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C-epic compactifications

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Abstract

Let K be a compactification of the Tychonoff space X, and $\rho_K : C(K) \to C(X)$ the map which restricts functions ($\rho_K(f) = f|X$, for $f \in C(K)$). In case ρ_K is an epimorphism in the category of Archimedean l-groups with unit (or equivalently, in Archimedean f-rings), we say that K is a C-epic compactification of X, or X is C-epic in K. This is "formally" equivalent to corresponding conditions in certain categories of frames, σ-frames, locales, and spaces with filter; from this some inferences can be drawn easily. Also, there is a workable criterion coming directly from the l-group theory which involves the canonical surjection $K \overset{\tau}{\leftarrow} \beta X$ from the Stone–Čech compactification βX. In any event, some specific results are: (1) if X is C-epic in K, then the restriction $K \overset{\tau}{\leftarrow} \upsilon X$ is one-to-one (υX being the Hewitt realcompactification) and conversely if υX is Lindelöf, (2) if X is zeroset-embedded in K, then X is C-epic in K, (3) if X is C-epic in K, then K and βX have the same basically disconnected cover, (4) X is C-epic in *each* of its compactifications if and only if X is almost Lindelöf. Various results related to these and various examples are presented. Many questions remain.

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1. Preliminaries

In this section we trace the shortest route to our basic definition and a few elementary facts. More background material will be sketched in Section 2.

Our references for topology and basic $C(X)$ are [20,21], and for general lattice ordered algebra [16,21].

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All topological spaces are Tychonoff (i.e., completely regular and Hausdorff). A compactification of the space X is a compact Hausdorff space K containing X densely. For two of these, K_1 and K_2, $K_1 \leqslant K_2$ means there is a continuous surjection $K_1 \leftarrow K_2$ which is the identity on X. The Stone–Čech compactification βX is the one (up to homeomorphism over X) for which $K \leqslant \beta X$ for each compactification K; the implied surjection $K \xleftarrow{\tau} \beta X$ is unique. We shall commonly say "let (K, τ) be a compactification of X".

$C(X)$ is the set of all real-valued continuous functions on X, and $C^*(X)$ the subset of bounded functions. All $f \in C^*(X)$ extend (of course we mean "continuously") over βX, with real values. All $f \in C(X)$ extend over $\beta(X)$ with values in $R \cup \{\pm\infty\}$. The largest subset of βX over which all $f \in C(X)$ extend with *real* values is the Hewitt realcompactification υX.

For category theory, see [31]. Not much will be needed.

W is the category of Archimedean l-groups with unit (meaning distinguished weak order unit) with homomorphisms the l-group homomorphisms preserving unit. This is a natural generalization of the category *Arf* of Archimedean f-rings with identity, with homomorphisms the l-ring homomorphisms preserving identity, which is itself a natural generalization of the category $\mathcal{C}$ of the rings $C(X)$ with identity the constant function 1, with rings homomorphisms preserving 1. We need not write an exposition of W here; it will suffice to describe the placement of the theory of $C(X)$ in W, and give some references.

A map $\varphi : C(X) \to C(X)$ which preserves 1's is a ring homomorphism if and only if it is an *Arf*-morphism [21, 1.6], and also if and only if it is a W-morphism [29]. Extending the description of ring homomorphisms in [21, Ch. 10], each W-homomorphism $\varphi : C(Y) \to C(X)$ is given by a unique continuous $\beta Y \xleftarrow{\sigma} \beta X$, as:

$$C(Y) \ni f \mapsto \beta f \mapsto (\beta f) \circ \sigma \mapsto (\beta f) \circ \sigma | X \in C(X), \tag{$*$}$$

where $\beta f : \beta Y \to R \cup \{\pm\infty\}$ is the extension of f. See [28,8].

Our interest in this paper is in the following specific class of homomorphisms (of W, *Arf*, or $\mathcal{C}$). Let (K, τ) be a compactification of X, and let $\rho_K : C(K) \to C(X)$ be the *restriction embedding*, $\rho_K(f) = f|X$ $(f \in C(K))$. (In terms of $(*)$ above, τ is the unique map there, and the description $(\beta f) \circ \tau | X$ reduces to $f|X$.) Since X is dense in K, $f|X = g|X$ implies $f = g$; so ρ_K is one-to-one.

In a category, a morphism e is called an epimorphism (or epi, or epic—abusing language) if, whenever r and s are morphisms with $re = se$, then $r = s$. A basic question about any category is "what are the epics?". For a concrete category, surjectives are epic, only sometimes conversely; for the category of sets with functions, epic = surjective; for Tychonoff spaces, with continuous maps, $X \xrightarrow{e} Y$ is epic if and only if $e(X)$ is dense in Y. For Hausdorff topological groups with continuous homomorphisms, the question is not answered. For W, there is a characterization of epimorphisms in [8] in terms of general situations like $(*)$ above, with ramifications in [10] (and elsewhere); we postpone discussion of this, preferring first to push around categories and generalities.

Bernhard Banaschewski has suggested to us the following principle of naming certain morphisms. Suppose $\mathcal{A}$ and $\mathcal{B}$ are categories, $F : \mathcal{A} \to \mathcal{B}$ is a functor, and $\mathcal{M}$ is a class of morphisms in $\mathcal{B}$. A morphism a of $\mathcal{A}$ will be called $F - \mathcal{M}$ if $F(a) \in \mathcal{M}$. We apply this principle to the contravariant functor $C : \mathit{Tych} \to W$ ($\mathit{Tych} =$ Tychonoff spaces with continuous maps) whose object action is $X \mapsto C(X)$, and morphism action is $X \xrightarrow{\varphi} Y \mapsto C(X) \xleftarrow{C(\varphi)} C(X)$ where $C(\varphi)(f) = f \circ \varphi$. For K a compactification of X, if we label the inclusion $X \subseteq K$ as φ, $C(\varphi)$ is the restriction embedding ρ_K defined above. Thus

1.1. Definition. Let K be a compactification of X, with $\rho_K : C(K) \to C(X)$ the restriction embedding. In case ρ_K is epic in W, we say that K is a C-epic compactification of X, or X is C-epic in K, or K is C-epic over X.

For any $Y \supseteq X$, let $K = \overline{X}^{\beta Y}$. It is easily seen that ρ_Y is epic in W if and only if ρ_K is epic in W. So our restriction in this paper to compactifications of X really covers all extensions of X.

Let T be a closed subset of Y. By Y/T we mean the quotient of Y obtained by identifying T to a point. Given X, and T closed in βX with $X \cap T = \emptyset$, we have the space $\beta X/T$. This is the one-point compactification of T' (the complement of T in βX), and $T' \supseteq X$. Thus $\beta X/T$ is a compactification of X. See [20, Ch. 2].

1.2. Proposition.

(a) *X is C-epic in βX.*

(b) *Let Y be a space with distinct points $p, q \in \beta Y - Y$, and let $X = \beta Y - \{p, q\}$. Then X is not C-epic in $\beta Y/\{p, q\}$.*

Proof. (a) means that $C(\beta X) \approx C^*(X) \subseteq C(X)$ is epic in W. This is shown in more generality in [8, 8.6.3]. Here is another proof: $C(X)$ is the ring of quotients of $C^*(X)$ formed by inverting all members of the multiplicative system $\{f \in C^*(X) \mid f \text{ is never } 0\}$ (since $g = (g/1 + g^2)/(1/1 + g^2)$) and any such construct is epic even in commutative semi-prime rings with identity (by an easy proof); thus $C^*(X) \subseteq C(X)$ is epic in *Arf*. But a W-morphism between *Arf*-objects is epic in W if and only if epic in *Arf*; see the discussion preceding 1.3 below.

(b) (This is a special case of 1.3(b) below.) Note that $\beta X = \beta Y$. Now, X is pseudocompact [21, 9D(3)], so $C(X) \approx C(\beta X)$, and $e_p : C(X) \to R$ defined by $e_p(f) = \beta f(p)$ is a W-homomorphism. Likewise, e_q, and $e_q \neq e_p$ since $q \neq p$. With $K = \beta X/\{p, q\}$, we have $e_p \rho_K = e_q \rho_K$, so ρ_K is not W-epic. □

We intend 1.2 simply as an immediate illustration of the basic definition. However, returning to the Basic Definition 1.1, one may well ask "why W?" The next few paragraphs address that.

First we are not fixated on W here. Since $C(X)$ is known as "the ring of continuous functions" (e.g., the *title* of [21]), we would just as soon work in *Arf*, but much relevant material in the references is developed in W. The point is, that the forgetful functor $\mathit{Arf} \to W$ has a left-adjoint, which permits construing *Arf* as a full monoreflective subcategory of W [28,29,17]. It follows by an easy argument that any of our ρ_K's is epic in W if and only if it is epic in *Arf*.

Now let us consider briefly the issue of epicity of our ρ_K with respect to several other natural categories. First, *Arch*: Archimedean l-groups (without distinguished weak unit) and l-group homomorphisms. By [8, 8.5.2], ρ_K is epic in W if and only if it is epic in *Arf*. Second, *lAb*: Abelian l-groups, with l-group homomorphisms. By [1], a *divisible* Abelian l-group is epicomplete in *lAb*, i.e., has no proper extension which is epic in *lAb*. Thus ρ_K is epic in *lAb* if and only if the extension $C(K) \subseteq C(X)$ is not proper, i.e., X is pseudocompact and $K = \beta X$.

More interestingly, let us consider the category $\mathcal{C}$ mentioned before, and the category *fRng*: commutative semi-prime f-rings with identity, with l-ring homomorphisms. As full subcategories, we have the inclusions *fRng* $\supseteq$ *Arf* $\supseteq \mathcal{C}$, so for any of our maps ρ_K, epic in *fRng* $\Rightarrow$ epic in *Arf* ($\Leftrightarrow$ epic in W) $\Rightarrow$ epic in $\mathcal{C}$.

1.3. Proposition. *Let (K, τ) be a compactification of X.*

(a) *ρ_K is epic in fRng if and only if τ is one-to-one, i.e., $K = \beta X$.*

(b) *ρ_K is epic in $\mathcal{C}$ if and only if the restriction $\tau|\upsilon X$ is one-to-one.*

Proof. For $p \in \beta X$, $M^p = \{f \in C(X) \mid p \in \overline{Z(f)}^{\beta}\}$ is an *fRng*-ideal and $C(X)/M^p$ is a totally ordered field which is the reals R if and only if $p \in \upsilon X$ [21, Chs. 7, 8]. Let $\pi_p : C(X) \to C(X)/M^p$ be the quotient; always $\pi_p \in$ *fRng*, and $\pi_p \in \mathcal{C}$ if and only if $p \in \upsilon X$ (since $R = C(\{0\})$).

(a) If $p \neq q$ in βX but $\tau(p) = \tau(q)$, let $i_p : C(X)/M^p \to C(X)/M^p + C(X)/M^q$ be the injection into the *fRng*-coproduct and likewise iq. Then, $i_p\pi_p \neq i_q\pi_q$ while $(i_p\pi_p)\rho_K = (i_q\pi_q)\rho_K$. The converse (and more) was noted in the proof of 1.2(a).

