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α-Specker spaces

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Abstract

This paper is motivated by work on Specker spaces and a recent article of the authors on the ring of α-quotients. Here α denotes an uncountable regular cardinal or else ∞, indicating no cardinal constraint whatsoever. All spaces are compact Hausdorff, and for the most part zero-dimensional.

Three strains of the "α-Specker" condition are studied: strong, weak, and one in between which is not qualified. One of the main results characterizes these conditions, for each α and each space X, in terms of the containment of $C(X)$ in the ring of α-quotients of $S(K_\alpha X)$, where the latter denotes the algebra of continuous functions with finite range, defined on an appropriate cover $K_\alpha X$ of X.

"Weakly $\mathfrak{c}^+$-Specker" is equivalent to "Specker". The paper examines the ω_1-Specker conditions, proving that "weakly ω_1-Specker" and "ω_1-Specker" are equivalent. In fact, it is shown that X is weakly ω_1-Specker if and only if for each $f \in C(X)$ there is a Baire set B with meagre complement such that $f(B)$ is countable.

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1. Introduction

The notion of a Specker space, over which continuous real valued functions are densely constant, first appeared in [25] and [31], and then received a concerted amount of attention in [4,24] and also [5]. Here we get into the subject of dense constancies, with cardinality constraints. Some of the motivation for this investigation comes from [14], where the

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laterally σ-complete reflection in $\boldsymbol{W}$, the category of all Archimedean ℓ-groups with designated unit, is described using countable partitions by Baire sets.

All topological spaces are Tychonoff. βX is the Stone–Čech compactification of X. $C(X)$ will denote, as usual, the ring of all continuous real valued functions on the space X, which is regarded as a lattice-ordered ring with the standard pointwise operations of addition, multiplication, supremum and infimum. Of some importance in any conversation about Archimedean ℓ-groups is $D(X)$, the lattice of all continuous functions f on X with values in the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$, for which $f^{-1}\mathbb{R}$ is a dense subset of X. In order for $D(X)$ to be a group under pointwise addition some assumptions need to be made about X; we need not consider this issue here.

Let us now recall the concept of a Specker space.

Definition 1.1. First recall (from [4]) that a space X is a *DC-space* if for each $f \in C(X)$ there is a quasi-partition of X by open sets $(U_\lambda)_{\lambda \in \Lambda}$—that is to say, a family of open sets which are pairwise disjoint so that the union is dense in X—such that f is constant when restricted to each U_λ. We say that X is *Specker* if it is a DC-space with a clopen π-base. (A family $\mathcal{B}$ of nonempty open sets in X is called a *π-base* if for each nonempty open set V there is a $U \in \mathcal{B}$ such that $U \subseteq V$.) If X is Specker then for each $f \in C(X)$ the quasi-partition referred to above may be chosen so that each U_λ is clopen.

We generalize the notion of a Specker space, by placing cardinality constraints on the quasi-partitions referred to above. A number of the hulls in the theory of Archimedean ℓ-groups play an important role.

Throughout the paper, α stands for an uncountable regular cardinal, or else the symbol ∞, which may be thought of in this context as a symbol larger than all cardinals, and as indicating the case in which no cardinality restrictions are placed.

Definition & Remarks 1.2. The ambient category in this discussion is $\boldsymbol{W}$, the category of Archimedean ℓ-groups with designated weak order unit, and the ℓ-homomorphisms which preserve the designated units.

(a) First, by an *extension* in $\boldsymbol{W}$ we mean the containment of one $\boldsymbol{W}$-object A in another, B, so that the inclusion mapping is a $\boldsymbol{W}$-morphism; we denote this by $A \leqslant B$.

Recall that the extension $A \leqslant B$ is *essential* if for each $0 < b \in B$ there is an $a \in A$ and a positive integer n, such that $0 < a \leqslant nb$. For a given $\boldsymbol{W}$-object A, eA denotes the maximum essential extension of A in $\boldsymbol{W}$; see [6]. We call this the *essential hull* of A. $p(\alpha)$, $c(\alpha)$ and $l(\alpha)$, stand for the operators which construct the α-projectable hull, the conditional α-completion and the lateral α-completion, respectively. All of these are extensions inside the essential hull. Let us now recall how these hulls are obtained.

(b) Let A be a $\boldsymbol{W}$-object and $S \subseteq A$. $S^\perp$ stands for the polar of S. We explain, briefly: recall that $P \subseteq A$ in a $\boldsymbol{W}$-object A is a *polar* if $P = S^\perp$ where $S \subseteq A$ and

$$S^\perp \equiv \{a \in A\colon\ |a| \wedge |s| = 0,\ \forall s \in S\}.$$

For $C(X)$ the polars may be viewed as follows; let $S \subseteq C(X)$. Let

$$\widehat{S} = \bigcup \{\operatorname{coz}(f)\colon\ f \in S\}.$$

Then $S^{\perp\perp} = \{h \in C(X)\colon \operatorname{coz}(h) \subseteq \operatorname{cl}_X \widehat{S}\}$, while $S^{\perp} = \{g \in C(X)\colon \operatorname{coz}(g) \cap \widehat{S} = \emptyset\}$.

We shall observe the notational conventions established in [12] and [15]. If $a \in S^{\perp\perp} + S^{\perp}$ we may express $a = a[S] + a[S^{\perp}]$, uniquely, with $a[S] \in S^{\perp\perp}$ and $a[S^{\perp}] \in S^{\perp}$. $a[S]$ will be referred to as the *projection of a on S*.

Recall that A is *α-projectable* if for each subset S of size $< \alpha$, $A = S^{\perp\perp} + S^{\perp}$. (On this occasion let us also consider ω; "ω-projectable" *is* "projectable": $A = a^{\perp\perp} + a^{\perp}$, for each $a \in A$.)

Now, let us begin with $p(\alpha)$: if A is a $\boldsymbol{W}$-object, $p(\alpha)A$ is obtained through a transfinite construction:

(i) $p^1(\alpha)A$ is the ℓ-subgroup of eA generated by all $a[S]$, ranging over all $a \in A$ and $S \subseteq A$, with $|S| < \alpha$.

(ii) If γ is an ordinal > 1,

$$p^{\gamma}(\alpha)A = p^1(\alpha)\left(\bigcup_{\beta<\gamma} p^{\beta}(\alpha)A\right).$$

For some ordinal γ, $p^{\gamma}(\alpha)A = p^{\gamma'}(\alpha)A$, for all $\gamma' > \gamma$. This is $p(\alpha)A$.

(c) Here we elect to describe the other two hulls for objects which are already α-projectable, because it is easier to do. This will suffice, for now; we have occasion to refer to the construction of $l(\alpha)$ in Remark 2.12.

Let A stand for an α-projectable $\boldsymbol{W}$-object.

$$l(\alpha)A^{+} = \left\{\bigvee_i x_i\colon\ 0 \leqslant x_i \in A,\ i \neq i' \Longrightarrow x_i \wedge x_{i'} = 0,\ i \in I,\ |I| < \alpha\right\},$$

$$c(\alpha)A^{+} = \left\{\bigvee_i x_i\colon\ 0 \leqslant x_i \in A,\ \{x_i\colon\ i \in I\} \text{ is } A\text{-bounded}, |I| < \alpha\right\}.$$

To conclude our introduction, let us review the Yosida representation. We refer the reader to [18] for more details.

