Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2012,**32B**(2):813–817
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

ON REDUCIBILITY OF THE SELF-HOMOTOPY EQUIVALENCES OF WEDGE SPACES*

Yu Haibo (俞海波) *Shen Wenhuai* (沈文淮)
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
E-mail: yuhaibo@scnu.edu.cn; shenn@scnu.edu.cn

Abstract Reducibility of the self-homotopy equivalences of wedge spaces is studied and some conditions implying the reducibility are obtained.

Key words Self-homotopy equivalence; reducibility; nilpotency; homologically distant

2000 MR Subject Classification 55P10

1 Introduction

Throughout this article, we work in the category of pointed and simply-connected spaces. Given a space X, we use $\mathrm{Aut}(X)$ to denote the set of homotopy classes of based self-maps of X which are homotopy equivalences. The operation induced by the composition of homotopy class makes $\mathrm{Aut}(X)$ into a group, which is normally called the group of self-homotopy equivalences of X. For a survey of the literatures about $\mathrm{Aut}(X)$ and related concepts, please refer to [2, 9].

In what follows, we do not distinguish a map from its homotopy class. Let us first fix some notions and notations. Suppose that we are given a map $f\colon X\vee Y \longrightarrow X\vee Y$ for two spaces X and Y. For $I, J \in \{X, Y\}$, we denote $f\circ i_I$ and $p_J\circ f\circ i_I$ by f_I and f_{JI}, respectively, where $i_I\colon I \longrightarrow X\vee Y$ is a coordinate inclusion and $p_J\colon X\vee Y \longrightarrow J$ is a coordinate project. Thus, there is $f = (f_X, f_Y)$ by the universal property of wedge spaces. The group $\mathrm{Aut}(X\vee Y)$ is called reducible if for any $f \in \mathrm{Aut}(X\vee Y)$ there are $f_{XX} \in \mathrm{Aut}(X)$ and $f_{YY} \in \mathrm{Aut}(Y)$. Let $\mathrm{Aut}_X(X\vee Y) = \{f \in \mathrm{Aut}(X\vee Y) | f\circ i_X = i_X\}$ and $\mathrm{Aut}_Y(X\vee Y) = \{f \in \mathrm{Aut}(X\vee Y) | f\circ i_Y = i_Y\}$.

As a dual notion to the the group of self-homotopy equivalences of product spaces, the group of self-homotopy equivalences of wedge spaces were widely studied. Let us recall some historical results on it. In 1970, Sieradski [10] obtained an exact sequence

$$1 \longrightarrow [X\wedge Y, X\flat Y] \longrightarrow \mathrm{Aut}(X\vee Y) \longrightarrow GL(2, \lambda_{IJ}) \longrightarrow 1$$

for two spaces X and Y. About the definitions of $X\flat Y$ and $GL(2, \lambda_{IJ})$, please refer to [10]. In [6], Oka-Sawashita-Sugawara studied the group $\mathrm{Aut}(Y\vee \Sigma X)$ when X is $(m-2)$-connected and $\dim(Y) \le m-1$ for $m \ge 3$. In 1977, the group $\mathrm{Aut}(S^1\vee S^n\vee S^{2n-1})$ was studied by Frank-Kahn [3]. In 1984, Maruyama-Mimura [4] calculated $\mathrm{Aut}(KP^2\vee S^m)$, where KP^2 is a

*Received April 8, 2010.

complex, quaternionic, or Caylay projective plane. In 1988, Rutter [8] calculated the group of self-homotopy equivalences of wedge of two Moore spaces.

It was shown in [12] that $\mathrm{Aut}_X(X \vee Y)$ and $\mathrm{Aut}_Y(X \vee Y)$ are subgroups of $\mathrm{Aut}(X \vee Y)$ and furthermore if $\mathrm{Aut}(X \vee Y)$ is reducible, then, there exists a factorization of group

$$\mathrm{Aut}(X \vee Y) = \mathrm{Aut}_X(X \vee Y) \cdot \mathrm{Aut}_Y(X \vee Y).$$

In general, the condition implying reducibility of $\mathrm{Aut}(X \vee Y)$ is strongly restrictive and is also hardly verified except for some special cases: (1) X is n-dimensional and Y is n-connected for $n > 0$; (2) at least one of the homomorphism groups $\mathrm{Hom}(H_n(X), H_n(Y))$ and $\mathrm{Hom}(H_n(Y), H_n(X))$ is trivial for all $n > 0$. In [7], Pavesic obtained a series of computable criteria which imply the reducibility of $\mathrm{Aut}(X \times Y)$. Dual to the results in [7], in this article, we study $\mathrm{Aut}(X \vee Y)$ and obtain some conditions implying the reducibility of $\mathrm{Aut}(X \vee Y)$. Our main results may be stated as follows.