(b) If $p \neq q$ in υX, but $\tau(p) = \tau(q)$, then $\pi_p \neq \pi_q$ while $\pi_p\rho_K = \pi_q\rho_K$. (Here, π_p and π_q range in R.)

For the converse, recall that $C(X) \approx C(\upsilon X)$ by $f \mapsto \upsilon f$, the extension of f in $C(\upsilon X)$, and that any homomorphism $C(X) \stackrel{G}{\to} C(Y)$ is induced by some unique $\beta X \stackrel{\sigma}{\leftarrow} \beta Y$, as $f \mapsto (\beta f) \circ \sigma|Y = G(f)$ ((∗) above). Since $\sigma(\upsilon Y) \subseteq \upsilon X$, the action of G can also be described as:

$$C(X) \approx C(\upsilon X) \ni g \mapsto g \circ \sigma \in C(\upsilon Y). \tag{$**$}$$

(That description is exactly [21, 10.9(a)].) So now, let $K \stackrel{\tau}{\leftarrow} \beta X$ have $\tau|\upsilon X$ one-to-one, and suppose $C(\upsilon X) \stackrel{G_i}{\to} C(\upsilon Y)$ $(i = 1, 2)$ have $G_1\rho_K = G_2\rho_K$. Let $\upsilon X \stackrel{\sigma_i}{\leftarrow} \upsilon Y$ be the map inducing G_i per (∗∗) (i.e., the restriction of the map inducing G_i per (∗)). Thus $\tau\sigma_i$ induces $G_1\rho_K = G_2\rho_K$, and by uniqueness of inducing maps, $\tau\sigma_1 = \tau\sigma_2$. Since $\tau|\upsilon X$ was one-to-one, $\sigma_1 = \sigma_2$ and thus $G_1 = G_2$. □

1.4. Corollary. *If (K, τ) is a C-epic compactification of X (i.e., ρ_K is epic in W), then $\tau|\upsilon X$ is one-to-one.*

Proof. If ρ_K is epic in W, it is epic in $\mathcal{C}$, and 1.3(b) applies. □

1.5. Examples. The converse of 1.4 is very far from true: We shall see later in 8.1 that any realcompact non-Lindelöf space has a compactification which is not C-epic. More specially, the following is shown in [8, 8.6.9]: If X is uncountable discrete (and then realcompact, when $|X|$ is is not measurable [21, 12.2]), then X is not C-epic in its one-point compactification αX.

The exact condition that a locally compact X is C-epic in αX is a special case of 8.1 below. When that occurs, X will be C-epic in every compactification, by the following:

1.6. Proposition. *Suppose $K \leqslant L$ as compactifications of X. If K is C-epic over X, then so is L.*

Proof. Let the surjection $K \overset{\sigma}{\leftarrow} L$ witness $K \leqslant L$. This induces an embedding $\widetilde{\sigma} : C(K) \to C(L)$ defined as $\widetilde{\sigma}(f) = f \circ \sigma$. We then have $\rho_K = \rho_L \circ \widetilde{\sigma}$; and the second factor of an epic is epic. □

At this point two questions might occur. (1) What about *minimum* C-epic compactifications? See 2.9(b) below for a non-definitive answer. (2) What about compact spaces which are C-epic over every dense subspace? Every extremally disconnected space K has this property, since $K = \beta X$ for every X dense in K [21, 6M], and 1.2(a). But not every *basically* disconnected space does, by 3.4 below. We shall discuss these spaces further in [27].

2. Interpretation in several categories

We are going to interpret the C-epic condition in the categories of frames, σ-frames, locales, and spaces with filter, respectively. With apologies, we shall explain little about these categories, though we try to provide decent references. Some knowledge of these categories and how they interact will yield, quite immediately, a number of understandable topological statements about C-epicity. We should like at least to pique the interest of topologists in these categories, and on the other hand, of some mathematicians not especially interested in compactifications. However, for most of the topological corollaries in this section, purely topological proofs (considerably more lengthy) are available and will be described later.

2.0. Spaces. We recall some notation and basic facts which shall be used constantly. Let X be a Tychonoff space.

For $f \in C(X)$, the cozeroset of f is $\operatorname{coz} f = \{x \mid f(x) \neq 0\}$ and $X - \operatorname{coz} f = Z(f)$ is the zeroset. $\operatorname{coz} X = \{\operatorname{coz} f \mid f \in C(X)\}$ and $\mathcal{Z}(X) = \{Z(f) \mid f \in C(X)\}$. When $X \subseteq K$, $\operatorname{coz}(K, X) = \{S \in \operatorname{coz} K \mid X \subseteq S \subseteq K\}$; the collection of complements is $\mathcal{Z}^*(K, X) = \{Z \in \mathcal{Z}(K) \mid Z \cap X = \emptyset\}$. For $K = \beta X$ and if X is clear from context, we abbreviate $\mathcal{Z}^*(\beta X, X)$ as $\mathcal{Z}^*$. Recall that $\upsilon X = \bigcap \operatorname{coz}(\beta X, X)$ [21].

For $\mathcal{A} \subseteq \mathcal{P}(X)$ (the power set), $\mathcal{A}_\sigma$ (respectively, $\mathcal{A}_\delta$) is the class of all sets $\bigcup_{n=1}^{\infty} A_n$ (respectively, $\bigcap_{n=1}^{\infty} A_n$), $A_1, A_2, \ldots \in \mathcal{A}$. Thus we have $\operatorname{coz}_\delta(\beta X, X)$ and its family of complements Z_σ^*. Crucial facts are: if $X \subseteq K$ and K is compact, then each member of $\mathrm{coz}(K, X)$ is σ-compact (since $\mathrm{coz}\, f = \bigcup_n \{Z \mid |f(Z)| \geqslant \frac{1}{n}\}$), and thus each member of $\mathrm{coz}_\delta(K, X)$ is Lindelöf [18,20, 3.8.F].

Of course, any Lindelöf space is realcompact. Further: Let $X \subseteq K$. X is said to be G_δ-closed, or regularly placed, in K if, whenever $p \in K - X$ there is a G_δ-set G (equivalently, $G \in \mathcal{Z}(K)$) with $p \in G$ and $G \cap X = \emptyset$. If K is realcompact and X is G_δ-closed in K, then X is realcompact. Also, X is realcompact if and only if X is G_δ-closed in βX (which is really the same theorem as "$\upsilon X = \bigcap \mathrm{coz}(\beta X, X)$"). See [20, 3.12.26].

Analogously: X is normally placed in K if, whenever F is closed in K with $F \cap X = \emptyset$, then there is $Z \in \mathcal{Z}^*(K, X)$ with $F \subseteq Z$. A useful theorem of Smirnov is: X is Lindelöf if and only if X is normally placed in βX [20, 3.12.25].

2.1. Frames and such, briefly. A frame is a complete lattice F which satisfies the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\} \quad (a \in F,\ S \subseteq F); \tag{$*$}$$

a frame homomorphism is a lattice homomorphism preserving all suprema. The category of *completely regular* frames is denoted *Frm*. A frame is Lindelöf if $\bigvee S =$ top implies a countable $S_0 \subseteq S$ with $\bigvee S_0 =$ top. The full subcategory of *Frm* whose objects are Lindelöf is *LFrm*. The formal opposite of *Frm* is the category of completely regular locales, denoted *Loc*. The opposite of *LFrm* is *LLoc*. See [33].

A σ-frame is a bounded lattice with all countable suprema satisfying the law $(*)$ for just countable S; a σ-frame homomorphism is a lattice homomorphism preserving countable suprema. The category of completely *regular* σ-frames is denoted σ*Frm*. See [3,35] and other references therein.

Frames model, among other things, the open set lattices ΩX, X a space. σ-frames model the lattices $\mathrm{coz}\, X$ of all cozerosets of X.

There is a functor $\mathrm{coz}: \mathit{Frm} \to \sigma\mathit{Frm}$ which extends the association of X (as ΩX) with $\mathrm{coz}\, X$, which restricts to a *categorical equivalence of LFrm with* σ*Frm*. See [6,37], and references therein.

For $G \in W$, $\ker G \equiv \{\ker \zeta \mid G \xrightarrow{\zeta} H \in W\}$ (here $\ker \zeta$ is the usual kernel). These ideals in $\ker G$ have a somewhat involved intrinsic description, but $\ker G$ is a frame in a natural way, which has the Lindelöf property. For $G \xrightarrow{\varphi} H \in W$, $\ker \varphi : \ker G \to \ker H$ is the map: $(\ker \varphi)(I)$ is the least ideal in $\ker H$ containing $\varphi(I)$. Thus is defined a functor $\ker : W \to \mathit{LFrm}$. Moreover, there is a subcategory c^3 or W_∞ of W which contains all $C(X)$'s, for which ker restricts to a *categorical equivalence* of c^3 with *LFrm*. See [35,12, 2.4], and references therein.

According to [36], *LFrm* is coreflective in *Frm*, or, oppositely, *LLoc* is reflective in *Loc*; see also [4]. We prefer the latter view, since it is more familiar in topology: To each $X \in \mathit{Loc}$ corresponds $\lambda X \in \mathit{LLoc}$ and a dense *Loc*-inclusion $X \subseteq \lambda X$ with the universal mapping property for reflections. One realization is $\lambda X = \bigcap^{\mathrm{loc}} \mathrm{coz}(\beta X, X)$. Here βX is the localic Stone–Čech compactification, $\mathrm{coz}(\beta X, X)$ denotes all cozero-subspaces of βX

which contain X, and $\cap^{\text{loc}}$ means localic intersection. Another realization (in *Frm*) is, $(\lambda X)^{\text{op}} = \ker C(X)$ ($C(X)$ meant localically of course), see [35,12].

Of course, a space is a locale. For *spaces* X, recall that $\upsilon X = \bigcap \operatorname{coz}(\beta X, X)$ (topological intersection). In fact we have: $X \subseteq \upsilon X = \operatorname{pt}(\lambda X) \subseteq \lambda X \subseteq \beta X$ (inclusions as locales), with $\upsilon X = \lambda X$ if and only if υX is Lindelöf. ("pt" is the "take the points" functors from frames to spaces; generally, a frame need have no points.)

Our last category of present interest is, on the face of it, more classically topological than the above. This is spaces with filter, or *SpFi*: the objects are pairs $(X, \mathcal{F})$, X a compact space and $\mathcal{F}$ a filter of dense open subsets, with morphisms $(X, \mathcal{F}) \overset{f}{\leftarrow} (Y, \mathcal{G})$ continuous f with $f^{-1}(F) \in \mathcal{G}$ for each $F \in \mathcal{F}$. This category was made explicit in [8, 8.3], with subsequent studies in [11,14]. The connection with our other categories is in various functors, which we explain briefly.

SpFi $\overset{\cap}{\to}$ *Loc* has object-action $\cap(X, \mathcal{F}) = \bigcap^{\text{loc}}\{F \mid F \in \mathcal{F}\}$; this is a dense sublocale of X. Given $(X, \mathcal{F})$, a closed subset T for which each $F \cap T$ is dense in $T (F \in \mathcal{F})$ is called a *subspace* of (X, F). *SpFi* $\overset{\text{cosub}}{\longrightarrow}$ *Frm* has object-action $\operatorname{cosub}(X, X) = \{X - T \mid T$ is a subspace of $(X, \mathcal{F})\}$ (which is a Frame in the inclusion order). Then [14] $\cap = \operatorname{op} \circ \operatorname{cosub}$.