Definition & Remarks 1.3. A stands for a $\boldsymbol{W}$-object. YA is the set of values of the designated unit e, that is to say, the set of all the convex ℓ-subgroups of A which are maximal with respect to not containing e. Relative to the hull-kernel topology YA is a compact Hausdorff space; this is well known. YA is the *Yosida space* of A.

If X is any compact Hausdorff space and $G \subseteq D(X)$ is a sublattice containing the constant 1, then *G is a $\boldsymbol{W}$-object in $D(X)$* if G is an ℓ-group and $(f+g)(x) = f(x)+g(x)$, for each $x \in U$, where U is a dense open subset of $f^{-1}\mathbb{R} \cap g^{-1}\mathbb{R}$.

Now back to our $\boldsymbol{W}$-object A with designated unit e; here is a formulation of the Yosida Representation Theorem:

There is a $\boldsymbol{W}$-object A' in $D(YA)$ and a $\boldsymbol{W}$-isomorphism $\theta : A \to A'$ such that $\theta(e) = 1$, which separates the points of YA; (that is, so that if $x \neq y$ in YA, then there is an $a \in A$ such that $\theta(a)(x) \neq \theta(a)(y)$).

YA is unique in the sense that if X is a compact Hausdorff space, and $\phi : A \to B$ is a $\boldsymbol{W}$-isomorphism of A onto the $\boldsymbol{W}$-object B in $D(X)$, such that $\phi(e) = 1$ and ϕ separates the points of X, then there is a homeomorphism $\tau : X \to YA$ such that $\theta(a)(\tau(x)) = \phi(a)(x)$, for each $x \in X$ and each $a \in A$.

The Yosida Representation is functorial. To see this the reader should note that if $g : A \to B$ is a $\boldsymbol{W}$-morphism, then there is induced a continuous function $Yg : YB \to YA$, such that $\theta_B(g(a))(x) = \theta_A(a)(Yg(x))$, for each $a \in A$ and $x \in YB$.

Remark 1.4. Some properties of hulls might be kept in mind. Let A be a $\boldsymbol{W}$-object.

(a) $p(\alpha)A \leqslant c(\alpha)A \cap l(\alpha)A$ [15].
(b) $c(\alpha)l(\alpha) = l(\alpha)c(\alpha)$. This is the hull for the class of $\boldsymbol{W}$-objects which are both conditionally and laterally α-complete [15, Theorem 4.1].
(c) If A is projectable then YA is zero-dimensional. (This is well known; we leave it as an exercise to the reader.)
(d) $Yp(\alpha)A$ is α-disconnected, that is, every union of fewer than α cozerosets has open closure [12, Corollary 2.4].
(e) $D(Yp(\alpha)A)$ is a group, and a $\boldsymbol{W}$-object, and the uniform closure of the divisible hull of $l(\alpha)A$ [15, Theorem 5.5(b)].

In the next section we shall define one of the principal "Specker" conditions in terms of the ring of α-quotients. Rather than give a general review of the concept, as discussed in [17], we shall appeal directly to Theorem 5.4 of that paper, which gives a working definition of the ring of quotients we want.

Definition & Remarks 1.5. (a) Recall some common notation first. In general, if $\mathcal{H}$ is a filter base of dense subsets of the space X, we use $C[\mathcal{H}]$ to denote the direct limit

$$C[\mathcal{H}] \equiv \varinjlim C(U), \quad U \in \mathcal{H},$$

with the understanding that the bonding maps are the restrictions $C(U) \to C(V)$, with $V \subseteq U$ in $\mathcal{H}$.

In the sequel $\mathcal{G}_\alpha(X)$ stands for the filter base of all dense α-cozerosets of X. (An *α-cozeroset* is a union of fewer than α cozerosets.)

(b) Suppose that A is an Archimedean f-ring with identity, and regard 1 as the designated unit. For each dense open subset U of YA, let A_U be the subring of $C(U)$ defined by the following condition: $f \in A_U \Longleftrightarrow f \in C(U)$ and, for each $x \in U$ there exists a neighborhood $V \subseteq U$ of x and $a, b \in A$ such that, for each $y \in V$,

$$f(y) = \frac{a(y)}{b(y)}.$$

Now put

$$Q_\alpha A \equiv \varinjlim A_U \quad \big(U \in \mathcal{G}_\alpha(YA)\big),$$

and it is understood, once more, that the bonding maps of the above direct limit are the restrictions $A_U \to A_V$, when $V \subseteq U$ in $\mathcal{G}_\alpha(YA)$. This is the ring of α-quotients, which is studied extensively in [17] and also in [27].

(c) We spell out what the description in (a) reduces to for $C(X)$, with X compact Hausdorff. Quite simply,

$$Q_\alpha C(X) = C\big[\mathcal{G}_\alpha(X)\big] \quad \text{[17, Proposition 5.7].}$$

We record the following special case for $\alpha = \omega_1$. The reader is reminded that qA denotes the classical ring of quotients of the ring A.

Proposition 1.6 [17, Proposition 5.8]. *For any space X,*

$$Q_{\omega_1} C(X) = qC(X) = C\big[\mathcal{G}_{\omega_1}(X)\big].$$

This concludes our introduction and general review.

2. Various α-Specker conditions

We remind the reader that α stands for a regular, uncountable cardinal, or the symbol ∞. In this section, X denotes a compact Hausdorff space, unless the contrary is stated. As we have already indicated, in considering Specker conditions linked to cardinal bounds, a number of subtleties appear, and at least three natural generalizations of Specker spaces deserve some attention.

Definition & Remarks 2.1. (a) A space X is *weakly α-Specker* if it has a clopen π-base and for each $f \in C(X)$ there is a quasi-partition $\{V_i\colon i \in I\}$ of X by open sets with $|I| < \alpha$ such that f restricted to each V_i is constant. Observe immediately that if $\alpha < \alpha'$ are cardinals, then any weakly α-Specker space is weakly α'-Specker.

By dropping the "clopen π-base" provision, one gets a definition of what probably ought to be called a *weakly $DC(\alpha)$-space*; we will not explore this concept here.

Suppose that X is a Specker space; let $f \in C(X)$ and $\mathcal{K}$ be a quasi-partition by clopen sets such that $f|_K$ is constant, for each $K \in \mathcal{K}$. Now let $\mathcal{K}'$ be the quasi-partition defined as follows: $K \in \mathcal{K}'$ if and only if K is the union of all $C \in \mathcal{K}$ which have the same image under f. This is indeed a quasi-partition by *open* sets, and it should be clear that $|\mathcal{K}'| \leqslant \mathfrak{c}$, where $\mathfrak{c}$ stands for the cardinality of the continuum. Thus, X is weakly $\mathfrak{c}^+$-Specker. (Note: α^+ denotes the successor cardinal of α.)

We emphasize:

"Specker" *and* "weakly $\mathfrak{c}^+$-Specker" *are equivalent.*

(b) X is *α-Specker* if $C(X) \leqslant l(\alpha)S(X)$. (Recall that $S(X)$ stands for the subalgebra of $C(X)$ consisting of all continuous functions of finite range.) As with the previous Specker condition, $\alpha < \alpha'$ implies that any α-Specker space is also α'-Specker.

We have not been able to settle whether "weakly $\mathfrak{c}^+$-Specker" implies "$\mathfrak{c}^+$-Specker".

