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SINGULARLY PERTURBED SEMI-LINEAR BOUNDARY VALUE PROBLEM WITH DISCONTINUOUS FUNCTION*

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Abstract A class of singularly perturbed semi-linear boundary value problems with discontinuous functions is examined in this article. Using the boundary layer function method, the asymptotic solution of such a problem is given and shown to be uniformly effective. The existence and uniqueness of the solution for the system is also proved. Numerical result is presented as an illustration to the theoretical result.

Key words Singular perturbation; asymptotic expansion; boundary layer function; invariant manifold

2000 MR Subject Classification 34B16

1 Introduction

Many articles studied the nonlinear singularly perturbed boundary value problems with continuous functions on the right side of the equation involved [1, 2]. While in practical application, we often encounter discontinuous functions in boundary value problems. For example, in convention-diffusion problem. We want to solve such problems. Article [3] studied the numerical method of the two-point boundary value problem of the form

$$\begin{cases} \mathrm{L}_\epsilon u_\epsilon(x) = -\epsilon {u_\epsilon}''(x) + a(x)u_\epsilon(x) = f(x), \quad x \in \Lambda = (0,1), \\ u_\epsilon(0) = u_0, \quad u_\epsilon(1) = u_1, \\ f(d_-) \neq f(d_+), \quad d \in (0,1), \quad a(x) > a^2 > 0, \quad a > 0. \end{cases}$$

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This article deals with the asymptotic solutions of the semi-linear problem.

$$\epsilon^2 \frac{d^2 y}{dx^2} = F(y,x), \quad 0 < x < 1, \tag{1.1}$$

$$y(0,\epsilon) = y^0, \quad y(1,\epsilon) = y^1. \tag{1.2}$$

where both y^0 and y^1 are n-dimensional vectors, y is an n-dimensional vector function, $F(y,x)$ is an $n \times n$ nonsingular matrix function.

We suppose that the following three assumptions are satisfied:

H1 $F(y,x)$ is sufficiently smooth in y and has a discontinuous point x_0 of the first kind on the interval $[0,1]$.

H2 Equation $F(y,x) = 0$ has a discontinuous isolated solution on the interval $[0,1]$.

$$y = \varphi(x) = \begin{cases} \varphi^{(-)}(x) & 0 \le x < x_0, \\ \varphi^{(+)}(x) & x_0 < x \le 1, \end{cases} \tag{1.3}$$

complementarily defining

$$\varphi^{(-)}(x_0) \triangleq \varphi^{(-)}(x_0 - 0), \quad \varphi^{(+)}(x_0) \triangleq \varphi^{(+)}(x_0 + 0).$$

H3 All the n eigenvalues of $F_y(\varphi(x), x)$ are neither zero or negative real numbers.

In the following, we would prove that, for problem (1.1)–(1.2), there exists an inner layer from $\varphi^-(x)$ to $\varphi^+(x)$ at $x = x_0$ as well as boundary layers at $x = 0$ and $x = 1$. This step-type contrast structure can be viewed as a smooth connection of two pure boundary-layer solutions of the problems:

$P^{(-)}$ $(0 \le x \le x_0)$

$$\epsilon \frac{dy^{(-)}}{dx} = z^{(-)}, \quad \epsilon \frac{dz^{(-)}}{dx} = F(y^{(-)}, x), \tag{1.4}$$

$$y^{(-)}(0,\epsilon) = y^0, \quad y^{(-)}(x_0,\epsilon) = \gamma^*, \tag{1.5}$$

$P^{(+)}$ $(x_0 \le x \le 1)$

$$\epsilon \frac{dy^{(+)}}{dx} = z^{(+)}, \quad \epsilon \frac{dz^{(+)}}{dx} = F(y^{(+)}, x), \tag{1.6}$$

$$y^{(+)}(x_0,\epsilon) = \gamma^*, \quad y^{(+)}(1,\epsilon) = y^1, \tag{1.7}$$

where $\gamma^* = \gamma_0 + \epsilon\gamma_1 + \epsilon^2\gamma_2 + \cdots$, $\gamma_k, k \ge 0$ are undetermined vector parameters.