LSpFi is the full subcategory of *SpFi* with objects $(X, \mathcal{F})$ for which the $F \in \mathcal{F}$ are Lindelöf (which means cozero). $W \overset{SY}{\to}$ *LSpFi* is the SpFic-Yosida functor: For $G \in W$, there is a unique compact YG and a W-isomorphism $G \to \widehat{G}$ onto a W-object in $\{f \in C(YG, R \cup \{\pm\infty\} \mid f^{-1}(R)$ is dense in $YG\}$. The object-action of SY is $SY(G) = (YG, \{\widehat{g}^{-1}R \mid g \in G\}$. It is to be noted that $SY(C(X)) = (\beta X, \operatorname{coz}(\beta X, X))$, and that $(*)$ in Section 1 manifests the morphism-action of SY.

From [11], the kernels of $G \in W$ are exactly the subsets $I(T) = \{g \in G \mid \widehat{g}|T = 0\}$, for subspaces T of $SY(G)$. This results in the formula $\ker = \operatorname{cosub} \circ SY$, whence $\operatorname{op} \circ \ker = \cap \circ SY$.

This largely completes our explanation of the various relevant categories. We have said less than is needed to totally understand the theorem below because we do not want to write a book here, but more than is needed just to state the theorem, because we want at least to indicate how intimately connected the various categories are. In any event:

We fix notation for the rest of this section. (K, τ) is a compactification of X (meaning K is the compactification and $K \overset{\tau}{\leftarrow} \beta X$ is the surjection extending the inclusion $X \hookrightarrow K$), and $\rho_K : C(K) \to C(X)$ is the W-embedding $\rho_K(f) = f|X$.

2.2. Theorem. *The following conditions are equivalent:*

(a) $\rho_K : C(K) \to C(X)$ *is epic in* W (*i.e.*, X *is* C-*epic in* K).
(b) $\ker \rho_K : \ker C(K) \to \ker C(X)$ *is epic in Frm* (*or in LFrm*).
(c) *The trace map* $t_K : \operatorname{coz} K \to \operatorname{coz} X$ ($t_K(S) \equiv S \cap X$) *is epic in* σ*Frm.*
(d) *The localic restriction* $K \overset{\tau}{\leftarrow} \lambda X$ *is monic in Loc* (*or in LLoc*).
(e) $(K, \{K\}) \overset{\tau}{\leftarrow} (\beta X, \operatorname{coz}(\beta X, X))$ *is monic in SpFi* (*or in LSpFi*).

Theorem 2.2 follows, more or less immediately, from the nature of the various functors mentioned in 2.1. More explicitly: In (b), the equivalence of epicity in *Frm* and *LFrm* is because the frames in question are Lindelöf, and *LFrm* is coreflective in *Frm*. (a) $\Leftrightarrow$ (b) because $\ker : c^3 \to \mathit{LFrm}$ is an equivalence. (d) is simply the opposite statement to (b). Regarding (c), one first notes that the action of the functor $\mathrm{coz} : \mathit{Frm} \to \sigma \mathit{Frm}$ on $\ker \rho_K$ really is the mentioned trace map. Then (b) $\Leftrightarrow$ (c) because $\mathrm{coz} : \mathit{LFrm} \to \sigma \mathit{Frm}$ is an equivalence. A proof that (d) $\Leftrightarrow$ (e) can be extracted from the fact that the functor $\cap$ restricted to a certain subcategory of *SpFi* is an equivalence with *LLoc*; see the remarks in [14, Section 4] which also comment on some unclear points. We can note also that (a) $\Leftrightarrow$ (e) is a special case of [11, 3.1] (which does not contain complete proofs).

We consider stronger properties of the maps in 2.2(a)–(e).

2.3. Proposition. *One of the five maps in* 2.2 *is an isomorphism in its category if and only if they all are, and this occurs if and only if* $K = \beta K$ *and* X *is pseudocompact.*

Regarding the proof: the equivalence of "isomorphism" for (a)–(d) follows from the categorical equivalences discussed before. The equivalence of "isomorphism" for (a), or for (e), with "$K = \beta X$ and X is pseudocompact" is obvious.

2.4. Proposition. *These are equivalent*:

(b) $\ker \rho_K$ *is surjective.*

(c) t_K *is surjective, i.e.,* X *is* z*-embedded in* K.

(d) *The localic restriction* $K \leftarrow \lambda X$ *is an inclusion in Loc.*

Regarding the proof: Since $\lambda X = (\ker C(X))^{\mathrm{op}}$, 2.4 (b) and (d) mean the same thing. For (b) $\Leftrightarrow$ (c), B. Banaschewski has pointed out (and we thank him) the relatively easy proof that the functor $\mathrm{coz} : \mathit{LFrm} \to \sigma \mathit{Frm}$ preserves and reflects surjections. (We had an earlier, laborious proof that (b) $\Leftrightarrow$ (c) proceeding through *SpFi*, using results in [11].)

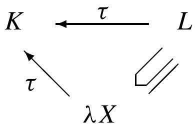
Since surjections are epic, we have

2.5. Corollary. *If* X *is* z*-embedded in* K, *then* X *is* C*-epic in* K.

2.5 gives considerable flesh to our present project. A lot is known about z-embedding (see [7]), and we shall see below (especially Sections 8, 9) that in several regards, z-embedding is "strong C-epic".

2.6. Corollary. *If there is a Lindelöf space* L *with* $X \subseteq L \subseteq \beta X$ *for which the restriction* $K \xleftarrow{\tau} L$ *is one-to-one, then* X *is* C*-epic in* K.

Proof. For such L, $L = \bigcap \mathrm{coz}(\beta X, L)$, so that λX is the localic intersection of all such L's. For any such L, we have the triangle



and if the top is one-to-one, it is monic in *Loc* and then so is the left side and 2.2(d) holds. □

2.7. Corollary. *X is C-epic in K if either* (a) *υX is Lindelöf and the restriction $K \stackrel{\tau}{\leftarrow} \upsilon X$ is one-to-one, or* (b) *there is $S \subseteq \mathrm{coz}_\delta(\beta X, X)$ for which the restriction $K \stackrel{\tau}{\leftarrow} S$ is one-to one.*

This follows from 2.6. (In (b), such S is Lindelöf, as noted in 2.0.) Note that (a) is a partial converse to 1.4.

Recall from just before 1.2, the notation $\beta X/T$ for the quotient obtained by collapsing closed T to a point, and that this is a compactification of X when $T \subseteq \beta X - X$.

2.8. Corollary. *If $F \in \mathrm{coz}(\beta X, X)$, then X is C-epic in $\beta X/F'$ (F' denoting $\beta X - F$).*

Proof. The quotient map $\beta X/F' \stackrel{\tau}{\leftarrow} \beta X$ is one-to-one on F. Apply 2.7(b). □

2.9. Corollary.

(a) *βX is the only C-epic compactification of X if and only if X is pseudocompact.*

(b) *If X is locally compact and realcompact, then the one-point compactification αX is the infimum* (*in the poset of compactifications of X*) *of C-epic compactifications of X.* (*So, with* 1.5*, the infimum of C-epic compactifications need not be C-epic.*)

Proof. (a) If X is not pseudocompact, there is proper $F \in \mathrm{coz}(\beta X, X)$, and $|F'| \geqslant 2$ [21, 9.5]. Then $\beta X/F' \neq \beta X$, while $\beta X/F'$ is C-epic by 2.8. If X is pseudocompact, then $\beta X = \upsilon X$. If K is a compactification with $K \neq \beta X$, then $K \stackrel{\tau}{\leftarrow} \beta X = \upsilon X$ is not one-to one, so 1.3 applies.

(b) We have $\alpha X \stackrel{\tau}{\leftarrow} \beta X$. Whenever (p, q) have $\tau(p) = \tau(q)$, choose $F(p, q) \in \mathrm{coz}(\beta X, X)$ with $p, q \notin F(p, q)$. (X is regularly placed in βX, and the intersection of two cozeros is cozero.) Then $\alpha X = \bigwedge_{(p,q)} \beta X/F(p, q)'$. □

3. The basically disconnected cover

A continuous surjection of compact spaces, $A \stackrel{\tau}{\leftarrow} Y$, is called *irreducible* if σ maps no proper closed subset of Y onto A; then (Y, σ) (or briefly, just Y), is called a *cover* of A. It is to be noted that, when (K, τ) is a compactification of X, then $(\beta X, \tau)$ is a cover of K.

For covers of a particular A, $(Y_1,\sigma_1) \leqslant (Y_1,\sigma_2)$ (or briefly, $Y_1 \leqslant Y_2$) means there is continuous $Y_1 \overset{f}{\leftarrow} Y_2$ with $\sigma_1 f = \sigma_2$ (and then f is unique, and irreducible). If f is a homeomorphism, the two covers are *equivalent* (and this if and only if $(Y_i,\sigma_i) \leqslant (Y_j,\sigma_j)$ for $i \neq j$); equivalent covers will be called "equal".

A property $\mathcal{P}$ of compact spaces is a *covering property* if each compact space has a cover minimum with $\mathcal{P}$ (up to equivalence), and this is called "the" $\mathcal{P}$ cover. Extremal disconnectivity (*ED*) is a covering property, and for each A, the *ED* cover *EA* is the maximum cover of A (which means *ED* is the least covering property, as a class of spaces). *EA* is also called the absolute of A, or Gleason's projective cover.

Two other covering properties (there are many) are basic disconnectivity (*BD*), with cover denoted *BA*, and the property quasi-$F(\mathcal{QF})$, with cover $\mathcal{Q}A$. We have $ED \subseteq BD \subseteq \mathcal{QF}$ (as classes of spaces), and then equivalently, for each A, $\mathcal{Q}A \leqslant BA \leqslant EA$.

See [23,38] for surveys of covering theory.

An extension $G \hookrightarrow H$ in W is *essential* (in W) if, whenever $H \overset{\varphi}{\to} H' \in W$ and $\varphi|G$ is one-to-one, then φ is one-to-one. It is to be noted that, when (K,τ) is a compactification of X, our extension $\rho_K : C(K) \hookrightarrow C(X)$ is essential, and this is equivalent to τ being irreducible. See [29].

A W-object is called *epicomplete* (in W) if it has no proper epic extension (in W), and this is equivalent to being of the form $D(Y)$ for some Y which is compact and *BD* [9]. (Here, $D(Y) = \{f \in C(Y, R \cup \{\pm\infty\}) \mid \{f^{-1}(R)$ dense in $Y\}$, which is a W-object exactly when Y is $\mathcal{QF}$.) A W-extension $G \hookrightarrow H$ is an *epicompletion* of G (in W) if it is an epic extension and H is epicomplete. [10] analyzes epicompletions in W and, among other things, shows that each G has a *unique* essential epicompletion which, when $G = C(X)$, is $D(B\beta X)$.