(c) We say that a space X is *strongly α-Specker* if $C(X) \subseteq Q_\alpha S(X)$. In 2.2 we give an example showing that, for each α, there is a space which is α^+-Specker but not strongly α^+-Specker. In particular then, "$\mathfrak{c}^+$-Specker" does not imply "strongly $\mathfrak{c}^+$-Specker".

As the terms suggest, "strongly α-Specker" implies "α-Specker", and the latter implies "weakly α-Specker", in turn. To get on with the proofs of these and other results, we need to describe the ring of α-quotients of $S(X)$, using the description in 1.5(b).

As we shall see, the weak Specker condition is connected to the absolute EX of the space of discourse X, while the "middle" Specker condition is connected to the minimum α-disconnected cover. We shall have more to say about minimum covers in the next section.

Section 5 concentrates on the case $\alpha = \omega_1$. For now, by way of illustration, let us record some basic features of $Q_{\omega_1} S(X)$, from §5 of [17].

Remark 2.2. Recall that X is an *almost P-space* provided it has no proper dense cozerosets. Proposition 5.10 of [17] states that $Q_{\omega_1} S(X) = S(X)$ precisely when X is an almost P-space. Thus, when $X = \alpha D$, the one-point compactification of the discrete space D, then $Q_{\omega_1} S(\alpha D) = S(\alpha D)$ if and only if D is uncountable. On the other hand, $Q_{\omega_1} S(\alpha\mathbb{N}) = C(\mathbb{N}) = QS(\alpha\mathbb{N})$. In particular, note that

$$Q_{\omega_1} S(\alpha\mathbb{N}) > qS(\alpha\mathbb{N}) = S(\alpha\mathbb{N}).$$

Now let D be uncountable and λD be the space obtained by adjoining a point λ to D, whose neighborhoods are the subsets containing λ, having a countable complement in D. Then note that $l(\omega_1)S(\alpha D) = S(\beta\lambda D)$. The point is that $C(\alpha D) \leqslant l(\omega_1)S(\alpha D)$, so that αD is ω_1-Specker, but it is not strongly ω_1-Specker.

More precisely, if $|D| = \alpha^+$, then αD, being ω_1-Specker, is α^+-Specker. However, any quasi-partition by clopen sets of αD must have size α^+, and so αD is not strongly α^+-Specker, as Proposition 2.3 will presently demonstrate.

Here is the description of $Q_\alpha S(X)$, with X zero-dimensional, following 1.5(b). Note that since X is assumed to be zero-dimensional, the continuous step functions on X separate the points of X, whence $YS(X) = X$.

Proposition 2.3. *Suppose X is zero-dimensional. Then the following are equivalent for a function $f \in C(X)$.*

(a) $f \in Q_\alpha S(X)$.
(b) *There is a family $\mathcal{K}$ of fewer than α clopen sets, such that, $\bigcup \mathcal{K}$ is dense and, for each $K \in \mathcal{K}$, $f|_K \in S(K)$.*
(c) *There is a quasi-partition $\mathcal{K}$ by clopen sets, with $|\mathcal{K}| < \alpha$, such that $f|_K$ is constant, for each $K \in \mathcal{K}$.*

Proof. The equivalence of (a) and (b) comes from Theorem 5.4 of [17]. (c) clearly implies (b), and as to the reverse, suppose there is a family $\mathcal{K}$ of clopen sets, of size $< \alpha$, whose union is dense, and such that $f|_K \in S(K)$, for each $K \in \mathcal{K}$. Refining, we may assume that the members of $\mathcal{K}$ are pairwise disjoint and $f|_K$ is constant, for each $K \in \mathcal{K}$. $\square$

At last, here is one of the expected implications between Specker conditions.

Corollary 2.4. *Let X be zero-dimensional.*

(a) *X is strongly α-Specker if and only if* (c) *in Proposition* 2.3 *holds for each $f \in C(X)$.*
(b) *If X is strongly α-Specker then it is also α-Specker.*

Proof. (a) is obvious from Proposition 2.3. As to (b), observe that $Q_\alpha S(X) \leqslant l(\alpha)S(X)$: using Proposition 2.3(c), if $f \in Q_\alpha S(X)$, and $\mathcal{K}$ is a quasi-partition which witnesses this, according to (c) of the proposition, one easily checks that f is the supremum (in the essential hull of $S(X)$) of the step functions g_K defined by $g_K(x) = f(x)$, if $x \in K$, while $g_K(x) = 0$, otherwise. This says that $f \in l(\alpha)S(X)$. □

Remark 2.5. The weak α-Specker condition is "internal", in the sense that it is articulated in the definition in terms of quasi-partitions of subsets of the space. Now Proposition 2.3 affords, likewise, an internal description of the strong α-Specker condition. We have not been able to find such an internal condition for the unqualified α-Specker condition, except for the two extreme cases, ω_1 and ∞. In the first case, it will be shown—Corollary 5.6—that "weakly ω_1-Specker" and "ω_1-Specker" mean the same thing; as for $\alpha = \infty$, the three Specker conditions are equivalent.

Here is an immediate consequence of Proposition 2.3.

Corollary 2.6. *Suppose that X is zero-dimensional, with a dense discrete subset of size α. Then X is strongly α^+-Specker.*

Remark 2.7. The situation in the preceding corollary is limiting, in the sense that if X is the one-point compactification of a discrete set D, with $|D| = \alpha$, then, as has already been argued before, X is not strongly α-Specker.

Next, we single out some special situations, both of which are obvious consequences of Proposition 2.3. Recall that a space is *extremally disconnected* if the closure of any open set is open. A (not necessarily compact) space X is a *P-space* if the family of open sets is closed under countable intersection. It is well known that X is a P-space if and only if every zeroset of X is open [8, 14.28].

Corollary 2.8.

(a) *If X is extremally disconnected and weakly α-Specker, then it is also strongly α-Specker.*
(b) *If Y is a P-space, then βY is strongly $\mathfrak{c}^+$-Specker.*

Proof. (a) is evident from Proposition 2.3. As to (b), $f \in C(\beta Y)$, then $f|_Y$ induces the partition of Y $\{f^{-1}\{r\}\colon r \in \mathbb{R}\}$, which consists of clopen sets and has cardinal $\leqslant \mathfrak{c}$. The closures of the members of such a partition of Y form a quasi-partition of βY. □

Example 2.9. Note that if $|D| = \omega_1$, and D is discrete, then βD is strongly $\mathfrak{c}^+$-Specker, but not ω_1-Specker in any of its flavors, on account of Corollary 2.8.

Of some importance in the next section is the following consequence of Proposition 2.3.