The boundary conditions (1.5) and (1.7) ensure that solutions of $P^{(-)}$ and $P^{(+)}$ are continuous at $x = x_0$. The smoothness is guaranteed under an extra condition

$$\frac{dy^{(-)}}{dx}(x_0,\epsilon) = \frac{dy^{(+)}}{dx}(x_0,\epsilon). \tag{1.8}$$

In the next section, we will see that condition (1.8) is exactly the equation determining parameters $\gamma_k, k \ge 0$.

2 Construction of the Formal Asymptotic Solution

We seek solutions of $P^{(\pm)}$ in the form

$$w^{(-)}(x,\epsilon)=\overline{w}^{(-)}(x,\epsilon)+\Pi w(\tau_0,\epsilon)+Qw^{(-)}(\tau,\epsilon),\quad 0\le x\le x_0, \tag{2.1}$$

$$w^{(+)}(x,\epsilon)=\overline{w}^{(+)}(x,\epsilon)+Qw^{(+)}(\tau,\epsilon)+Rw(\tau_1,\epsilon),\quad x_0\le x\le 1, \tag{2.2}$$

where $w=(z,y)^T$,

$$\overline{w}^{(\pm)}(x,\epsilon)=\overline{w}_0^{(\pm)}(x)+\epsilon\overline{w}_1^{(\pm)}(x)+\cdots \tag{2.3}$$

are regular series,

$$\Pi w(\tau_0,\epsilon)=\Pi_0 w(\tau_0)+\epsilon\Pi_1 w(\tau_0)+\cdots,\quad \tau_0=x/\epsilon, \tag{2.4}$$

is a left boundary series in a neighborhood of $x=0$,

$$Rw(\tau_1,\epsilon)=R_0 w(\tau_1)+\epsilon R_1 w(\tau_1)+\cdots,\quad \tau_1=(x-1)/\epsilon, \tag{2.5}$$

is a right boundary series in a neighborhood of $x=1$, and

$$Qw^{(\pm)}(\tau,\epsilon)=Q_0 w^{(\pm)}(\tau)+\epsilon Q_1 w^{(\pm)}(\tau)+\cdots,\quad \tau=(x-x_0)/\epsilon, \tag{2.6}$$

are inner layer series in a neighborhood of $x=x_0$, $Qw^{(-)}$corresponds to $\tau\le 0$ whereas $Qw^{(+)}$ corresponds to $\tau\ge 0$.

The coefficients of asymptotic solutions (2.1) and (2.2) are determined by the boundary layer function method. We consider first the solution of problem $P^{(-)}$ by rewriting function $F(y,x)$ as $F(y,x)=F(\overline{y}^{(-)}(x),x)+\Pi F(\tau_0,\epsilon)+QF^{(-)}(\tau,\epsilon)$, where

$$QF^{(-)}(\tau,\epsilon)=F(\overline{y}^{(-)}(x_0+\tau\epsilon)+Qy^{(-)}(\tau),x_0+\tau\epsilon)-F(\overline{y}^{(-)}(x_0+\tau\epsilon),x_0+\tau\epsilon),$$

$$\Pi F(\tau_0,\epsilon)=F(\overline{y}^{(-)}(\tau_0\epsilon)+Qy^{(-)}+\Pi y(\tau_0),\tau_0\epsilon)-F(\overline{y}^{(-)}+Qy^{(-)},\tau_0\epsilon).$$

We thereby reduce the problem $P^{(-)}$ to three problems, namely, regular problem $\overline{y}^{(-)}$, left boundary layer problem Πy, and right boundary layer problem $Qy^{(-)}$.