3.1. Theorem. *If X is C-epic in the compactification K, then $BK = B\beta X$.*

Proof. If $\rho_K : C(K) \to C(X)$ is epic, then, with $e : C(X) \hookrightarrow D(B\beta X)$ denoting the essential epicompletion of $C(X)$, the composite embedding $e\rho_K = C(K) \hookrightarrow D(B\beta X)$ is an essential epicompletion of $C(K)$. By uniqueness, this is isomorphic over $C(K)$ to $D(BK)$. This implies that BK and $B\beta X$ are homeomorphic "over K", i.e., equivalent covers of K, by the basics of the W-representation theory as described in many places, e.g., [28,8]. □

3.2. Example. The converse of 3.1 fails. Let Y be realcompact in Example 1.2(b), $X = \beta Y - \{p,q\}$ and $K = \beta Y/\{p,q\}$; so X is not C-epic in K. Then there is $S \in \operatorname{coz}(\beta Y, Y)$ with $p,q \notin S$ (as noted in the proof of 2.9(b)), so $K \leqslant \beta S$ as covers of K. For compact L, $\mathcal{Q}L = \lim_{\leftarrow}\{\beta S \mid S$ is dense cozero in $L\}$ [19]. Thus, $\mathcal{Q}K = \mathcal{Q}\beta X$. This implies $BK = B\beta X$, since $BD \subseteq \mathcal{QF}$.

Further examples, perhaps more fulfilling, will be presented in 8.8.

3.3. Corollary. *K is a basically disconnected C-epic compactification of X if and only if X is basically disconnected and $K = \beta X$.*

Proof. X is *BD* if and only if βX is [21, 6M], and βX is C-epic by 1.2. If K is *BD* and C-epic, then as covers of $K, BK = K \leqslant \beta X \leqslant B\beta X = BK$ (the last using 3.1). So $K = \beta X$, βX is *BD* and so is X. □

3.4. Examples. Here is a class of large non-C-epic compactifications of certain spaces: if K is a *BD* compactification of X and $K \neq \beta X$, then K is not C-epic over X (by 3.3, since $BK = K < \beta X \leqslant B\beta X$ as covers of X). To construct (all) such things, let $\mathcal{A}$ be any point-separating σ-field of subsets of the set X, and let $M = \{f \in R^X \mid f^{-1}(a,b) \in \mathcal{A}\ \forall a, b\}$ (the $\mathcal{A}$-measurable functions). Give X the topology with $\mathcal{A}$ as basis; now X is a P-space, and $M \subseteq C(X)$. Let K be the minimum compactification of X over which all $f \in M^*$ extend. See [20, 3.12.22]. (K is also the maximal ideal space of M, and also the Stone space of the Boolean algebra $\mathcal{A}$). Then K is *BD* (e.g., [39]) and $M \neq C(X)$ if and only if $\mathcal{A} \neq \mathcal{Z}(X)$ if and only if $K \neq \beta X$ if and only if K is not C-epic over X. (Another view of this non-C-epicity is that

$$C(K) \approx M^* \subseteq M \subseteq C(X),$$

and were $C(K)$ epic in $C(X)$ then M would be, while M has no proper epic extension; see [9,10].)

Specifically take $\mathcal{A} =$ the usual Baire field on R; here $|\mathcal{A}| = c$. Then X is discrete R, and $\mathcal{Z}(X)$ is the power set of R, so $|\mathcal{Z}(X)| = 2^c$.

4. Weak Z-embedding and the quasi-F cover

X is *weakly z-embedded* in K if, for each cozeroset S of X there is a cozeroset T of K with $T \cap X$ contained densely in S. (Saying "$T \cap X = S$" means z-embedding. The weak z-embedding of X in K is equivalent to the $Z^{\#}$-embedding in [30] (the first source) and the R-embedding in [25].)

For a compactification K of X, z-embedding implies both C-epic and weak z-embedding (the first being 2.5, the second obvious). Neither implication reverses, and in fact, for C-epic and weak z-embedding, neither implies the other, nor does the conjunction of the two imply z-embedding. To justify this most expeditiously now, we shall quote some theorems (including one of our main theorems here), though some examples which are perhaps more penetrating will be put forward in Section 6.

4.1. Definition (*and more*). X is

(a) almost compact if $|\beta X - X| \leqslant 1$, equivalently, of each pair of disjoint zerosets, at least one is compact (see [21]),
(b) almost Lindelöf if υX is Lindelöf and $|\upsilon X - X| \leqslant 1$, equivalently, of each pair of disjoint zerosets, at least one is Lindelöf (see [24] and Section 9 below),
(c) weakly Lindelöf if each open cover has a countable subfamily with dense union.

Let $\mathcal{E}$ be a property that an embedding might have. We say that X is *absolutely $\mathcal{E}$-embedded* if for each compactification K of X, the embedding $X \subseteq K$ has $\mathcal{E}$.

4.2. Theorem.

(a) (Hewitt–Smirnov; see [21].) *X is absolutely C^*-embedded if and only if X is almost compact.*

(b) (Jerison et al.; see [24,7].) *X is absolutely z-embedded if and only if X is either almost compact or Lindelöf.*

(c) ([30,25].) *X is absolutely weakly z-embedded if and only if X is either almost compact or weakly Lindelöf.*

(d) (Theorem 9.1, below.) *X is absolutely C-epically embedded if and only if X is almost Lindelöf.*

4.3. Examples of almost Lindelöf spaces. Of course, any Lindelöf space is almost Lindelöf. Also, any almost compact X is almost Lindelöf; in this case, $|\beta X - X| \leqslant 1$ and $\beta X = \upsilon X$, so υX is compact.

For other examples: Let Y be any space which is neither pseudocompact nor Lindelöf, and let S be any Lindelöf subset of βY with $Y \nsubseteq S \nsubseteq \beta Y$. Take $p \in S - Y$, and let $X = S - \{p\}$. Then X is C^*-embedded in S (since $Y \subseteq X \subseteq \beta Y$) and G_δ-dense in S (since $\{p\}$ is not a G_δ-set [21, 9.6, 9.7]), therefore $\upsilon X = S$ (by [21, 1.18] or [7, 3.5]).

Specifically here, we could use Y uncountable discrete and $S \in \operatorname{coz}(pY, Y)$. Other choices will be necessary in 4.4.

4.4. Examples. We prove the existence of various examples by comparing the conditions in 4.2.

(a) C-epic $\not\Rightarrow$ z-embedding: By 4.2 (b) and (d), there will be such a compactification K for any X which is almost Lindelöf, but neither Lindelöf nor almost compact. (No such X can be realcompact.) The spaces $X = S - \{p\}$ in 4.3 are exactly the spaces with this property. For certain of these X, we can identify easily such a K: Choose S to be cozero. Then X is locally compact, and has the one-point compactification $\alpha X \equiv K$, in which X is not z-embedded since X is neither almost compact nor Lindelöf.

(b) C-epic $\not\Rightarrow$ weak z-embedding: (Of course, any example here is also an example for (a), but (a) is easier.) By 4.2 (d) and (c), there will be such a compactification K of any X which is almost Lindelöf, but neither weakly Lindelöf nor almost compact. (Note that no such X can be realcompact.) If S is a Lindelöf P-space with a nonisolated point p for which $X = S - \{p\}$ is C^*-embedded in S, then X is an example. W.W. Comfort has shown us that such an S is the P-space coreflection of the ordinals $\leqslant \omega_2$, with $p = \omega_2$. (This is not obvious.) A further stock of examples X is constructed as follows. Begin with any nonrealcompact space A. Let $Y = \Delta(A, A)$ be the space from [19, 5.6]: Let $\alpha D = D \cup \{\infty\}$ be the one-point compactification of uncountable discrete D, and $Y = A \times \alpha D$ with all points $(a, z), z \neq \infty$, made isolated. Then Y is an almost-P-space (no proper dense cozerosets), is not pseudocompact (clearly), and is not realcompact since Y contains a copy of A as a closed set. Now (proceeding as in 4.3), take any Lindelöf S with $Y \nsubseteq S \nsubseteq \beta Y$ and any $p \in \upsilon Y - Y$, and let $X = S - \{p\}$. As in 4.3, X is almost Lindelöf and not almost compact. Our special choice of Y and p make X not weakly

Lindelöf: For each $x \in X$ select a cozeroset U_x of βY which contains x and excludes p. Then $\mathcal{U} \equiv \{U_x \cap X \mid x \in X\}$ is an open cover of X. If $\mathcal{U}_0 \subseteq \mathcal{U}$ were countable with $\bigcup \mathcal{U}_0$ dense in X, then $Y \cap \bigcup \mathcal{U}_0$ would be a dense cozeroset of Y, hence equal to Y; so $Y \subseteq \bigcup \mathcal{U}_0$. Then $T \equiv \bigcup\{U_x \mid U_x \cap X \in \mathcal{U}_0\}$ is a cozeroset of βY containing Y, hence $\upsilon Y \subseteq T$. But $p \notin T$.

(c) weakly z-embedded $\not\Rightarrow$ C-epic: By 4.2(c) and (d), there will be such a K for any X which is weakly Lindelöf but not almost Lindelöf. For example, take X separable realcompact but not Lindelöf, such as the Sorgenfry plane.

(d) weakly z-embedded + C-epic $\not\Rightarrow$ z-embedded: By 4.2(b)–(d), there will be such a K for any X which is weakly Lindelöf and almost Lindelöf, but not almost compact and not Lindelöf. (Note that no such X can be realcompact.) Such X can be constructed by the method in 4.3, starting with Y separable, not pseudocompact and not Lindelöf, e.g., the Sorgenfry plane.

This method of constructing examples by combining the results in 4.2 cannot, as we said, construct realcompact examples for (a), (b), and (d) above, and in (b) and (d) it is not so visible what the compactifications K are. In Section 6 below we shall construct realcompact examples with reasonably explicit compactifications.

We pursue further connections with covers. The following is implied quickly by [30, 2.13] and [25, 4.1].

4.5. Theorem. *For a compactification K of X, X is weakly z-embedded in K if and only if $\mathcal{Q}K = \mathcal{Q}\beta X$.*

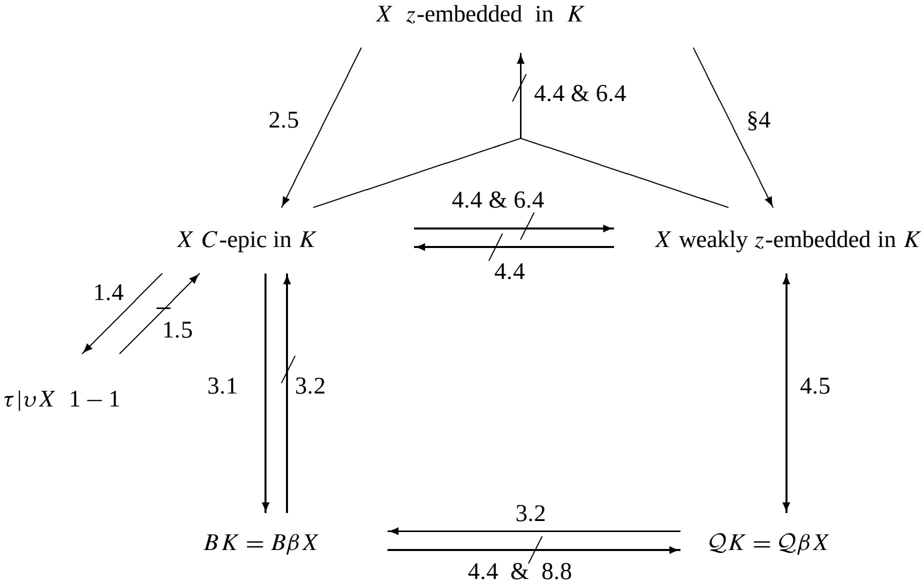
Since $\mathcal{Q}F \supseteq BD$, the covers satisfy $A \leqslant \mathcal{Q}A \leqslant BA$ for each A, and thus for a compactification K of X, $\mathcal{Q}K = \mathcal{Q}\beta X$ implies $BK = B\beta X$. Of course, the converse fails. In view of 3.1 and 4.5, there are examples of this in 4.4(b) above, also in 8.8 below.

