection of a family of congruences of some algebra A is the minimum congruence, A is a subdirect product of the corresponding quotient algebras. □

Finally, before proving completeness of ΠL we need some results relating linearly ordered product algebras and ordered abelian groups.

Theorem 2 *Let $\mathcal{A}$ be a linearly ordered product algebra. Then there is an ordered abelian group $\mathcal{G} = (G, +_G, 0_G, \leq_G)$ and an isomorphism ι of the non-positive part $N = \{g \in G, g \leq 0_G\}$ and $A - \{0\}$, i.e. a one-to-one mapping from N to $A - \{0\}$ such that*

$$\begin{gathered} \iota(0_G) = 1 \\ \iota(g +_G h) = \iota(g) \odot \iota(h) \\ g \leq_G h \textit{ iff } \iota(g) \leq \iota(h) \end{gathered}$$

Proof: (Hint) $A - \{0\}$ is a linearly ordered semigroup with cancellation whose neutral element is maximal; thus it is the non-positive part of an ordered abelian group as desired. □

Theorem 3 *If an identity $\tau = \sigma$, in the language of product algebras, is valid in the unit interval algebra then it is valid in all linearly ordered product algebras.*

Proof: (Hint) To transfer a counterexample from an arbitrary product algebra $\mathcal{A}$ into $[0, 1]$ use the theorem (see e.g. [9]) saying that if $\mathcal{G}$ is an ordered abelian group and D a finite subset of it, then there is a finite subset D' of the multiplicative group $(0, +\infty)$ of the positive reals and an isomorphism between the subgroups which are finitely generated by D' and D respectively. Clearly, if all elements of D are non-positive in $\mathcal{G}$ then $D' \subseteq (0, 1]$. Take extra care of the zero element of $\mathcal{A}$ and send it to 0. □

Theorem 4 (Completeness) *If φ is a 1-tautology then φ is provable in ΠL.*

Proof: If φ is a 1-tautology then the corresponding identity $\varphi = 1$ is valid in $[0,1]$, hence in all linearly ordered product algebras (due to theorem 3), hence in *all* product algebras (due to Corollary 1), hence, in particular, in the algebra of classes of equivalent formulas of ΠL , hence $\Pi L \vdash \varphi \leftrightarrow 1$, hence $\Pi L \vdash \varphi$. $\square$

4 Concluding remarks

The next problem is: what happens if we add the "classical" fuzzy negation $1 - x$ and thus the MYCIN disjunction $x + y - xy$? Note that a completeness theorem for a fuzzy logic containing all these, but with respect to an infinitary system, is contained in [19]. Another problem reads: how far can we develop a "graded" variant of product logic, similar to Pavelka's logic, cf. [14, 7] ? Complete analogy is impossible since our implication is not continuous in [0, 1] (see [14]), but a partial analogy seems possible. This is a subject of current research.

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