Proposition 2.10. *For each zero-dimensional space X, $Q_\alpha S(X)$ is projectable.*

Proof. Suppose that $f, g \in Q_\alpha S(X)$, and suppose that $\mathcal{K}_f$ and $\mathcal{K}_g$ are quasi-partitions by clopen sets, with $|\mathcal{K}_f|, |\mathcal{K}_g| < \alpha$ that witness, respectively, this membership. Consider the family $\mathcal{L}$ of all $K \cap L$, with $K \in \mathcal{K}_f$ and $L \in \mathcal{K}_g$. This defines a quasi-partition by clopen sets with fewer than α sets. We proceed to write $f = f[g] + f[g^\perp]$. These components of f are to be defined, piecewise, on $\bigcup \mathcal{L}$. Put $f[g](x) = f(x)$, provided that $x \in K \cap L$, with $K \in \mathcal{K}_f$ and $L \in \mathcal{K}_g$, and L such that the constant $g|_L \neq 0$. $f[g](x) = 0$, if $x \in K \cap L$, with $K \in \mathcal{K}_f$ and $L \in \mathcal{K}_g$, and L such that $g|_L = 0$. $f[g^\perp] \equiv f - f[g]$. Since the members of $\mathcal{K}_f$ and those of $\mathcal{K}_g$ are pairwise disjoint these elements are unambiguously defined, and they are continuous on $\bigcup \mathcal{L}$, because those members are clopen sets. It should be evident that $f[g] \in g^{\perp\perp}$ and $f[g^\perp] \in g^\perp$. □

Remark 2.11. One cannot conclude from Proposition 2.10 that $Q_\alpha S(X)$ is α-projectable. Consider the example in Remark 2.2: let D be an uncountable discrete set. As indicated in 2.2, $Q_{\omega_1} S(\alpha D) = S(\alpha D)$. This is not ω_1-projectable; for if N is any countably infinite subset of D and

$$P = \{ f \in S(\alpha D): \ f(d) = 0, \ \forall d \in D \setminus N \},$$

then P is an ω_1-polar which is not a summand.

Finally, in this section, we should like to record an observation about condition (c) in Proposition 2.3 which we find intriguing.

Remark 2.12. For any cardinal α and any $\boldsymbol{W}$-object A, the construction of $l(\alpha)$ goes through a transfinite iteration, the first step of which is $l^1(\alpha)A$, the ℓ-subgroup of eA generated by the suprema of all pairwise disjoint subsets $S \subseteq A$, such that $|S| < \alpha$. With this in mind, (c) in Proposition 2.3, may be interpreted as saying that $f \in l^1(\alpha)S(X)$. Thus, $Q_\alpha S(X) \leqslant l^1(\alpha)S(X)$.

In the next section we review the matter of covers of compact spaces. The analysis of these covers gives some insight into the relationship between the various Specker conditions; we shall express this in terms of the containment of $C(X)$ in $Q_\alpha S(K_\alpha X)$, where $K_\alpha X$ is a particular minimum cover of X (Theorem 4.7).

3. Minimum covers

Once more, in this section, all spaces are assumed to be compact and Hausdorff. We review the aspects of the theory of minimum covers with specified conditions that will be needed further on in the paper. The work of Gleason marks the origin of this subject; the reader may read about this in Chapter 10 of [30]. For general information on covering classes, see also [10] and [29].

Definition & Remarks 3.1. (a) Suppose that $f : Y \to X$ is a continuous surjection. f is said to be *irreducible* if X is not the image of a proper closed subset of Y. It is well known that if f is irreducible then for each open set $U \subseteq Y$ there is an open set $V \subseteq X$ such that $f^{-1}V$ is dense in U. Also if f is irreducible then it induces a Boolean isomorphism from $\mathfrak{R}(X)$, the algebra of regular closed sets of X, onto $\mathfrak{R}(Y)$, by the assignment $A \mapsto \mathrm{cl}_Y\, f^{-1}(\mathrm{int}_X\, A)$.

Now fix the space X. Let $\mathrm{Cov}(X)$ denote the set of all irreducible surjections $f : Y \to X$, modulo the equivalence relation defined by $f \sim f'$ (where $f' : Y' \to X$ is an irreducible surjection) if there is a homeomorphism $h : Y \to Y'$ such that $f' \cdot h = f$. It is convenient, especially where the notation is concerned, to identify an irreducible surjection with its equivalence class in $\mathrm{Cov}(X)$; we believe that no confusion will ensue from this identification.

One can partially order $\mathrm{Cov}(X)$ by setting $f \leqslant g$ (with $f : Y \to X$ and $g : Z \to X$) if there is a continuous surjection $g^* : Z \to Y$ (necessarily irreducible) such that $f \cdot g^* = g$.

(b) Suppose that $\mathcal{T}$ is a class of spaces which is closed under formation of homeomorphic copies. $\mathcal{T}$ is called a *covering class* if, for each space X, the set $\mathrm{Cov}(X) \cap \mathcal{T}$ has a minimum.

We briefly review some well-known covering classes next, in the context of more general, cardinal-related properties. We give appropriate references as we go.

Definition & Remarks 3.2. Let X be a space, and α be a regular uncountable cardinal, or else ∞. We review the concept of a quasi F_α-space. The reader is referred to [3] for additional discussion of the material reviewed here.

Suppose that $f : Y \to X$ is an irreducible surjection. We say that f is *α-irreducible* if for each α-cozeroset U in Y there is an α-cozeroset V in X such that $f^{-1}V$ is dense in U. On the other hand, we say that X is a *quasi F_α-space* if every dense α-Lindelöf subspace of X is C^*-embedded. ($Z \subseteq X$ is *α-Lindelöf* if every open cover of Z has a subcover of fewer than α sets.)

Theorems 4.9 and 4.11 of [3] assert the following:

For each compact Hausdorff space X there is an α-irreducible surjection $q_\alpha : qF_\alpha X \to X$, with $qF_\alpha X$ a quasi F_α-space, least among $g : Z \to X$ in $\mathrm{Cov}(X)$ *with Z quasi F_α. Thus, the class of quasi F_α-spaces is a covering class. We refer to $qF_\alpha X$ as the minimum* quasi F_α- cover *of X.*

When $\alpha = \omega_1$ we have the quasi F-spaces. These have received considerable attention, and the reader is referred to [7] and [19]; for a treatment of the same subject from an ideal-theoretic point of view, see [21]. When $\alpha = \infty$ we have the extremally disconnected spaces. The existence of the qF_∞- cover was established by Gleason; it is commonly called the *absolute* of a space X, and denoted EX. The reader is referred to [30]. A different development of the absolute may be found in [10].

There is an annoying gap in our understanding of quasi F_α-spaces for cardinal $\alpha > \omega_1$. The following observation records what we know about the matter.

Remark 3.3. The definition of a quasi F-space, traditionally, is that every dense cozeroset is C^*-embedded. By Theorem 3.6 of [9], it then follows that every dense Lindelöf subspace is also C^*-embedded. For a regular cardinal $\alpha > \omega_1$, it is not known whether every dense α-Lindelöf subspace of X is C^*-embedded, if it assumed that every dense α-cozeroset of X has this property. The reader is referred to the related discussion in §6 of [3], concerning the difficulties in obtaining the quasi F_α-cover as a space of ultrafilters, for cardinals $\alpha > \omega_1$.

If X is assumed to be zero-dimensional then Theorem 7.4 of [17] characterizes the Yosida space of a certain subalgebra of $Q_\alpha S(X)$, as the minimum α-cloz cover. The discussion now turns to α-cloz spaces.

Definition & Remarks 3.4. (a) In [20] the authors introduce the concept of a cloz space; we explain the term. For now X is compact and Hausdorff, but not necessarily zero-dimensional. First, recall that a cozeroset U is said to be *complemented* if there is a cozeroset V such that $U \cap V = \emptyset$ and $U \cup V$ is dense in X. Then X is called a *cloz space* if every complemented cozeroset has clopen closure. Generalizing now, with the symbol α used as above, we say that an α-cozeroset U is *α-complemented* if there is an α-cozeroset V, disjoint from U, such that $U \cup V$ is dense in X. Observe that the ω_1-complemented ω_1-cozerosets are none other than the complemented cozerosets.