The equation of regular problem $P^{(-)}$ is

$$\epsilon\frac{d\overline{y}^{(-)}}{dx}=\overline{z}^{(-)},\quad \epsilon\frac{d\overline{z}^{(-)}}{dx}=F(\overline{y}^{(-)},x). \tag{2.7}$$

Comparing the same power of ϵ, we obtain

$$F(\overline{y}_0^{(-)},x)=0,\quad \overline{z}_0^{(-)}(x)=0, \tag{2.8}$$

$$\overline{z}_{2k-1}^{(-)}=\frac{d\overline{y}_{2k-2}^{(-)}}{dx},\quad k\ge 1, \tag{2.9}$$

$$\overline{y}_{2k}^{(-)}=F_y(\overline{y}_0^{(-)},x))^{-1}h_k(\overline{y}_0^{(-)},\overline{z}_1^{(-)},\cdots,\overline{y}_{2k-2}^{(-)},\overline{z}_{2k-1}^{(-)}),k\ge 1, \tag{2.10}$$

$$\overline{y}_{2k-1}^{(-)}\equiv 0,\quad \overline{z}_{2k}^{(-)}\equiv 0,\quad k\ge 1. \tag{2.11}$$

So, we obtain $\overline{y}_0^{(-)}=\varphi^{(-)}(x)$ and $\overline{y}_k^{(-)}(x),\overline{z}_k^{(-)}(x)$ for $k\ge 1$.

Secondly, we consider the $Qy^{(-)}$ problem

$$\frac{\mathrm{d}^2Qy^{(-)}}{\mathrm{d}\tau^2}=F(\overline{y}^{(-)}+Qy^{(-)},x_0+\tau\epsilon)-F(\overline{y}^{(-)},x_0+\tau\epsilon), \tag{2.12}$$

$$Qy^{(-)}(0)=\gamma^*-\overline{y}^{(-)}(x_0),\quad Qy^{(-)}(-\infty)=0,\quad Qz^{(-)}(-\infty)=0. \tag{2.13}$$

Comparing the ϵ^0 of both sides of (2.12), (2.13) and rewriting (2.12) in equivalent equations

$$\frac{\mathrm{d}Q_0y^{(-)}}{\mathrm{d}\tau}=Q_0z^{(-)},\frac{\mathrm{d}Q_0z^{(-)}}{\mathrm{d}\tau}=F(\varphi^{(-)}(x_0)+Q_0y^{(-)}(\tau),x_0), \tag{2.14}$$

$$Q_0y^{(-)}(0)=\gamma_0-\varphi^{(-)}(x_0),Q_0y^{(-)}(-\infty)=0,Q_0z^{(-)}(-\infty)=0. \tag{2.15}$$

System (2.14) has a singular point $(0,0)$ and its corresponding linear system is $\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}Q_0y^{(-)}\\ Q_0z^{(-)}\end{pmatrix}=A\begin{pmatrix}Q_0y^{(-)}\\ Q_0z^{(-)}\end{pmatrix}$, where $A=\begin{pmatrix}0 & I_n\\ F_y(\varphi^{(-)}(x_0),x_0) & 0\end{pmatrix}$.

The characteristic polynomial of matrix A is

$$\begin{aligned}|A-\lambda I_n|&=\begin{vmatrix}-\lambda I_n & I_n\\ F_y(\varphi^{(-)}(x_0),x_0) & -\lambda I_n\end{vmatrix}=\begin{vmatrix}0 & I_n\\ F_y(\varphi^{(-)}(x_0),x_0)-\lambda^2I_n & -\lambda I_n\end{vmatrix}\\ &=(-1)^n|I_n||\ F_y(\varphi^{(-)}(x_0),x_0)-\lambda^2I_n|.\end{aligned} \tag{2.16}$$

(2.16) tells us that, if λ is a characteristic root of $F_y(\varphi^{(-)}(x_0),x_0)$, then $\pm\sqrt{\lambda}$ are both characteristic roots of A. According to **H3**, each of the characteristic root of $F_y(\varphi^{(-)}(x_0),x_0)$ is neither zero nor a negative real number, matrix A therefore has n characteristic roots with positive real part and n characteristic roots with negative real part. Hence, in the neighborhood of the singular point $(0,0)$, system (2.14) has an n-dimensional stable differential manifold and an n-dimensional unstable differential manifold denoted as $U:Q_0y^{(-)}=\Phi_1(Q_0z^{(-)})$. When $\frac{\partial\Phi_1}{\partial Q_0z}\neq 0$, U can also be written as $U:Q_0z^{(-)}=\Psi_1(Q_0y^{(-)})$ with domain G_1.