In view of 4.5 and 3.1 one asks: are there covering properties $\mathcal{P}$ whose covers PA satisfy (1) X is z-embedded in K if and only if $PK = P\beta X$?, or (2) X is C-epic in K if and only if $PK = P\beta X$? The answer to (2) would seem to be "no" since a condition like $PK = P\beta X$ would seem to only speak of how $C(K)$ sits in $C^*(X)$ (not $C(X)$). But of course, there are $\mathcal{P}$ such that (†): (X is C-epic in $K \Rightarrow PK = P\beta X$), namely $\mathcal{P} = BD$ with $P = B$. So, seeking to improve that result, one asks (3) is there $\mathcal{P} \not\supseteq BD$ satisfying (†)? ($\mathcal{P} = QF$ does *not* satisfy (†), by virtue of 4.5 and 4.4(a).)

On the other hand, 4.5 and our attention to the BD-cover, suggest this: (4) is there (or "describe") a property $\mathcal{E}$ of embeddings in compactifications such that $X \subseteq K$ has $\mathcal{E}$ if and only if $BK = B\beta X$? One description of the condition $BK = B\beta X$ is: For each Baire set E of βX, there is a Baire set E, of K such that the symmetric difference $E \Delta \tau^{-1} E$, is meager. (This follows from information on B in [40,41,10].) This seems not very illuminating, but perhaps it is hard to do better.

5. A summary chart

Let (K, τ) be a compactification of X. The following chart displays the relation between the various positionings of X in K that we are considering. The symbol $\overset{n}{\to}$ (or $\overset{n}{\not\to}$) means there is a valid (or, invalid) implication as indicated, and this fact is recorded in this paper at location n.



6. A class of examples

We construct some C-epic compactifications of various X, which compared to some of those in 4.3, represent sharper examples and are more easily visualized. The idea is based on this restricted version of 2.7(b).

6.1. Proposition. *Let (K, τ) be a compactification of X for which there is $S \in \operatorname{coz}(\beta X, X)$ with $\tau|S$ one-to-one. Then X is C-epic in K.*

We now shall construct such things, in a simple and general way, and then tailor the X to fit our needs.

6.2. Construction. Let Y be any locally compact space. Let $\alpha N = N \cup \{\infty\}$ be the one-point compactification of N, and $d\alpha N$ the set αN given the discrete topology.

Let $X = X(Y)$ be the space $d\alpha N \times Y$; this is the topological sum of countably many copies of Y. Let X_1 be the space obtained by refining the topology of the product space $\alpha N \times \beta Y$ to include all sets $\{\infty\} \times G$ for G open in Y. This makes X a dense subspace of X_1, and the subspace $\{\infty\} \times (\beta Y - Y)$ of X_1 is still homeomorphic to $\beta Y - Y$. (We have used the local compactness.) Let $K(Y) = \beta X_1$. This is a compactification of X.

The identity map $X \leftarrow X$ extends continuously to the identity function $X_1 \leftarrow d\alpha N \times \beta Y$, which extends continuously to a surjection $K(Y) = \beta X_1 \overset{\tau}{\leftarrow} \beta(d\alpha N \times \beta Y) = \beta X$. $S \equiv d\alpha N \times \beta Y$ is σ-compact and locally compact. Such a space is an open F_σ in any space in which it is dense, thus cozero is any normal space in which it is dense [20], thus $S \in \operatorname{coz}(\beta X, X)$. Clearly $\tau|S$ is one-to-one.

6.3. Theorem. *Let Y be any locally compact space, $X(Y)$ and $K(Y)$ as just described.*

(a) *$K(Y)$ is a C-epic compactification of $X(Y)$.*

(b) *If Y is realcompact, then $X(Y)$ is realcompact and G_δ-closed in $K(Y)$.*

(c) *$X(Y)$ is z-embedded* (*respectively, weakly z-embedded*) *in $K(Y)$ if and only if Y is Lindelöf* (*respectively, weakly Lindelöf*).

Proof. (a) follows from 6.1.

(b) When Y is realcompact, it is even true that $\alpha N \times Y$ is G_δ-closed in $\alpha N \times \beta Y$ *a fortiori* $X(Y)$ is G_δ-closed in X_1. But X_1 is realcompact also and therefore so is $X(Y)$, and also X_1 is G_δ-closed in $\beta X_1 = K(Y)$, so X is G_δ-closed in $K(Y)$ (by transitivity of "G_δ-closed"). (X_1 is the union of the compact set $\{\infty\} \times (\beta Y - Y) \equiv F$, and its complement F' in X_1; F' is the topological sum of the realcompact spaces $\{\infty\} \times Y$ and all the $\{n\} \times \beta Y (n \in N)$, so F' is realcompact [21, 12G]; the union of a compact set and a realcompact set is realcompact [21, 8.16].)

(c) Clearly, if Y is Lindelöf (respectively, weakly Lindelöf) then so is $X(Y)$ and therefore z-embedded (respectively, weakly z-embedded) in any superspace, by 4.2.

Conversely, suppose Y is not Lindelöf (respectively, weakly Lindelöf). Let $C \equiv \{\infty\} \times Y$; this is a cozeroset of $X(Y)$. Suppose C_0 is a cozeroset of K with $C_0 \cap X \subseteq C$. Then, $C_0 \cap (N \times Y) = \emptyset$, hence $C_0 \cap (N \times \beta Y) = \emptyset$, hence $C_0 \cap X_1 \subseteq \{\infty\} \times \beta Y$, hence $C_0 \cap X_1 \subseteq \{\infty\} \times Y$ (because $\{\infty\} \times (\beta Y - Y)$ is not "isolated" from $N \times (\beta Y - Y)$). Since X_1 is Lindelöf (as a continuous image of $d\alpha N \times \beta Y$), $C_0 \cap X_1$ is also Lindelöf. Since Y is not Lindelöf (respectively, weakly Lindelöf), $C_0 \cap X$ cannot be equal to (respectively, dense in) $\{\infty\} \times Y$. □

6.4. Examples. (*Compare* 4.4.) (a) Realcompact X with a compactification K in which X is G_δ-closed, C-epic, not weakly z-embedded. Take any Y which is locally compact, realcompact, and not weakly Lindelöf, and use the $X(Y)$ and $K(Y)$ in 6.3. For example, Y discrete of uncountable nonmeasurable cardinal, and then $X(Y) \approx Y$.

(b) Realcompact X with a compactification K in which X is G_δ-closed, C-epic, weakly z-embedded, not z-embedded. Take any Y which is locally compact, realcompact, weakly Lindelöf, and not Lindelöf, and use the $X(Y)$ and $K(Y)$ in 6.3. (Note that such Y cannot be paracompact, by [20, 5.1.27].)

The reader who knows such Y should skip to Section 7. Otherwise, to construct such Y, it suffices to find compact W with a closed set F which is the union of zerosets, is not G_δ, and with complement F' separable. Then $Y \equiv F'$ is locally compact (since F was closed), realcompact (as the intersection of cozerosets (see 2.0)), weakly Lindelöf (since separable), and not Lindelöf (since F was not G_δ).

The reader who knows such W should skip to Section 7. Otherwise, first take W_1 compact with points G_δ, with a closed set F not G_δ with weight $W_1 = \omega_1$. Here, we can in, Alexandrov's manner, double $[0,1]$ with one of its dense sets E of power ω_1: Take a copy E^* of E, with points $x^* \in E^*$ associated with points $x \in E$. Let $W_1 \equiv [0,1] \cup E^*$, with all points x^* isolated, and neighborhoods of $z \in [0,1]$ are $G \cup (G \cap E - \{z\})^*$, for neighborhoods G of z in $[0,1]$. See [20, 3.1.26]. Then by Parovičenvo's Theorem [20, p. 236], there is a compactification W of N with $W - N = W_1$. In W, F is still not G_δ, and all points are G_δ-whence F is the union of zerosets- and F' is separable (since it densely contains N).

7. A criterion for C-epicity

We now present, and shall subsequently work with, a description of C-epicity in terms of sets and points and functions, in contrast to the conditions in 2.2. Using this, proofs are available for most of the earlier propositions which were derived, one might say, *via* the higher mysticism; we shall sketch some of these proofs, indicating some technical independence from frames and so forth. Recall that $\mathcal{Z}^*_\sigma$ denotes the collection of all countable unions from $\mathcal{Z}^* = \{Z \in \mathcal{Z}(\beta X) \mid Z \cap X = \emptyset\}$.

7.1. Theorem. *The following are equivalent for a compactification (K, τ) of X.*

(a) *X is C-epic in K.*

(b) *$\forall h \in C(\beta X), \exists Z \in \mathcal{Z}^*_\sigma$ such that $(h(p) \neq h(q)$ and $\tau(p) = \tau(q))$ implies p OR $q \in Z$.*

(c) *$\forall$ disjoint pair $Z_1, Z_2 \in \mathcal{Z}(\beta X), \exists Z \in \mathcal{Z}^*_\sigma$ such that $(p \in Z_1, q \in Z_2$ and $\tau(p) = \tau(q))$ implies p OR $q \in Z$.*

(d) *$\forall p \neq q$ in βX, $\exists$ disjoint neighborhoods U, V and $\exists Z \in \mathcal{Z}^*_G$ such that $\tau(U \cap Z') \cap \tau(V \cap U') = \emptyset$ ("prime" being complement in βX).*

(b) is just the statement for our special circumstances of the characterization of W-epics in [8, 8.3.2] (see also 8.2.6 and 8.6.4, and perhaps [11, 2.7]). Elementary topological arguments prove the equivalence of (b)–(d); (d) is explicit in [11].

In 7.1, one sees the word "OR" capitalized for emphasis. This is explained in the next section.

We now sketch some proofs from 7.1.

7.2 (= 1.3). *If X is C-epic in K, then $\tau|\upsilon X$ is one-to-one.*

Proof. Any $Z \in \mathcal{Z}^*_\sigma$ misses υX, so if $p \neq q$ in υX with $\tau(p) = \tau(q)$, 7.1(b) must fail. □

7.3 (= 2.5). *If X is z-embedded in K, then X is C-epic in K.*

Proof. Let $h \in C(\beta X)$. By [22, 3.6], for each n there are $S_n \in \operatorname{coz}(K, X)$ and $f_n \in C(K)$ such that $|h(x) - f_n(x)| < \frac{1}{n}$ for each $x \in X$. Then $Z = \bigcup_n \tau^{-1}(S_n')$ works in 7.1(b) (S_n' being the complement in K). □

7.4 (= 2.7(b)). If there is $S \in \operatorname{coz}_\delta(\beta X, X)$ with $\tau|S$ one-to-one, then X is C-epic in K.