X is an *α-cloz space* if every α-complemented α-cozeroset has clopen closure. Note that the ∞-cloz spaces are again the extremally disconnected ones. "ω_1-cloz" is "cloz".

(b) Using, essentially, the argument in Theorem 3.4 of [20], one can prove that X is an α-cloz space if and only if every dense α-cozeroset U of X is 2-*embedded*; that is, every continuous function of U into the two-element discrete space can be extended continuously to a 2-valued function on X. This gives us Proposition 3.5 below, which is already mentioned in the comments of 3.5, [11].

(c) General considerations, discussed by Vermeer in [29], show that the class of α-cloz spaces is a covering class. It can be shown by techniques from [20], using the family $\Gamma_\alpha(X)$ consisting of the closures of α-complemented α-cozerosets of X, that a model of the minimum α-cloz cover, $Cz_\alpha X$, of a space X may be constructed. Theorem 7.4, [17], presents an alternate approach.

Proposition 3.5. *Every quasi F_α-space is α-cloz. Conversely, in every zero-dimensional α-cloz space every dense α-cozeroset is C^*-embedded.*

We resume the discussion of α-cloz spaces.

Remark 3.6. For any commutative ring A, let $Q_\alpha^{\mathrm{s}}A$ denote the subalgebra of $Q_\alpha A$ generated by A and the idempotents of $Q_\alpha A$. This subalgebra is described in Proposition 3.9 of [17] as follows; $\mathcal{E}(A)$ stands for the algebra of idempotents of A:

$$Q_\alpha^{\mathrm{s}}A = \left\{ \sum_{i=1}^{n} a_i e_i \colon a_i \in A, e_i \in \mathcal{E}(Q_\alpha A) \right\}.$$

Theorem 3.7. *Suppose that X is zero-dimensional. Then $Cz_\alpha X$ is zero-dimensional as well, and*

$$Cz_\alpha X = YQ_\alpha S(X).$$

Proof. By Proposition 2.10, $Q_\alpha S(X)$ is projectable, and hence its Yosida space is zero-dimensional by Remark 1.4(c). Now Theorem 7.4 and Corollary 7.5 in [17] insure that

$$Cz_\alpha X = YQ_\alpha^{\mathrm{s}}S(X) = YQ_\alpha S(X),$$

and we emphasize that it is the zero-dimensionality of $YQ_\alpha S(X)$ which furnishes us with the second identity above. □

We close the section with a retrospective of the state of affairs regarding the Yosida space of the ring $Q_\alpha S(X)$.

Proposition 3.8. *Assume here that X is zero-dimensional. Then*

$$Cz_\alpha X = YQ_\alpha^{\mathrm{s}}S(X) = YQ_\alpha S(X) \leqslant qF_\alpha X. \tag{†}$$

We distinguish the following special situations:

(i) *For $\alpha = \infty$: we get identities throughout in* (†), *and $Cz_\infty X = EX$, while $Q_\infty = Q$, where QA denotes the maximum ring of quotients of A.*
(ii) *For $\alpha = \omega_1$: any zero-dimensional cloz space is quasi F* [20, *Corollary* 3.5(b)]. *Thus, $Cz_{\omega_1}X = YQ_{\omega_1}S(X) = qFX$.*

Proof. It is just the inequality in (†) that needs explaining. On the one hand, [17, Corollary 7.2] tells us that $YQ_\alpha S(X)$ is an α-cloz space. Then, as a consequence of Theorem 4.9 of [3], it is enough to show that the inclusion $S(X) \leqslant Q_\alpha S(X)$ induces an α-irreducible surjection $YQ_\alpha S(X) \to X$. The α-irreducibility comes out of the first paragraph in the proof of Theorem 3.4 in [17], which shows the following: if $b \in Q_\alpha A$ (for any semiprime ring A), then $b^{\perp\perp} = \bigvee_{i\in I} b_i^{\perp\perp}$, for some $b_i \in A$, with $|I| < \alpha$. □

4. Uniformly complete lateral completions

In this section we examine the relationship between the α-Specker condition and its weak counterpart. This discussion is facilitated by an investigation of when the various lateral completions are uniformly complete. The assumptions about the symbol α are as earlier in the paper. Without mention to the contrary, all spaces are compact and Hausdorff.

Definition & Remarks 4.1. The $\boldsymbol{W}$-object A is called a $UL(\alpha)$-*object* (or said to *be* $UL(\alpha)$) if $l(\alpha)A$ is uniformly complete. Here are some preliminary remarks about this concept.

(a) (Theorem 5.5(b), [15]) If A is a divisible $\boldsymbol{W}$-object, then A is a $UL(\alpha)$ object if and only if $l(\alpha)A = l(\alpha)c(\alpha)A$, that is, if and only if $c(\alpha)A \leqslant l(\alpha)A$. (We will devote most of the discussion in the sequel to divisible $\boldsymbol{W}$-objects.)

(b) Recall that the f-ring A satisfies the *bounded inversion property* if $a \geqslant 1$ implies that a is invertible. bA denotes the ring of quotients of A defined by

$$bA \equiv \{a/s\colon\ a, s \in A,\ s \geqslant 1\}.$$

Indeed, $bA \leqslant qA$. The operator b is functorial, defining a monoreflection, although that will not come into play. Now, if the Archimedean f-ring A satisfies the bounded inversion property, then A is $UL(\infty)$ if and only if QA, the maximum ring of quotients, is uniformly complete. This is so because $b \cdot l(\infty) = l(\infty) \cdot b = Q$ in **Arf**; see [16], Theorem 1.3. The commuting of the operators for arbitrary α will be discussed elsewhere.

We note that $A = C(X)$ is a $UL(\infty)$-object precisely when the underlying space X is a uniform quotients space, as discussed in [23].

(c) Let us also briefly consider the antithesis of divisibility. The $\boldsymbol{W}$-object A is *singular* if the designated unit e is singular, which is to say that $a \wedge (e - a) = 0$, for each $0 \leqslant a \leqslant e$. Then A is singular and laterally α-complete if and only if YA is α-disconnected and $A = D(YA, \mathbb{Z})$ [13, Theorem 7.4]. Moreover, any singular $\boldsymbol{W}$-object is uniformly complete, as any uniformly Cauchy sequence is eventually constant, and therefore converges. Thus, every singular object is $UL(\alpha)$.

The following motivates studying the α-projectable $UL(\alpha)$-objects first.

Proposition 4.2. *A is $UL(\alpha)$ if and only if $p(\alpha)A$ is $UL(\alpha)$.*

Proof. $l(\alpha)A = l(\alpha)p(\alpha)A$ (1.4). □

We identify $Y(p(\alpha)S(X))$ next.

Remark 4.3. Recall that X is α-disconnected if every α-cozeroset has clopen closure. By Vermeer's principles the class of α-disconnected spaces is also a covering class. We denote the minimum α-disconnected cover of X by $E_\alpha X$, and note that $E_\infty = E$, while $E_{\omega_1} X$ is the basically disconnected cover.

The next proposition gives two models for $E_\alpha X$.