Theorem 1.1 Given two spaces X and Y. The group $\mathrm{Aut}(X \vee Y)$ is reducible if and only if any element $f \in \mathrm{Aut}(X \vee Y)$ implies $f_{XX} \in \mathrm{Aut}(X)$.

By introducing the notion of homological distant in Section 2, we obtain another condition implying the reducibility. We state it as follows.

Theorem 1.2 Given two spaces X and Y. If X and Y are homological distant, then, $\mathrm{Aut}(X \vee Y)$ is reducible.

For the wedge of finite spaces $X_1 \vee \cdots \vee X_n$, we also study the reducibility of $\mathrm{Aut}(X_1 \vee \cdots \vee X_n)$ by introducing the notion of atomic space in Section 3. We state our result as follows.

Theorem 1.3 Given spaces $X_1, \cdots, X_n$. If every space X_i is a pairwise non-equivalent atomic space, then, $\mathrm{Aut}(X_1 \vee \cdots \vee X_n)$ is reducible.

This article is organized as follows. In Section 2, we prove Theorem 1.1 first and then introduce the notions of quasi-regularity and nilpotency. The relations among quasi-regularity, reducibility, and nilpotency are considered. As a consequence, Theorem 1.2 is proven. In Section 3, we introduce the notion of atomic space and then give the proof of Theorem 1.3.

2 Conditions Implying Reducibility

First, we give the proof of Theorem 1.1. Our proof needs the following lemma.

Lemma 2.1 [12] Given any $f \in \mathrm{Aut}(X \vee Y)$, $(f_X, i_Y) \in \mathrm{Aut}_Y(X \vee Y)$ if and only if $p_X \circ f_X \in \mathrm{Aut}(X)$.

Proof of Theorem 1.1 For any element $f \in \mathrm{Aut}(X \vee Y)$ with $f_{XX} \in \mathrm{Aut}(X)$, we must show that $f_{YY} \in \mathrm{Aut}(Y)$. By Lemma 2.1, $f_{XX} \in \mathrm{Aut}(X)$ means that (f_X, i_Y) is a self-homotopy equivalence of $X \vee Y$. Let f' denote the inverse of f. Then, $f' \in \mathrm{Aut}(X \vee Y)$ and hence, $f'_{XX} \in \mathrm{Aut}(X)$ by assumption. Therefore, there is $(f'_X, i_Y) \in \mathrm{Aut}(X \vee Y)$. Thus, we have

$$(i_X, f_Y) = (f \circ f' \circ i_X, f_Y) = ((f_X, f_Y) \circ f'_X, f_Y) = (f_X, f_Y) \circ (f'_X, i_Y) \in \mathrm{Aut}(X \vee Y).$$

This implies $f_{YY} \in \mathrm{Aut}(Y)$ by Lemma 2.1.

In what follows, we give the definition of quasi-regular endomorphism for a group and the definition of quasi-regular self-map for a space.

Definition 2.2 Let G be a group. An endomorphism $\phi: G \longrightarrow G$ is called quasi-regular if the function $1-\phi: G \longrightarrow G$ defined by $(1-\phi)(g) = g - \phi(g)$ is an automorphism of G. A self map $f: X \longrightarrow X$ is called quasi-regular if the endomorphism $H_n(f): H_n(X) \longrightarrow H_n(X)$ is quasi-regular for any $n > 0$.

Proposition 2.3 The group $\mathrm{Aut}(X \vee Y)$ is reducible, if and only if the composite $f_{XY} \circ f'_{YX}$ is quasi-regular for any $f \in \mathrm{Aut}(X \vee Y)$, where f' is the inverse of f.