Proof. $Z = S' \in \mathcal{Z}_\sigma^*$, and works in 7.1(b) for *every* $h \in C(\beta X)$. □

7.5. Remarks. (a) It seems likely that 2.6 (thus 2.7(a)) can be derived from 7.1 and 7.3, in view of 4.2. But we do not see the argument. (b) Our proof of 3.1 did not use "the other categories", in fact, derived from 7.1, though this is well disguised.

8. More on Z-embedding versus C-epic embedding

An item of some interest is the relation explored in [13,26] between W-epicity and relative uniform density, some aspects of which we now describe.

For A an Archimedean l-group with a, a_n's, $u \in A$, the sequence (a_n) converges relatively uniformly to a with regulator u, written $a_n \to a(u)$, if, given k, there is $n(k)$ such that $n \geqslant n(k)$ implies $k|a_n - a| \leqslant u$. (Think of $A = C(X),\ u = 1,\ \varepsilon = \frac{1}{k}$.) For $S \subseteq A$, $r_0(S, A) = S$, for an ordinal $\beta = \alpha + 1$, $r_\beta(S, A) \equiv \{a \mid \exists (a_n) \subseteq r_\alpha(S, A)$ and $u \in A$ with $a_n \to a(u)\}$, and for a limit ordinal β, $r_p(S, A) \equiv \bigcup_{\alpha<\beta} r_\alpha(S, A)$; then $r_{\omega_1}(S, A) = r_{\omega_{1+1}}(S, A)$, and is called the *relative uniform closure* of S in A, denoted $r(S, A)$. One says that S is *relative uniformly dense* (r.u.d.) in A if $r(S, A) = A$. (These ideas are explicated briefly in the above references.) It is easy to see that an r.u.d. embedding is epic in Archimedean l-groups [13, 3.2.3.].

We recall the epicomplete monoreflection in W, from [9,8]: For each $A \in W$, there is an epic W-embedding $A \subseteq \beta A$, with βA epicomplete, such that each W-homomorphism to an epicomplete W-object extends uniquely over βA. (W-epicompleteness was described briefly in Section 3.) It is easy to see that an embedding $A \subseteq B$ is epic if and only if the composite embedding $A \subseteq \beta B$ is epic, and it is shown in [13] that an epic embedding $A \subseteq B$ in an *epicomplete* B has A r.u.d. in B. Thus,

8.1. Theorem.

(a) *In W, $A \subseteq B$ is epic if and only if $A \subseteq \beta B$ is r.u.d.*
(b) *For a compactification K of X, X is C-epic in K if and only if $C(K) \subseteq \beta C(X)$ is r.u.d.*

Theorem 8.1(b) is compelling, but we have not been able to see it as a productive point of analysis for C-epicity: $\beta C(X)$ is complicated (see [10,15]), and so is relative uniform density. But the question arises: What does it mean that $C(K)$ is r.u.d. in $C(X)$?

Another question, perhaps more facetious: What happens if "OR" is changed to "AND" in 7.1(b)?

8.2. Theorem. *For a compactification K of X, the following are equivalent*:

(a) $C(K)$ *is r.u.d. in* $C(X)$.

(b) $\forall h \in C(\beta X), \exists Z \in \mathcal{Z}^*_\sigma$ *such that* $(h(p) \neq h(q)$ *and* $\tau(p) = \tau(q))$ *implies* p *AND* $q \in Z$.

(c) X *is* z*-embedded in* K.

The proof will be presented as two separate more general theorems, one yielding (a) ⇔ (c), the other (b) ⇔ (c). The following is preliminary.

8.3. Zeroset factorizations. This material is from [22], and is needed in the proof of 8.2.

For K a compactification of X, recall the notation $\mathcal{Z}^*(K) = \{Z \in \mathcal{Z}(K) \mid Z \cap X = \emptyset\}$ and $\mathcal{Z}^*_\sigma(K) = \{\bigcup_n Z_n \mid Z_1, Z_2, \ldots \in \mathcal{Z}^*(K)\}$. Let $C_1 = \bigcup\{C(Z')|X \mid Z \in \mathcal{Z}^*(K)\}$, $C_2 = \bigcup\{C(Z')|X \mid Z \in \mathcal{Z}^*_\sigma(K)\}$. Then, $C(K) \approx C(K)|X \subseteq C_1 \subseteq C_2 \subseteq C(X)$ (and these are inclusions as W-subobjects and as l-subrings), and

(a) C_2 is the (usual) uniform closure of C_1;

(b) $C_2 = \{f \in C(X) \mid \forall a < b,\ f^{-1}(a,b) \in (\operatorname{coz} K) \cap X\}$;

(c) X is z-embedded in K if and only if $C_2 = C(X)$.

We will not need (b) in our proofs, but it provides some insight and perspective; likewise, the following:

(d) the maximal ideal space of C_2—call it K_2—is the minimum compactification of X over which all $f \in C_2^*$ extend continuously, is the Wallman space of $\mathcal{Z}(K) \cap X$-ultrafilters, and is the maximum among compactifications M of X for which $\mathcal{Z}(M) \cap X = \mathcal{Z}(K) \cap X$. A sometimes useful fact is that $K_2 = K$ if and only if each $S \in \operatorname{coz}(K, X)$ is C^*-embedded in K (i.e., $K = \beta S$).

Thus we have the "zeroset factorizations": The W-factorization $C(K) \subseteq C_2 \subseteq C(X)$, and the factorization of compactifications $K \leqslant K_2 \leqslant \beta X$. We comment on this further in 8.9 below.

The following, coupled with 8.3(c), proves the equivalence of (a) and (c) in 8.2.

8.4. Theorem. *For any compactification K of X,*

$$r\big(C(K), C(X)\big) = r_3\big(C(K), C(X)\big) = C_2.$$

Proof. We shall show that

(I) $C_1^* \subseteq r_1(C(K), C_1)$;

(II) $C_1 \subseteq r_1(C_1^*, C_1)$;

(III) $C_2 \subseteq r_1(C_1, C_2)$.

Then, $C_2 \subseteq r_1(C_1, C_2) \subseteq r_1(r_1(C_1^*, C_1), C_2) \subseteq r_2(C_1^*, C_2) \subseteq r_2(r_1(C(K), C_1), C_2) \subseteq r_3(C(K), C_2) \subseteq r_3(C(K), C(X)) \subseteq r(C(K), C(X))$.

To prove (I): Let $f \in C_1^*$ say $|f| \leqslant B$ with $B \geqslant 1$, and $f \in C(Z')|X$ for $Z \in \mathcal{Z}^*(K)$. Take $h \in C(K)$ with $0 \leqslant h \leqslant 1$ for which $Z(h) = Z$, and let $u = 1/(h|X)$. Let $Y_n = \{x \in K \mid h(x) \geqslant \frac{1}{n}\}$, so $Z' = \bigcup_n Y_n$. These Y_n's are compact, thus C^*-embedded in K, and we choose $g_n \in C(K)$ with $g_n|Y_n = f|Y_n$ and $|g_n| \leqslant B$. We then have $g_n \to f(u)$: Given k, choose $n(k) \geqslant 2Bk$, and let $n \geqslant n(k)$. If $x \in Y_n$, then $k|g_n(x) - f(x)| = 0 \leqslant u(x)$, and if

$x \notin Y_n$, then $n < u(x)$ and $k|g_n(x) - f(x)| \leqslant k(|g_n(x)| + |f(x)| \leqslant k(B + B) = 2Bk \leqslant n(k) \leqslant n < u(x)$.

(II) is immediate from the convergence $(f \wedge n) \vee (-n) \to f(f^2)$, valid in any Archimedean f-ring with identity. See [8].

(III) follows from 8.3(a).

We now show that $r(C(K), C(X)) \subseteq C_2$. Let $\mathcal{S}$ be the collection of all $\bigcap_n T_n$, where $T_1, T_2, \ldots$ are σ-compact subsets of K which contain X. We have

(IV) $\cup\{C(S)|X \mid S \in \mathcal{S}\} = C_2$.

("$\supseteq$" because $\mathcal{S} \supseteq \{Z' \mid Z \in \mathcal{Z}^*_\sigma(K)\}$. "$\subseteq$" because the members of $\mathcal{S}$ are Lindelöf (2.0); then, if $f \in C(S)$, S is z-embedded in K (4.2). f extends to a function on a set $\bigcap_n S_n$ for $S_1, S_2, \ldots \in \mathrm{coz}(K, S)$ by 8.3(c); but then $f \in C_2$.

Since $r(C(K), C(X)) = \bigcup_{\alpha<\omega_1} r_\alpha(C(K), C(X))$, it now suffices to show

(V) For each α, if $f \in r_\alpha(C(K), C(X))$, then f extends over some $S \in \mathcal{S}$.

We prove this by induction on α. For $\alpha = 0$, $S = K$ works. Suppose the statement is true for each $\alpha < \beta < \omega_1$, and let $f \in r_\beta(C(K), C(X))$. This means there is $u \in C(X), \alpha_n$'s $< \beta$ and $g_n \in r_{\alpha_n}(C(K), C(X))$ with $g_n \to f(u)$. By the induction hypothesis, for each n there is $S_n \in \mathcal{S}$ with $g_n \in C(S_n)|X$. Let $S = \bigcap S_n$; so $S \in \mathcal{S}$. Let $E_n = \{x \in X \mid u(x) \leqslant n\}$. Then $g_n \to f$ uniformly on E_n (in the usual sense), since the regulator u is bounded on E_n. Since all g_n extend continuously over S, it follows that f extends continuously over $\overline{E}_n^S = \overline{E}_n^K \cap S$, for each n. Then, by [21, 6H] f extends continuously over

$$\bigcup_n \overline{E}_n^S = \bigcup_n (\overline{E}_n^K \cap S) = \left(\bigcup_n \overline{E}_n^K\right) \cap S,$$

which set is a member of $\mathcal{S}$. Invoking induction completes this proof.

That completes the proof of 8.4. □

We now address the equivalence of (b) and (c) in 8.2. For $f \in C(X)$ (not necessarily bounded), there is the continuous extension $\beta f : \beta X \to R \cup \{\pm\infty\}$, and for $f \in C^*(X), \beta f \in C(\beta X)$. Let (K, τ) be a compactification of X. For the nonce, let us write $f \in \alpha C(K)$ (for $f \in C(X)$) to mean: there is $Z \in \mathcal{Z}^*_\sigma$ such that ($\beta f(p) \neq \beta f(q)$ and $\tau(p) = \tau(q)$) implies p *AND* $q \in Z$. (α stands for *AND*. Thus, 8.2(b) holds if and only if $C^*(X) \subseteq \alpha C(K)$.)

8.5. Lemma.

(a) *Let* $f \in C(X)$. $f \in \alpha C(X)$ *if and only if for each* n, $(f \wedge n) \vee (-n) \in \alpha C(K)$.

(b) 8.2(b) *holds if and only if* $\alpha C(K) = C(X)$.

Proof. (a). "$\Rightarrow$" is clear. Conversely, if the memberships $(f \wedge n) \vee (-n) \in \alpha C(K)$ are witnessed by Z_n's $\in \mathcal{Z}^*_\sigma$, then $Z \equiv \bigcup_n Z_n \in \mathcal{Z}^*_\sigma$ witnesses $f \in \alpha C(K)$.