Proposition 4.4. *For any $\boldsymbol{W}$-object A, $Yp(\alpha)A = E_\alpha YA$. In particular, if X is zero-dimensional, we have that*

$$E_\alpha X = Yp(\alpha)C(X) = Yp(\alpha)S(X).$$

Proof. From Remark 1.4(d), $Yp(\alpha)A$ is α-disconnected; this means that $E_\alpha YA \leqslant Yp(\alpha)A$.

On the other hand, the Yosida representation together with the observation that $D(E_\alpha YA)$ is α-projectable, produces the embeddings

$$A \leqslant p(\alpha)A \leqslant D(E_\alpha YA),$$

from which we extract the dual of the second inclusion:

$$YD(E_\alpha YA) = E_\alpha YA \to Yp(\alpha)A,$$

which establishes that $Yp(\alpha)A \leqslant E_\alpha YA$ and proves the first assertion.

The second assertion is an immediate consequence of the first. □

Corollary 4.5. *Suppose that X is zero-dimensional. If $S(X)$ is a $UL(\alpha)$-object then $l(\alpha)S(X) = D(E_\alpha X)$.*

Proof. From [15, Theorem 5.5(b)], and the preceding proposition we deduce that

$$c(\alpha)l(\alpha)S(X) = D(E_\alpha X).$$

By the comments in 4.1(a), we have that

$$l(\alpha)S(X) = D(E_\alpha X). \qquad \square$$

Now here is the tie-in with the Specker conditions.

Theorem 4.6. *Suppose that X is zero-dimensional. Then the following are equivalent statements.*

(i) *$S(X)$ is $UL(\alpha)$.*
(ii) *$S(E_\alpha X)$ is $UL(\alpha)$.*
(iii) *$E_\alpha X$ is α-Specker.*
(iv) *$C(X)$ is $UL(\alpha)$ and X is α-Specker.*

Proof. To begin, observe that if $S(X)$ is $UL(\alpha)$, then X is α-Specker. For, according to Corollary 4.5,

$$D(E_\alpha X) = l(\alpha)S(X) \leqslant l(\alpha)C(X) = D(E_\alpha X).$$

Thus, $C(X) \leqslant l(\alpha)S(X)$ and X is α-Specker.

Next, we note that $S(E_\alpha X) = p(\alpha)S(X)$, so that $S(E_\alpha X) \leqslant l(\alpha)S(X)$. Thus, $l(\alpha)S(X) = l(\alpha)S(E_\alpha X)$; this makes it clear that (i) and (ii) are equivalent. The first paragraph of the proof tells us that (ii) implies (iii).

Assuming (iii), we have the following:

$$C(X) \leqslant c(\alpha)C(X) = C(E_\alpha X) \leqslant l(\alpha)S(E_\alpha X) = l(\alpha)S(X) \leqslant l(\alpha)C(X), \qquad (*)$$

whence $C(X)$ is $UL(\alpha)$, and X is α-Specker, which proves (iv).

Finally, with (iv) we have the following inclusions:

$$C(X) \leqslant l(\alpha)S(X) = l(\alpha)S(E_\alpha X) \leqslant l(\alpha)C(X) = D(E_\alpha X).$$

Applying $l(\alpha)$ to the above string of inclusions, we get that $l(\alpha)S(X) = D(E_\alpha X)$, whence $S(X)$ is $UL(\alpha)$. This proves that (iv) implies (i), and finishes the proof. □

To conclude this section, we give the promised characterization of the α-Specker conditions in terms of α-quotients, from which it easily follows that "α-Specker" implies "weakly α-Specker".

Theorem 4.7. *Suppose that X is zero-dimensional.*

(a) *For each $\alpha \leqslant \alpha'$, $Q_\alpha S(E_{\alpha'}X) = l(\alpha)S(E_{\alpha'}X)$.*
(b) *X is strongly α-Specker if and only if $C(X) \leqslant Q_\alpha S(Cz_\alpha X)$.*
(c) *X is α-Specker if and only if $C(X) \leqslant Q_\alpha S(E_\alpha X)$.*
(d) *X is weakly α-Specker if and only if $C(X) \leqslant Q_\alpha S(EX)$.*

Thus, a zero-dimensional α-Specker space is necessarily weakly α-Specker.

Proof. (a) From Corollary 3.11 of [15] we may conclude that $l(\alpha)S(E_{\alpha'}X) = l^1(\alpha)S(E_{\alpha'}X)$, because $S(E_{\alpha'}X)$ is α-projectable; then apply either Proposition 2.3(c) or 2.12.

(b) All the idempotents of $Q_\alpha S(X)$ reside in $S(Cz_\alpha X)$, and conversely. Thus, $S(Cz_\alpha X) \leqslant Q_\alpha S(X)$, and therefore $Q_\alpha S(Cz_\alpha X) = Q_\alpha S(X)$. Now apply Proposition 2.3(c).

(c) In (a) take $\alpha = \alpha'$, and observe, as in the proof of Theorem 4.6, that $l(\alpha)S(X) = l(\alpha)S(E_\alpha X)$.

(d) It should be clear that X is weakly α-Specker precisely when, for each $f \in C(X)$, there is a quasi-partition $\mathcal{S}$ by regular closed sets, with $|\mathcal{S}| < \alpha$, such that for each $B \in \mathcal{S}$, $f|_B = r_B \in \mathbb{R}$, identically. Now let $e_X : EX \to X$ stand for the irreducible surjection of the absolute EX onto X. As pointed out in, 3.1(a), e_X induces a Boolean isomorphism of the algebra $\mathfrak{R}(EX)$ of regular closed subsets of EX onto $\mathfrak{R}(X)$. Since EX is extremally disconnected, $\mathfrak{R}(EX)$ is just the algebra of clopen sets.

Now if $f \in C(X)$ has an associated quasi-partition $\mathcal{S}$ as specified above, then

$$e_X^{-1}\mathcal{S} \equiv \{e_X^{-1}B\colon\ B \in \mathcal{S}\}$$

is a quasi-partition by clopen subsets of EX, and $f = \bigvee_{B\in\mathcal{S}} r_B \chi_{e_X^{-1}B}$. That is to say, $f \in l(\alpha)S(EX) = Q_\alpha S(EX)$. The converse is just as easy, and we leave it to the reader.

The final claim is then, indeed, obvious. □

We conclude this section with some comments on the heels of Theorem 4.7.

Remarks 4.8. (a) In [11] the author defines the notion of an *α-fraction dense space*; for compact spaces, X is α-fraction dense if and only if $Cz_\alpha X = EX$. Evidently, if X is α-fraction dense and zero-dimensional it is weakly α-Specker if and only if it is strongly α-Specker.

(b) Assume X is zero-dimensional. If every α-cozeroset of X is α-complemented then $Cz_\alpha X = E_\alpha X$, and so, if X is α-Specker, then it is also strongly α-Specker.

(c) In Corollary 5.6 it is shown that a weakly ω_1-Specker space is ω_1-Specker. About $\alpha = \infty$ we have already remarked (2.5). Whether "weakly α-Specker" implies "α-Specker" for all α is an open question. Evidently, if X is weakly α-Specker and

$E_\alpha X = EX$ then X is α-Specker; this includes all spaces for which every Borel set differs from an α-Borel set by a meagre set. (An *α-Borel set* is a member of the σ-algebra generated by the α-cozerosets.)