H4 $Q_0y^{(-)}(0)=\gamma_0-\varphi^-(x_0)\in G_1$.

Under the assumption **H4**, the zero-order approximation problem (2.14)–(2.15) has solutions $Q_0y^-(\tau),Q_0z^-(\tau)$. They decay exponentially as $\tau\to-\infty$.

Let us proceed to the analysis of the first-order approximation:

$$\frac{\mathrm{d}Q_1y^{(-)}}{\mathrm{d}\tau}=Q_1z^{(-)}, \tag{2.17}$$

$$\frac{\mathrm{d}Q_1z^{(-)}}{\mathrm{d}\tau}=F_y(\varphi^{(-)}(x_0)+Q_0y^{(-)}(\tau),x_0)Q_1y^{(-)}(\tau)+g_1(\tau), \tag{2.18}$$

$$Q_1y^{(-)}(0)=\gamma_1,\quad Q_1y^{(-)}(-\infty)=0,\quad Q_1z^{(-)}(-\infty)=0, \tag{2.19}$$

where

$$\begin{aligned}g_1(\tau)=&(F_y(\varphi^{(-)}(x_0)+Q_0y^{(-)}(\tau),x_0)-F_y(\varphi^{(-)}(x_0),x_0))(\varphi^{(-)'}(x_0)\tau)\\ &+(F_x(\varphi^{(-)}(x_0)+Q_0y^{(-)}(\tau),x_0)-F_x(\varphi^{(-)}(x_0),x_0))\tau.\end{aligned}$$

Equations (2.17)–(2.18) are linear differential equations. Their corresponding homogeneous linear differential equation has a solution $\frac{\mathrm{d}Q_0y^{(-)}}{\mathrm{d}\tau}$. According to [2] Lemma 4.5, when $Q_1z^{(-)}(0)$

takes value $\frac{\partial \Psi_1}{\partial Q_0 y}\big|_{\tau=0} \cdot Q_1 y^{(-)}(0) + \delta_1$, where δ_1 is a determined vector by integrating known functions $Q_0 y^-, \Psi_1, g_1$, problem (2.17)–(2.19) has a unique solution decaying exponentially.

By analogy with the first approximation, we claim that the kth-order approximation problem has exponentially decayed solution when $Q_k z^{(-)}(0)$ takes value $\frac{\partial \Psi_1}{\partial Q_0 y}\big|_{\tau=0} \cdot Q_k y^{(-)}(0) + \delta_k$, where δ_k is a determined vector.

Thirdly, we consider the Πy problem. Because $Q_k y^{(-)} (k \geq 0)$ decays exponentially, ΠF is equivalent to $F(\overline{y}^{(-)}(\tau_0 \epsilon) + \Pi y(\tau_0), \tau_0 \epsilon) - F(\overline{y}^{(-)}(\tau_0 \epsilon), \tau_0 \epsilon)$, when comparing the same order of ϵ. The analysis of the left boundary series $\sum\limits_{k=0}^{n} \Pi_k y(\tau_0)$ is analogous to $\sum\limits_{k=0}^{n} Q_k y^{(-)}(\tau)$.

Now, we have calculated the approximation series of $\overline{y}^{(-)}, \Pi y(\tau_0)$, and $Q y^{(-)}(\tau)$, while the expression of $Q y^{(-)}(\tau)$ still has undetermined parameters γ_k $(k \geq 0)$.