(b) follows using (a). □

Now the following, combined with 8.3(c), proves (b) $\Leftrightarrow$ (c) in 8.2.

8.6. Theorem. *For any compactification* K *of* X, $\alpha C(K) = C_2$.

Our proof here is an exercise in the use of oscillations; see [20, 4.3]. Restricting to the case of a compactification K of X, we recall: For $f \in C(X)$, and $t \in K$, the oscillation of f at t is $\omega(f,t) \equiv \inf\{df(G \cap X) \mid G$ a neighborhood of $t\}$ (where d is the diameter). Then, f extends continuously to t (with real values) if and only if $\omega(f,t) = 0$, and $c(f) \equiv \{t \mid \omega(f,t) = 0\}$ is the largest subset of K over which f so extends. So, e.g., $f \in C_2$ if and only if $Z' \subseteq c(f)$ for some $Z \in \mathcal{Z}^*_\sigma(K)$.

Now (for $f \in C(X)$), let $\delta f \equiv \{p \in \beta X \mid \exists q \in \beta X$ with $\beta f(p) \neq \beta f(q)$ and $\tau(p) = \tau(q)\}$. So, e.g., $f \in \alpha C(K)$ if and only if there is $Z \in \mathcal{Z}^*_\sigma$ with $\delta f \subseteq Z$. The following shows how δf witnesses nonextendability over points of K. We omit the proof; it requires some calculation, but contains no surprises.

8.7. Lemma. *For a compactification (K,τ) of X, and $f \in C(X)$, $t \in K$, the following are equivalent*:

(a) $\omega(f,t) = 0$;

(b) $\beta f|\tau^{-1}(t)$ *is constant*;

(c) $t \in \tau((\delta f)')(()'$ *being complement in* βX).

Thus, $\tau^{-1}(c(f)) = (\delta f)'$.

Proof of 8.6. Let $f \in C_2$, say $f \in C(Z')|X$ for $Z \in \mathcal{Z}^*_\sigma(K)$. Thus $Z' \subseteq c(f)$, so $\tau^{-1}(Z') \subseteq \tau^{-1}(c(f)) = (\delta f)'$, hence $\delta f \subseteq (\tau^{-1}(Z'))' = \tau^{-1}(Z) \in \mathcal{Z}^*_\sigma$. Thus $f \in \alpha C(K)$.

Let $f \in \alpha C(K)$ so $\delta f \subseteq Z$ for $Z \in \mathcal{Z}^*_\sigma$. Then, $(\delta f)' \supseteq Z'$, so $\tau((\delta f)') \supseteq \tau(Z')$ and by 8.7, f extends continuously over $\tau(Z')$ to $f^\#$. Since Z' is Lindelöf (2.0), so is $\tau(Z')$, so by 4.2(b) and 5.3(c), $f^\#$ extends further to a function on a set $\bigcap_n S_n$ for $S_1, S_2, \ldots \in \operatorname{coz}(K, \tau(Z'))$. But this means $f \in C_2$. □

8.8. Two examples. (a) We mentioned before 8.1 that a W-epic embedding $A \subseteq B$ with A divisible and B epicomplete has A r.u.d. in B [13, 4.1.1]. An *ad hoc* construction of some length [13, Section 4.3] shows the need for the hypothesis "B epicomplete". The present development provides many such examples in a natural way: By 8.2, whenever K is a compactification of X in which X is C-epic but not z-embedded, then $C(K) \subseteq C(X)$ is W-epic but not r.u.d. Various such X, K are exhibited in 4.4, Section 6, and (b) below.

(b) Some rather striking examples of compactifications K of X for which $BK = B\beta X$ but $\mathcal{Q}K \neq \mathcal{Q}\beta X$ (because they exhibit the failure (C-epic $\not\Rightarrow$ weak z-embedding)) are obtained by combining the construction in Section 6 with the zeroset factorization of 8.3.

Begin with any X which is almost P (i.e., no proper dense cozeros) and not Lindelöf (e.g., uncountable discrete X) and any compactification K in which X is C-epic but not weakly z-embedded (e.g., use 6.3). Now consider the zeroset factorization $K \leqslant K_2 < \beta X$. Then, any L in the interval of compactifications $[K_1, K_2]$ has X C-epic in L (1.6), and $\mathcal{Z}(L) \cap X = \mathcal{Z}(K) \cap X$, so X is not weakly z-embedded in L. Moreover, due to features of K_2 described in 8.3 and the fact that X was almost P, K_2 is quasi-F and $\mathcal{Q}L = K_2 < \beta X$, thus $\mathcal{Q}L < \mathcal{Q}\beta X$ as covers, while $BL = B\beta X$ by 3.1. (Such K_2 thus provide a large, recognizable, stock of spaces which are $\mathcal{Q}F$, not *BD*.) Note that if X is a P-space, then βX is actually *BD*, and if X is discrete, then βX is actually *ED*.

8.9. Several questions. (a) The situation described in 8.2, 8.4, and 8.6 cries out for generalization to W, perhaps even to *Arch*. This would seem to represent a considerable project.

(b) The zeroset factorizations (for a compactification K of X) described in 8.3, $K \leqslant K_2 \leqslant \beta X$ and $C(X) \subseteq C_2 \subseteq C(X)$, are surely manifestations of the injective $\circ$ surjective factorization in σFrm of $t_k : \operatorname{coz} K \to \operatorname{coz} X$. For example, K_2 is surely the σFrm Stone–Čech compactification (in opposite description) of $t_k(\operatorname{coz} K) = \operatorname{coz} K \cap X$. One would like to see this situation put on firm footing.

(c) Analogously, W has (extremal monic) $\circ$ epic factorizations, as noted in [9]. So accordingly, $C(K) \subseteq C(X)$ can be factored as $C(K) \leqslant E \leqslant C(X)$, with first factor epic and the second extremal monic (which means E has no further epic extension in $C(X)$). Then the maximal ideal space K_E of E is a compactification of X, with $K \leqslant K_E \leqslant \beta X$. It is also fairly clear that $C(K) \subseteq C_2 \subseteq E \subseteq C(X)$ and $K \leqslant K_2 \leqslant K_E \leqslant \beta X$. While various further things can be said, the basic questions are: What *are* E and K_E? Of course, this amounts to a special case of the more general issue in W of describing the (extremal monic) $\circ$ epic factorizations; the issues raised in (a) are part of that.

(d) Next, we look at an aspect of the connection of these considerations with locales. There is a theorem of Isbell characterizing the W-objects which are of the form $C(L)$ for a locale L [32], further described in [36,12], among other places; L may as well be Lindelöf and then it is unique. Given a compactification K of X, the W-objects C_2 and E satisfy that criterion, so $C_2 = C(L_2)$ and $E = C(L_E)$ for unique Lindelöf locales L_2 and L_E, and $K_2 = \beta L_2$, $K_E = \beta L_E$, β being localic Stone–Čech compactification. The question is: What are L_2 and L_E? Of course, X is z-embedded in K if and only if $L_2 = \lambda X$ if and only if $\beta L_2 = \beta X$ and X is C-epic in K if and only if $L_E = \lambda X$ if and only if $\beta L_E = \beta X$ (λX discussed in Section 2). Almost certainly, $L_2 = \bigcap^{\mathrm{loc}} \operatorname{coz}(K, X)$, but we have not proved this. We have absolutely no idea what L_E might be.

9. Absolutely C-epic spaces

We now shall prove the following (stated earlier as 4.2(d)).

9.1. Theorem. *X is C-epic in each of its compactifications ($\equiv$ X is absolutely C-epic) if and only if υX is Lindelöf and $|\upsilon X - X| \leqslant 1$ ($\equiv$ X is almost Lindelöf).*

These spaces were called "almost Lindelöf" by R.L. Blair, by analogy with the term "almost compact" (see 4.1). The following internal description of these spaces seems more-or-less due independently to R.L. Blair (unpublished, we think) and Hager–Johnson [24, 4.2] (no proofs provided). One might compare 9.1 and 9.2 together with the descriptions of absolutely z-embedded spaces given in 4.2(b), [24,7].

9.2. Theorem. *The following are equivalent about X.*

(a) *υX is Lindelöf and $|\upsilon X - X| \leqslant 1$.*

(b) *Of each pair of disjoint zerosets, at least one is Lindelöf.*
(c) *For each zeroset Z, either Z or its complement Z' is Lindelöf.*
(d) *Either X is Lindelöf, or, $\{Z \in \mathcal{Z}(X) \mid Z' \text{ is Lindelöf}\}$ is the unique free z-ultrafilter with the countable intersection property.*

In the proofs of 9.2 and 9.1, we shall need to refer to the bijection association between z-ultrafilters $\mathcal{F}$ and the points of βX: Every $\mathcal{F}$ is uniquely of the form $\mathcal{F}_p = \{Z \in \mathcal{Z}(X) \mid p \in \overline{Z}^{\beta}\}$, and $\mathcal{F}_p$ has the countable intersection property if and only if $p \in \upsilon X$. See [21, Chs. 6 and 8].

Proof. (a) $\Rightarrow$ (b). Suppose υX is Lindelöf, and there are disjoint non-Lindelöf zerosets $Z_i = Z(f_i)$ $(i = 1, 2)$. Then $Z(\upsilon f_i)$ are disjoint and Lindelöf (υf_i being the extension of f_i over υX). Thus there is $p_i \in Z(\upsilon f_i) - X$, so $|\upsilon X - X| \geqslant 2$.

(b) $\Leftrightarrow$ (c). Suppose (b) and that $Z = Z(f)$ is not Lindelöf. Let $Z_n = \{x \mid |f(x)| \geqslant \frac{1}{n}\}$. These are zerosets disjoint from Z, thus Lindelöf, and so is $Z' = \bigcup_n Z_n$. Conversely, if Z_1 and Z_2 are disjoint zerosets with Z_1 not Lindelöf, then assuming (c), Z_1' is Lindelöf and so is its closed subset Z_2.

(c) $\Rightarrow$ (d). Assume (c), and suppose X is not Lindelöf. Let $\mathcal{F}^{\mathrm{L}} = \{Z \mid Z' \text{ is Lindelöf}\} = \{Z \mid Z \text{ is not Lindelöf}\}$ (since X is not Lindelöf). Then $\emptyset \notin \mathcal{F}^{\mathrm{L}}$ (since X is not Lindelöf); $\mathcal{F}^{\mathrm{L}} \ni Z \subseteq Z_1$ implies $Z' \supseteq Z_1'$, so Z_1' is cozero in Lindelöf Z', hence Lindelöf, hence $Z_1 \in \mathcal{F}^{\mathrm{L}}$; if $Z_1, Z_2, \ldots \in \mathcal{F}^{\mathrm{L}}$, then $(\bigcap Z_n)' = \bigcup Z_n'$ is Lindelöf, as the countable union of Lindelöf sets. Thus $\mathcal{F}^{\mathrm{L}}$ is a z-filter with cip. Now, X is locally Lindelöf, i.e., every point of X has a Lindelöf cozero neighborhood. (If x does not, then every zeroset not containing x is Lindelöf (by (c).) Then, for any cozero cover $\mathcal{U}$ of X, there is $U \in \mathcal{U}$ with $x \in U$; U' is Lindelöf, so $U' \subseteq \cup\mathcal{U}_0$ for some countable $\mathcal{U}_0 \subseteq \mathcal{U}$. Then $\{U\} \cup \mathcal{U}_0$ covers X. But X is not Lindelöf.) This means $X = \bigcup\{Z' \mid Z' \text{ is Lindelöf}\}$, hence $\cap\mathcal{F}^{\mathrm{L}} = \emptyset$, so $\mathcal{F}^{\mathrm{L}}$ is free.