Proof Because $f \circ f' = id_{X\vee Y}$, there is $H_n(f_{XX}) \circ H_n(f'_{XX}) + H_n(f_{XY}) \circ H_n(f'_{YX}) = id$. It follows that

$$H_n(f_{XX}) \circ H_n(f'_{XX}) = id_{H_n(X)} - H_n(f_{XY}) \circ H_n(f'_{YX}) = id - H_n(f_{XY} \circ f'_{YX}).$$

If $\mathrm{Aut}(X \vee Y)$ is reducible, then, both $H_n(f_{XX})$ and $H_n(f'_{XX})$ are automorphisms of $H_n(X)$ for all $n > 0$. By the above equation, the map $f_{XY} \circ f'_{YX}$ is quasi-regular.

Conversely, suppose that $f_{XY} \circ f'_{YX}$ is quasi-regular for any $f \in \mathrm{Aut}(X \vee Y)$. Thus, both $f_{XY} \circ f'_{YX}$ and $f'_{XY} \circ f_{YX}$ are quasi-regular. Hence, $id - H_n(f_{XY}) \circ H_n(f'_{YX})$ has a right inverse, or equivalently, $H_n(f_{XX}) \circ H_n(f'_{XX})$ has a right inverse. It follows that $H_n(f_{XX})$ has a right inverse. In the same way we can show that $H_n(f_{XX})$ has a left inverse. Thus, $H_n(f_{XX})$ has an inverse and then $f_{XX} \in \mathrm{Aut}(X)$ by Whitehead theorem. This shows that $\mathrm{Aut}(X \vee Y)$ is reducible by Theorem 1.1.

Definition 2.4 For an abelian group G, we say that an endomorphism $\phi: G \longrightarrow G$ is nilpotent if $\phi^n = \phi \circ \cdots \circ \phi$ is trivial but ϕ^{n-1} is nontrivial for some $n > 0$.

Lemma 2.5 Given an abelian group G, any nilpotent endomorphism $\phi: G \longrightarrow G$ is quasi-regular.

Proof We claim that $1 - \phi$ is a homomorphism. Indeed, for $g_1, g_2 \in G$, there is

$$(1-\phi)(g_1+g_2) = g_1+g_2-(\phi(g_1)+\phi(g_2)) = (g_1-\phi(g_1))+(g_2-\phi(g_2)) = (1-\phi)(g_1)+(1-\phi)(g_2).$$

Assume that $\phi^{k+1} = 0$ but $\phi^k \neq 0$. Because for any $g \in G$, there is

$$\begin{aligned}
&(1-\phi)(1+\phi+\phi^2+\cdots+\phi^k)(g)\\
&= g + \phi(g) + \phi^2(g) + \cdots + \phi^k(g) + \phi(-(g + \phi(g) + \phi^2(g) + \cdots + \phi^k(g)))\\
&= g + \phi(g) + \phi^2(g) + \cdots + \phi^k(g) - \phi(g) - \phi^2(g) - \cdots - \phi^k(g) - \phi^{k+1}(g)\\
&= g
\end{aligned}$$

and similarly $(1+\phi+\phi^2+\cdots+\phi^k)(1-\phi)(g) = g$. It follows that $1-\phi$ is an automorphism of G.

Definition 2.6 Given two spaces X and Y, they are called homologically distant if any self-map of X which can factor through Y induces nilpotent endomorphism on all homology groups of X.

By the above definition, if X and Y are homologically distant, then $H_n(f_{XY} \circ f'_{YX})$ is an nilpotent endomorphism of $H_n(X)$ for all $n > 0$. It follows that $H_n(f_{XY} \circ f'_{YX})$ is quasi-regular by Lemma 2.5. Thus, $\mathrm{Aut}(X \vee Y)$ is reducible by Proposition 2.3. Hence, we have the following theorem.

Theorem 2.7 (Theorem 1.2) If X and Y are homologically distant, then, $\mathrm{Aut}(X \vee Y)$ is reducible.

Example 2.8 Given a complex projective space $\mathbb{C}P^n$ localized at $p=3$, by [5], there is a stable splitting $\Sigma\mathbb{C}P^n \simeq A_1 \vee A_2$, where A_i satisfies

$$\widetilde{H}_*(A_i;\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}\Big\{x_{2i+1}, x_{2i+5}, \cdots, x_{2i+4(r_i-1)+1} | r_i = \Big[\frac{n-i}{2}\Big] + 1\Big\},$$

where x_j is a generator of degree j and $[\frac{n-i}{2}]$ denotes the greatest integer which is less than $\frac{n-i}{2}$. By checking the homology, we see that A_1 and A_2 are homologically distant. It follows that $Aut(A_1 \vee A_2)$ is reducible and hence there exists a group factorization

$$Aut(\Sigma\mathbb{C}P^n) \cong Aut(A_1 \vee A_2) \cong \mathrm{Aut}_{A_1}(A_1 \vee A_2) \cdot \mathrm{Aut}_{A_2}(A_1 \vee A_2).$$

3 Atomic Spaces Implying Reducibility

In this section, we will find more conditions implying the reducibility of $\mathrm{Aut}(X \vee Y)$ by studying atomic spaces.