As to the problem $P^{(+)}$, the analysis of $\overline{y}^{(+)}, Q y^{(+)}(\tau)$, and $R y^{(+)}(\tau_1)$ is similar to that in $P^{(-)}$. So, we will not calculate them concretely. We would concentrate our attention to how to determine the parameters γ_k $(k \geq 0)$.

For the zero-order approximation of $Q y^{(+)}(\tau)$,we have the problem

$$\frac{\mathrm{d} Q_0 y^{(+)}}{\mathrm{d}\tau} = Q_0 z^{(+)}, \frac{\mathrm{d} Q_0 z^{(+)}}{\mathrm{d}\tau} = F(\varphi^{(+)}(x_0) + Q_0 y^{(+)}(\tau), x_0), \tag{2.20}$$

$$Q_0 y^{(+)}(0) = \gamma_0 - \varphi^{(+)}(x_0), \quad Q_0 y^{(+)}(+\infty) = 0, \quad Q_0 z^{(+)}(+\infty) = 0. \tag{2.21}$$

Under the condition **H3**, there exists an n-dimensional stable differential manifold S, denoted as $Q_0 z^{(+)} = \Phi_2(Q_0 y^{(+)})$ with the domain of G_2.

H5 $Q_0 y^{(+)}(0) = \gamma_0 - \varphi^{(+)}(x_0) \in G_2$.

If **H5** is satisfied and $Q_0 z^{(+)}(0)$ takes the value of $\Phi_2(\gamma_0 - \varphi^{(+)}(x_0)$, problem (2.20)–(2.21) has a unique solution $Q_0 y^{(+)}(\tau), Q_0 z^{(+)}(\tau)$, decaying exponentially as $\tau \to +\infty$. For the kth-order approximation $Q_k y^{(+)}(\tau)$, we can calculate that if $Q_k z^{(+)}(0)$ takes the value $\frac{\partial \Phi_2}{\partial Q_0 y^{(+)}} |_{\tau=0} \cdot Q_k y^{(+)}(0) + \sigma_k$ (where σ_k is a definite vector concerning k), then, $Q_k y^{(+)}(\tau)$ and $Q_k z^{(+)}(\tau)$ exist and decay exponentially.

Now, let us determine the parameter γ_k $(k \geq 0)$. According to (1.8), $z^{(-)}(x_0)$ must equal to $z^{(+)}(x_0)$, that is,

$$\begin{aligned} \Delta z &= \overline{z}_0^{(-)}(x_0) + Q_0 z^{(-)}(0) + \epsilon(\overline{z}_1^{(-)}(x_0) + Q_1 z^{(-)}(0)) + \cdots \\ &\quad -(\overline{z}_0^{(+)}(x_0) + Q_0 z^{(+)}(0) + \epsilon(\overline{z}_1^{(+)}(x_0) + Q_0 z^{(+)}(0)) + \cdots) \\ &= 0. \end{aligned}$$

Substituting the values of $Q_0 z^{(\pm)}(0), Q_1 z^{(\pm)}(0), \cdots$ in the formula and comparing the same power of ϵ, we obtain

$$\Psi_1(\gamma_0 - \varphi^{(-)}(x_0)) = \Phi_2(\gamma_0 - \varphi^{(+)}(x_0)), \tag{2.22}$$

$$\begin{aligned} &(\varphi^{(-)})'(x_0) + \frac{\partial \Psi_1}{\partial Q_0 y^{(-)}}\bigg|_{\tau=0} \cdot Q_1 y^{(-)}(0) + \delta_1(F_y(\varphi^{(-)}(x_0) + Q_0 y^{(-)}, x_0), \Psi_1, g_1) \\ &= (\varphi^{(+)})'(x_0) + \frac{\partial \Phi_2}{\partial Q_0 y^{(+)}}\bigg|_{\tau=0} \cdot Q_1 y^{(+)}(0) + \sigma_1(F_y(\varphi^{(+)}(x_0) + Q_0 y^{(+)}, x_0), \Phi_2, g_2). \end{aligned} \tag{2.23}$$

$$\cdots\cdots$$

Let

$$G(\gamma)=\Psi_1(\gamma-\varphi^{(-)}(x_0))-\Phi_2(\gamma-\varphi^{(+)}(x_0)). \tag{2.24}$$

H6 Equation $G(\gamma)=0$ has a unique solution γ_0, and $\frac{\partial G}{\partial\gamma}\Big|_{\gamma=\gamma_0}\neq 0$.