For maximality, suppose $E \in \mathcal{Z}(X)$ with $E \cap Z \neq \emptyset$ $\forall Z \in \mathcal{F}^{\mathrm{L}}$. Were E Lindelöf, then $E \subseteq C$ for some Lindelöf cozeroset C (since X is locally Lindelöf). Thus $C' \in \mathcal{F}^{\mathrm{L}}$ and $E \cap C' = \emptyset$, which is a contradiction.

Finally, suppose $\mathcal{F}$ is another z-ultrafilter with cip. There are $Z_1 \in \mathcal{F}^{\mathrm{L}}$ and $Z_2 \in \mathcal{F}$ with $Z_1 \cap Z_2 = \emptyset$. Since Z_1 is not Lindelöf, Z_2 is (by (b)). Then $\{Z \cap Z_2 \mid Z \in \mathcal{F}\}$ is a z-filter with cip on the Lindelöf space Z_2, thus fixed. Thus $\mathcal{F}$ is fixed.

(d) $\Rightarrow$ (a). If X is Lindelöf, we are done, so suppose otherwise, and suppose (d). Since points of $\upsilon X - X$ correspond to free z-ultrafilters, $|\upsilon X - X| = 1$ with $\upsilon X = X \cup \{p_L\}$ and $Z \in \mathcal{F}^{\mathrm{L}}$ if and only if $p_L \in \overline{Z}^{\upsilon}$. Let $\mathcal{U}$ be a cozero cover of υX. Choose $U \in \mathcal{U}$ with $p_L \in U$, write $U = \bigcup Z_n$ as a union of zerosets of υX, and choose n with $p_L \in Z_n$. Now, $Z_n - \{p_L\} \in \mathcal{Z}(X)$ and $Z_n = \overline{Z_n - \{p_L\}}^{\upsilon}$ (by [21, 8.8(b)]). Thus $Z - \{p_L\} \in \mathcal{F}^{\mathrm{L}}$, hence $(Z_n - \{p_L\})'$ is Lindelöf, hence $(U - \{p_L\})'$ is Lindelöf, and thus contained in $\cup\mathcal{U}_0$ for some countable $\mathcal{U}_0 \subseteq \mathcal{U}$. So $\{U\} \cup \mathcal{U}_0$ covers υX.

Proof of 9.1. Almost Lindelöf implies absolutely C-epic: Suppose X is almost Lindelöf, and K is a compactification with canonical surjection $K \overset{\tau}{\leftarrow} \beta X$.

First proof from the frame-theoretic 2.7(b): υX is Lindelöf by hypothesis, and $\tau|\upsilon X$ is one-to-one since $\tau(\beta X - X) \subseteq K - X$ [21, 6.11], $\tau|X$ is the identity, and $|\upsilon X - X| \leqslant 1$. So 2.7(b) applies.

Second proof from 7.1(b): This is more laborious, and we sketch it. We have $\upsilon X = X \cup \{p_L\}$. Let $Y = \tau(\upsilon X) = X \cup \{\tau(p_L)\}$; Y is Lindelöf and K is a compactification of Y. Consider the embedding $\rho_K : C(K) \hookrightarrow C(X)$ factored as $C(K) \overset{\gamma}{\hookrightarrow} C(Y) \overset{\delta}{\hookrightarrow} C(X)$ (each embedding being the appropriate restriction homomorphism). We show that γ and δ are epic, thus too $\delta\gamma = \rho_K$. Since Y is Lindelöf and therefore z-embedded in K (4.2(b)), γ is epic by 7.3. Now we have $C(\beta Y) \approx C^*(Y) \hookrightarrow C(Y) \overset{\delta}{\hookrightarrow} C(X)$; it suffices to show that $C(\beta Y)$ is epic in $C(X)$, for then the second factor δ will be epic too. That is, we want to see that X is C-epic in βY. This follows from (i) and (ii) below.

(i) Whenever (any) X is dense in a space $X \cup \{p\}$, there is a closed set F in βX, $F \subseteq \beta X - X$, with $\beta(X \cup \{p\}) = \beta X/F$. (We omit this standard proof.)

(ii) If X is almost Lindelöf, with $\upsilon X = X \cup \{p_L\}$, and F is a closed set in βX with $p_L \in F \subseteq \beta X - X$, then X is C-epic in $\beta X/F$. (To satisfy 7.1(b) here, given $h \in C(\beta X)$, it suffices to surround $\{p \in F \mid h(p) \neq h(p_L)\}$ by a $Z \in \mathcal{Z}^*_\sigma$. For each n, let $F_n = F \cap \{p \mid |h(p) - h(p_L)| \geqslant \frac{1}{n}\}$. This is closed in $\beta X = \beta(\upsilon X)$ and $F_n \cap \upsilon X = \emptyset$. Since υX is Lindelöf, by Smirnov's theorem in 2.0, there is $Z_n \in \mathcal{Z}^*$ with $Z_n \supseteq F_n$. Then $Z = \bigcup Z_n \in \mathcal{Z}^{*\sigma}$ is as required. □

Comment. The degree of complication of the second proof above argues for the frame-theoretic approach. On the other hand, it seems likely that some argument vaguely like the second proof would prove 2.7(b) from 7.1(b); but we do not see it.

Absolutely C-epic implies almost Lindelöf: If $p, q \in \upsilon X - X$ with $p \neq q$, then $\beta X/_{\{p,q\}}$ is not C-epic over X (1.4). Thus absolutely C-epic implies $|\upsilon X - X| \leqslant 1$.

To see that υX is Lindelöf, note that for a compactification K of υX, K is C-epic over υX if and only if K is C-epic over X. So it suffices to show that a realcompact non-Lindelöf Y has a non$C$-epic compactification. This follows from (II), (V), and (VI) below.

A closed subset F of βY with $F \subseteq \beta Y - Y$, for which there is *no* $Z \in \mathcal{Z}^*$ with $Z \supseteq F$, will be called an *obstacle* for Y (an obstacle to the normal placement of Y in βY; see 2.0).

- (I) Y is realcompact if and only if no point is an obstacle.
- (II) Y is Lindelöf if and only if Y has no obstacles.
- (III) Let F be closed in βY with $F \subseteq \beta Y - Y$. F is an obstacle if and only if there is no $Z \in \mathcal{Z}^*_\sigma$ with $Z \supseteq F$.
- (IV) If F is an obstacle for realcompact Y, then the set $F^\# \equiv \{p \in F \mid \forall$ neighborhood G of p, $F \cap \overline{G}$ is an obstacle$\}$ is uncountable.
- (V) If realcompact Y has an obstacle, then Y has two disjoint obstacles.
- (VI) If F_0 and F_1 are disjoint obstacles for Y, then $\beta Y/F_0 \cup F_1$ is not C-epic over Y.

(I) and (II) are restatements of facts noted in 2.0.

To prove (III): $\Leftarrow$ is clear. Conversely, suppose there is $Z \in \mathcal{Z}^*_\sigma$ with $Z \supseteq F$. Since Z' is Lindelöf (see 2.0), $Z' \cap F = \emptyset$ and $\beta Z' = \beta Y$, (II) asserts $Z_1 \in \mathcal{Z}^*$ with $Z_1 \cap Z' = \emptyset$ and $Z_1 \supseteq F$.

To prove (IV): If $F^\#$ is countable, as $F^\# = \{p_n\}$, then by (I) for each n there is $Z_n \in \mathcal{Z}^*$ with $p_n \in Z_n$, and $Z \equiv \bigcup Z_n \supseteq F^\#$. Now, Z' is Lindelöf (again 2.0), and its closed subset $F \cap Z'$ is too, and $F \cap Z' \subseteq F - F^\#$. Thus, for each $q \in F \cap Z'$, there is a neighborhood G_q such that $F \cap \overline{G}_q$ is no obstacle. This means there is $Z_q \in \mathcal{Z}^*$ with $Z_q \supseteq F \cap \overline{G}_q$. Now let $\{G_{q_n}\}$ cover the Lindelöf set $F \cap Z'$. We have

$$F \cap Z' \subseteq \left(\bigcup_n G_{q_n}\right) \cap F = \bigcup_n (G_{q_n} \cap F) \subseteq U_n Z_{q_n}.$$

Let $Z_1 = \bigcup_n Z_{q_n} \in \mathcal{Z}^*_\sigma$. Now we have

$$F = (F \cap Z) \cup \big(F \cap Z'\big) \subseteq Z \cup Z_1 \in \mathcal{Z}^*_\sigma,$$

which, using (III), contradicts that F was an obstacle.

To prove (V): By (IV), there are distinct $p_0, p_1 \in F^\#$. Take neighborhoods $G_0,\ G_1$ with disjoint closures. Then $F_i = F \cap \overline{G}_i$ $(i = 0, 1)$ are disjoint obstacles.

To prove (VI): Let $K \equiv \beta Y / F_0 \cup F_1 \xleftarrow{\tau} \beta X$ be the surjection. Take $h \in C(\beta Y)$ with $h | F_i = i$ $(i = 0, 1)$. Then, for every choice $x_i \in F_i$ we have $x_0 \neq x_1$ and $\tau(x_0) = \tau(x_1)$. If these were $Z \in \mathcal{Z}^*_\sigma$ satisfying 7.1(b) for our h, then there would be $x_0 \in F_0 - Z$ (for otherwise $Z \supseteq F_0$, and F_0 is no obstacle, using (III)), and then for every $x_1 \in F_1,\ x_1 \in Z$ (since $x_0 \notin Z$), so F_1 is no obstacle. Thus there is no such Z, so K is not C-epic.

Comment. The proof of 9.1 has been formulated to avoid direct reference to the frame theory in Theorem 2.2. However, the frame theory *is* there. Just to illustrate: It can be shown that the $F^\#$ in (IV) is the largest closed subspace of F which is a *SpFi*-subspace of $(\beta Y, \mathrm{coz}(\beta Y, Y))$ and $\{f \in C(X) | \beta f \mid F^\# = 0\}$ is the least ideal in $\ker C(X)$ which contains $\{f \in C(X) | \beta f \mid F = 0\}$. (See Section 2 and [11].) Of course, 9.1 is equivalent, *via* 2.2, to a number of (new) theorems of frame theory, and our frame-free proof may be construed as a proof of theorems of point-free topology using the traditional topological objects, points, sets, and functions.

Just for example, we have shown

9.1σ. Theorem. *The space X has the property that, for each compactification K, the trace $t_k : \mathrm{coz}\, K \to \mathrm{coz}\, X$ is epic in σFrm, if and only if X is almost Lindelöf.*

Doubtless, 9.1σ has a generalization to frames X analogous to the generalization of 4.2(b) carried out elegantly in [6].

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