5. ω_1-Specker conditions

In this section we examine the ω_1-Specker conditions. Unless the contrary is specified, *all spaces here are assumed to be compact, Hausdorff and zero-dimensional*. The culminating result is that any weakly ω_1-Specker space is ω_1-Specker. By contrast, recall that if D is any uncountable discrete space, then αD, the one-point compactification of D, is an ω_1-Specker space that is not strongly ω_1-Specker (Remark 2.2).

We begin with a characterization of strongly ω_1-Specker spaces which uses Proposition 2.3. We leave the details to the reader.

Proposition 5.1. *For a space X the following are equivalent.*

(a) *X is strongly ω_1-Specker.*
(b) *For each $f \in C(X)$ there is a quasi-partition by clopen sets $\{V_n\colon n \in \mathbb{N}\}$ such that $f|_{V_n}$ is constant, for each $n \in \mathbb{N}$.*
(c) $Q_{\omega_1} S(X) = qC(X)$.

Next, we aim for an internal characterization of the ω_1-Specker condition. We first review some material which describes a construction of $l(\omega_1)A$ in terms of Baire sets. The principal references here are [2] and [14]. We sketch the basic references on epicompletion from [2], leaving it to the interested reader to appeal to that article for more detail. The term "σ-ideal" refers to an ℓ-ideal which is closed under existing countable suprema and infima.

Definition & Remarks 5.2. Suppose that X is a compact Hausdorff space, but not necessarily zero-dimensional.

(a) A *Baire set* is a member of $\mathcal{B}(X)$, the σ-subalgebra of subsets of X generated by the zerosets of X. $\mathcal{M}(X)$ denotes the σ-ideal of all *meagre* sets. (Recall that $M \in \mathcal{B}(X)$ is meagre if it is a countable union of nowhere dense sets.) $B(X)$ stands for the algebra of all *Baire functions*: the real-valued functions f on X for which $f^{-1}I$ is a Baire set, for each interval I in $\mathbb{R}$. $M(X)$ is the ℓ-ideal and σ-ideal of all functions $f \in B(X)$ such that $\operatorname{coz}(f)$ is meagre.

(b) The $\boldsymbol{W}$-object B is called *epicomplete* (in $\boldsymbol{W}$) if it has no proper $\boldsymbol{W}$-epimorphic extensions. It is shown in [1] that B is epicomplete if and only if B is isomorphic to some $D(Y)$, with Y compact and basically disconnected. Then in [2] the following are carried out: for a $\boldsymbol{W}$-object A, let

$$N(A) \equiv \left\{ f \in B(YA)\colon \operatorname{coz}(f) \subseteq \bigcup_n a_n^{-1}\bigl(\{\pm\infty\}\bigr),\ \text{for some } a_1, a_2, \ldots \in A \right\}.$$

This is a σ-ideal in $B(YA)$, and the construct

$$\beta A \equiv B(YA)/N(A)$$

is the epicomplete monoreflection of A in $\boldsymbol{W}$ [2, §5].

Since $N(A) \subseteq M(YA)$, there is the quotient

$$h_A : \beta A \to B(YA)/M(YA) \equiv \lambda A,$$

and λA is the unique *essential* epicompletion of A [2, §9]. It then follows easily that $\lambda A = D(E_{\omega_1} A)$.

(c) Next, we recall from [14] a construction of the laterally σ-complete (i.e., $l(\omega_1)$-complete) monoreflection, σA, of the $\boldsymbol{W}$-object A. First, $B_{\omega,A}(YA)$ consists, by definition, of those $f \in B(YA)$ for which there is a countable set $\{Y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(YA)$, with $Y_n \cap Y_m = \emptyset$, for $n \neq m$, and $YA = \bigcup_n Y_n$, and there is also a sequence $a_1, a_2, \ldots$ in A, for which $f|_{Y_n} = a_n|_{Y_n}$, for each $n \in \mathbb{N}$.

Then

$$\sigma A = B_{\omega,A}(YA)/\big(B_{\omega,A}(YA) \cap N(A)\big) \leqslant \beta A \quad [14, §4].$$

Next, observe that $l(\omega_1)A \leqslant \lambda A$ and that the embedding $A \leqslant l(\omega_1)A$ has a unique extension to σA. Moreover, the class of $l(\omega_1)$-complete $\boldsymbol{W}$-objects is closed under formation of images under $\boldsymbol{W}$-morphisms. It then follows that

$$h_A(\sigma A) = l(\omega_1)A = B_{\omega,A}(YA)/\big(B_{\omega,A}(YA) \cap M(YA)\big).$$

For each $f \in B(YA)$ abbreviate $h_A(f + N(A)) = \bar{f}$. We then have $\bar{f} \in l(\omega_1)A$ if and only if there are countably many pairwise disjoint Baire sets in YA, say $Y_1, Y_2, \ldots$ such that $YA = \bigcup_n Y_n$, and a meagre set M in YA, and also $a_1, a_2, \ldots \in A$, such that $f|_{Y_n \setminus M} = a_n|_{Y_n \setminus M}$, for each $n \in \mathbb{N}$.

The comments in 5.2 apply immediately to give, first, a description of the elements of $l(\omega_1)S(X)$, and then a characterization of ω_1-Specker spaces. Note that $YS(X) = X$, for each zero-dimensional X.

In the results that follow, all spaces are once again compact and zero-dimensional, as announced at the start of this section.

Lemma 5.3. *For each $f \in B(X)$, $\bar{f} \in l(\omega_1)S(X)$ precisely when there exist pairwise disjoint Baire sets $Y_1, Y_2, \ldots$ such that $X = \bigcup_n Y_n$, and a meagre set M such that $f|_{Y_n \setminus M}$ is constant, for each $n \in \mathbb{N}$.*

Recall that a Baire set is *comeagre* if its complement is meagre.

Theorem 5.4. *The following are equivalent.*

(a) *X is ω_1-Specker.*
(b) *For each $f \in C(X)$ there is a countable set of pairwise disjoint Baire sets $Y_1, Y_2, \ldots$ such that $\bigcup_n Y_n$ is comeagre and $f|_{Y_n}$ is constant.*

(c) *For each $f \in C(X)$ there is a countable set of pairwise disjoint zerosets $Y_1, Y_2, \ldots$ such that $\bigcup_n Y_n$ is comeagre and $f|_{Y_n}$ is constant.*
(d) *For each $f \in C(X)$ there is a comeagre Baire set $B \subseteq X$ such that $f(B)$ is countable.*

Proof. That (a) is equivalent to (b) is immediate from Lemma 5.3, and that (c) implies (d) is obvious. Suppose (b) holds and $f \in C(X)$; pick $Y_1, Y_2, \ldots,$ a sequence of Baire sets witnessing (b) for f, then it is clear that $Z_n = f^{-1}f(Y_n)$, defines the sequence of zerosets we want.

Finally, suppose that each continuous real-valued function on X has countable range on a suitable comeagre Baire set. Let $f \in C(X)$, and M be a meagre set such that $f(X \setminus M)$ is countable. Enumerate these images: $\{r_1, \ldots, r_n, \ldots\}$, and let $Z_n = f^{-1}\{r_n\}$. Then the $(Z_n)_{n\in\mathbb{N}}$ form a sequence of disjoint zerosets, and letting $Y_n = Z_n \setminus M$, we get a partition $M, Y_1, \ldots, Y_n, \ldots$ by Baire sets which witnesses the stipulations of the lemma for f. Thus, $\bar{f} \in l(\omega_1)S(X)$, and we have shown that (d) implies (a). □

For our first corollary to Theorem 5.4, recall that a space X is *scattered* if every nontrivial closed subspace of X contains an isolated point. Note that a compact Hausdorff scattered space is necessarily zero-dimensional.