Under the condition **H6**, (2.22) has a unique solution γ_0. (2.23) can be rewritten as

$$\frac{\partial G}{\partial\gamma}\Big|_{\gamma=\gamma_0}\cdot\gamma_1=\varphi^{(+)'}(x_0)+\sigma_1-\varphi^{(-)'}(x_0)-\delta_1,$$

which is a linear equation. Parameter γ_1 is determined uniquely. In a similar way, γ_k $(k\geq 2)$ can also be determined. Thus, we have constructed all asymptotic solutions to problems $P^{(-)}$ and $P^{(+)}$.

At the end of this section, we will prove that there exists γ^{**} connecting the two problems $P^{(\pm)}$ smoothly. For a given k $(k\geq 1)$, let $\gamma^{**}=\gamma_0+\epsilon\gamma_1+\cdots+\epsilon^k\gamma_k+\epsilon^{k+1}M$,

$$\begin{aligned}\Delta z(\gamma^{**})&=\epsilon^{k+1}[G'(\gamma_0)M-(\overline{z_k}^{(+)}(x_0)+\delta_k-\overline{z_k}^{(-)}(x_0)-t_k)]+O(\epsilon^{(k+2)})\\&=\epsilon^{k+1}[G'(\gamma_0)M-(\overline{z_k}^{(+)}(x_0)+\delta_k-\overline{z_k}^{(-)}(x_0)-t_k)+c(\epsilon)\epsilon].\end{aligned}$$

When M takes the value $[G'(\gamma_0)]^{-1}[\overline{z_k}^{(+)}(x_0)+\delta_k-\overline{z_k}^{(-)}(x_0)-t_k-c(\epsilon)\epsilon]$, $\Delta z(\gamma^{**})=0$, the two problems connect smoothly.

3 Main Theorem

Theorem Under the conditions H_1–H_6, problem (1.1)–(1.2) has a unique solution $y(x,\epsilon)$ and there exists constants $\epsilon^0>0$, $c>0$, subject to the following inequality $\parallel y(x,\epsilon)-Y_N(x,\epsilon)\parallel\leq c\epsilon^{N+1}$, $0\leq x\leq 1, 0<\epsilon\leq\epsilon^0$, where

$$Y_N(x,\varepsilon)=\begin{cases}\displaystyle\sum_{k=0}^{N}\epsilon^k[\varphi_k^{(-)}(x)+\Pi_k y(\tau_0)+Q_k y^{(-)}(\tau)], & 0\leq x\leq x_0;\\ \displaystyle\sum_{k=0}^{N}\epsilon^k[\varphi_k^{(+)}(x)+Q_k y^{(+)}(\tau)+R_k y(\tau_1)], & x_0\leq x\leq 1.\end{cases}$$

Proof Problem (1.1)–(1.2) can be seen connected by $P^{(-)}, P^{(+)}$. The two problems are smoothly connected at the point x_0. According to [6], problem $P^{(-)}$ is the boundary value problem of the following kind $R(y(0,\epsilon),y(x_0,\epsilon))=0$. To prove our theorem, we need to verify the condition 6 in [6]. In problem $P^{(-)}$,

$$R_0=\begin{pmatrix}\varphi^{(-)}(0)+\Pi_0 y(0)-y^0\\ \varphi^{(-)}(x_0)+Q_0 y(0)-\gamma_0\end{pmatrix}=0,$$