Definition 3.1 [1] A space X is called atomic if either every self-map $f\colon X \longrightarrow X$ is a homotopy equivalence or some iterated composite of f is null homotopic.

Recall that X is a homotopy retract of Y if there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $g \circ f \simeq id_X$.

Lemma 3.2 Let X be an atomic space. Then, X and Y are homologically distant, if and only if X is not a homotopy retract of Y.

Proof Assume that X is a homotopy retract of Y. Then, there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$, such that $g \circ f \simeq id_X$. This means that id_X is not a nilpotent self-map which can factor through Y. Therefore, X and Y are not homologically distant.

Conversely, assume that X and Y are not homologically distant. Then, there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$, such that $g \circ f$ induces a non-nilpotent endomorphism on $H_n(X)$ for some $n > 0$. Hence, $g \circ f\colon X \longrightarrow X$ is not nilpotent and therefore is invertible due to the fact that X is atomic. Let $h\colon X \longrightarrow X$ be the inverse of $g \circ f$. Then, $h \circ g$ is the left inverse of f. It follows that X is a homotopy retract of Y.

If the atomic space X is not a homotopy retract of Y, then, X and Y are homologically distant by Lemma 3.2. This implies the reducibility of $\mathrm{Aut}(X \vee Y)$ by Theorem 2.7. Thus, we have the following result.

Theorem 3.3 Assume that X is an atomic space. If X is not a homotopy retract of Y, then, $\mathrm{Aut}(X \vee Y)$ is reducible.

In what follows, we consider the reducibility of $\mathrm{Aut}(X_1 \vee \cdots \vee X_n)$ for finite wedge spaces.

Lemma 3.4 Assume that X is atomic. If X is a homotopy retract of $Y \vee Z$, then, X is a homotopy retract of Y or Z.

Proof Because X is a homotopy retract of $Y \vee Z$, there exist maps $f\colon X \longrightarrow Y \vee Z$ and $g\colon Y \vee Z \longrightarrow X$, such that $g \circ f \simeq id_X$. Then, we have $H_n(g \circ i_Y \circ p_Y \circ f) + H_n(g \circ i_Z \circ p_Z \circ f) = id$ in homology. If $g \circ i_Y \circ p_Y \circ f$ is invertible, then, $p_Y \circ f$ has a left inverse. It follows that X is a homotopy retract of Y. Otherwise, $g \circ i_Y \circ p_Y \circ f$ is nilpotent due to the fact that X is atomic. Thus, by Lemma 2.5, $g \circ i_Y \circ p_Y \circ f$ is quasi-regular and then, $H_n(id_X) - H_n(g \circ i_Y \circ p_Y \circ f)$ is an automorphism of $H_n(X)$ for all $n > 0$. It follows that $H_n(g \circ i_Z \circ p_Z \circ f)$ is invertible. Therefore, $g \circ i_Z \circ p_Z \circ f$ is invertible as X is atomic. Hence, X is a homotopy retract of Z.

Given spaces $X_1, \cdots, X_n$, let $\Omega_i = X_1 \vee \cdots \vee \hat{X}_i \vee \cdots \vee X_n$, where $\hat{X}_i$ means X_i is omitted. The following theorem shows that if a space can decompose as a wedge of some pairwise non-equivalent atomic spaces, then the reducibility of $\mathrm{Aut}(X_1 \vee \cdots \vee X_n)$ follows.

Theorem 3.5 (Theorem 1.3) Given spaces $X_1, \cdots, X_n$. If any X_i is a pairwise non-equivalent atomic space, then, $\mathrm{Aut}(X_1 \vee \cdots \vee X_n)$ is reducible.

Proof If X_i is a pairwise non-equivalent space, then, X_i is not a homotopy retract of Ω_i by iterated application of Lemma 3.4. Then, the result follows from Theorem 3.3.

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