$$x_0^*=\begin{pmatrix}\varphi^{(-)}(0)+\Pi_0 y(0)\\ \bar{z}_0^{(-)}(x_0)+Q_0 z^{(-)}(0)\end{pmatrix},\quad x_0^0=\begin{pmatrix}y^0\\ \Psi_1(\gamma_0-\varphi^{(-)}(x_0))\end{pmatrix},$$

$$\Delta_0^0=\left(\frac{\partial R_0}{\partial x_0^*}\right)\Big|_{x_0^*=x_0^0}=\begin{vmatrix}I_n & 0\\ 0 & \dfrac{\partial\Phi_1}{\partial Q_0 z}\end{vmatrix}_{(y^0,\Psi_1(\gamma_0-\varphi^{(-)}(x_0))}\neq 0.$$

The mentioned condition 6 is satisfied. So, on the interval $[0,x_0]$, the theorem is proved. In the same way, we can prove it on the internal $[x_0,1]$.

4 Numerical Example

For convenience, we illustrate a example of linear equation on $n = 2$. Consider the following boundary value problem

$$\epsilon^2 y_1'' = \begin{cases} 4y_1(x) - 6y_2(x) - 4x, \ 0 < x < 0.5; \\ 4y_1(x) - 6y_2(x) - 6x, \ 0.5 < x < 1, \end{cases}$$

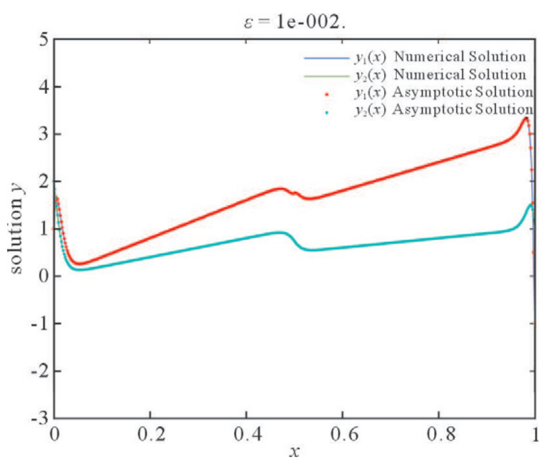
$$\epsilon^2 y_2'' = y_1(x) - y_2(x) - 2x, \quad 0 < x < 1, \quad y_1^0 = 1, \quad y_1^1 = -1, \quad y_2^0 = 2, \quad y_2^1 = 1.$$

The solution to the degenerate equation is discontinuous in both components. The zero-order approximation is

$$y_{10} = \begin{cases} 4x + [10e^{-\frac{x}{\epsilon}} - 9e^{-\sqrt{2}\frac{x}{\epsilon}}] + [-e^{\frac{x-0.5}{\epsilon}} + 0.75e^{\frac{x-0.5}{\epsilon}}], \quad 0 < x < 0.5, \\ 3x + [e^{-\frac{x-0.5}{\epsilon}} - 0.75e^{-\frac{x-0.5}{\epsilon}}] + [8e^{\frac{x-1}{\epsilon}} - 12e^{\sqrt{2}\frac{x-1}{\epsilon}}], \ 0.5 < x < 1. \end{cases}$$

$$y_{20} = \begin{cases} 2x + [5e^{-\frac{x}{\epsilon}} - 3e^{-\sqrt{2}\frac{x}{\epsilon}}] + [-0.5e^{\frac{x-0.5}{\epsilon}} + 0.25e^{\frac{x-0.5}{\epsilon}}], \quad 0 < x < 0.5, \\ x + [0.5e^{-\frac{x-0.5}{\epsilon}} - 0.25e^{-\frac{x-0.5}{\epsilon}}] + [4e^{\frac{x-1}{\epsilon}} - 4e^{\sqrt{2}\frac{x-1}{\epsilon}}], \ 0.5 < x < 1. \end{cases}$$

Comparing the asymptotic solution to the numerical solution, we have the following figure.



From the figure, we can see that the asymptotic solution calculated by the boundary layer function method is close to the numerical result.

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