The equivalence of (a) and (d) in Corollary 5.5 may be found in [26].

Corollary 5.5. *Every compact Hausdorff, scattered space is ω_1-Specker. In fact, the following are equivalent for a space X.*

(a) *X is scattered.*
(b) *For each $f \in C(X)$ there is a partition of X by countably many Baire sets $Y_1, Y_2, \ldots$ such that $f|_{Y_n}$ is constant.*
(c) *For each $f \in C(X)$ there is a partition of X by countably many zerosets $Y_1, Y_2, \ldots$ such that $f|_{Y_n}$ is constant.*
(d) *$f(X)$ is countable, for each $f \in C(X)$.*

Finally, if X is also an almost P-space, it is ω_1-Specker if and only if it is scattered.

Proof. It suffices to establish the equivalence of (b), (c) and (d) in the corollary. Now, (c) is obtained from (b) in the same manner as indicated in the proof of Theorem 5.4 for the corresponding implication. The implications (c) ⇒ (d) ⇒ (b) are obvious.

In particular, it is clear that a scattered space is ω_1-Specker. To conclude, if X is ω_1-Specker and almost P, then there are no nonempty meagre sets, proving that X is scattered. □

At last we have the result advertised at the beginning of this section.

Corollary 5.6. *Every weakly ω_1-Specker space is ω_1-Specker.*

Proof. Suppose that X is weakly ω_1-Specker, and let $f \in C(X)$. Since there is a countable quasi-partition by open sets $V_1, \ldots, V_n, \ldots$ such that $f|_{V_n}$ is constant, there is also a quasi-partition by the zerosets $Z_n \equiv f^{-1}f(V_n)$, and it is obvious that $f|_{Z_n}$ is constant. Thus, since $M = X \setminus (\bigcup_n Z_n)$ is a meagre Baire set, we have satisfied the conditions of Theorem 5.4. Thus, $f \in l(\omega_1)S(X)$ and we are done. □

We close out the section with a number of special observations. Recall that a Tychonoff space X is *cozero complemented* if every cozeroset is complemented. In this case (with X not necessarily zero-dimensional) we have that $qFX = E_{\omega_1}X$ (see [19]). Recall that *with* the assumption of zero-dimensionality we also have $CzX = qFX$ (3.8(ii)). We then have the following consequence of Theorem 4.7—which, admittedly, could have been stated immediately after that result.

Proposition 5.7. *Suppose X is cozero complemented. Then if X is ω_1-Specker it is also strongly ω_1-Specker.*

Remarks 5.8. This list ought to kill off a number of conjectures.

(i) A strongly ω_1-Specker space need not be scattered: think of $\beta\mathbb{N}$. In fact, any compactification of $\mathbb{N}$ is strongly ω_1-Specker.

(ii) Scattered spaces need not be strongly ω_1-Specker: just look at αD, with D discrete and uncountable. This also shows that cozero complementarity cannot be dropped in Proposition 5.7.

(iii) Call a space X *cozero scattered* if for each $f \in C(X)$ there is a dense cozeroset V such that $f(V)$ is countable. It is not hard to see that any strongly ω_1-Specker space is cozero scattered; clearly every scattered space is cozero scattered. Theorem 5.4 also shows that every cozero-scattered space is ω_1-Specker. αD (with D discrete and uncountable) is scattered, and therefore cozero scattered, but not strongly ω_1-Specker. We do not know whether ω_1-Specker spaces are necessarily cozero scattered.

(iv) A strongly ω_1-Specker space need not have countable cellularity: take the totally ordered space $\omega_1 + 1$ of all ordinals not exceeding ω_1, with the interval topology. Proposition 5.1 shows this space is strongly ω_1-Specker. (Note: X has *countable cellularity* if every family of pairwise disjoint nonempty open sets is countable.)

If X does have countable cellularity then all three ω_1-Specker conditions coincide, as X is then necessarily cozero complemented. In fact, if X has countable cellularity these conditions hold if and only if X is Specker.

6. Remnants

Again in this section, all spaces are assumed to be compact, Hausdorff and zero-dimensional, unless the contrary is stipulated. Note that α denotes, as before, an uncountable, regular cardinal or else the symbol ∞. In (a) of Proposition 6.1 below the regularity of the cardinal will be used. As the omission is rather conspicuous, let us admit in advance that we have no counterpart to the content of the proposition for the "middle" α-Specker condition.

Proposition 6.1. *Suppose $g: Y \to X$ is an irreducible surjection. Then we have*:

(a) *If Y is strongly α-Specker and g is α-irreducible, then X is also strongly α-Specker.*
(b) *If Y is weakly α-Specker then X is as well.*

Proof. We prove (a), leaving the proof of (b) to the reader, as it is very similar.

Suppose that Y is strongly α-Specker and $f \in C(X)$; then $f \cdot g \in C(Y)$, and so there is a quasi-partition $\mathcal{K}$ of Y by fewer than α clopen sets—Proposition 2.3—such that $(f \cdot g)|_K$ is constant, for each $K \in \mathcal{K}$. Now since f is α-irreducible, there is, for each $K \in \mathcal{K}$, an α-cozeroset V_K of X, such that $f^{-1}V_K$ is dense in K. Then it is easy to see that the $(V_K)_{K\in\mathcal{K}}$ form a quasi-partition by fewer than α α-cozerosets of X, so that $f|_{V_K}$ is constant. As X is zero-dimensional, and α is regular, one may refine these V_K once again, and obtain a quasi-partition of size $< \alpha$ by clopen sets of X, such that f is constant on each member. This means that X is strongly α-Specker. □

The following corollary stands in analogy to Theorem 4.6.

Corollary 6.2. *If $qF_\alpha X$ is strongly α-Specker, then so is X. Likewise, if EX is weakly α-Specker, the same is true of X.*

The converses of the statements in Corollary 6.2 are intriguing, but are left to be discussed elsewhere. Now, in conclusion, we have a comment about extremally disconnected ω_1-Specker spaces under certain set-theoretic assumptions.

Lemma 6.3. *If X is a compact strongly ω_1-Specker space and K is regular closed in X, then K too is strongly ω_1-Specker.*

Proof. Suppose that $f \in C(K)$. Note that $K = \mathrm{cl}_X V$, for a suitable open set V. (Evidently, we assume that K and V are nonempty, as there is nothing to prove otherwise.) Now f has a continuous extension to $g \in C(X)$. On account of Proposition 5.1(b), there is a countable quasi-partition of X by clopen sets U_n $(n \in \mathbb{N})$ such that $g|_{U_n}$ is constant. Note that V must intersect the union of the U_n. Enumerate the indices $i_1, \ldots, i_k, \ldots$ for which $V \cap U_{i_k} \neq \emptyset$. It is then easy to check that the $W_k = K \cap U_{i_k}$ form a quasi-partition by clopen subsets of K, and that $f|_{W_k}$ is constant. This shows that K is strongly ω_1-Specker. □

We need a lemma which refers to Souslin lines. For background on Souslin lines and their existence, we refer the reader to [22,28].

Lemma 6.4. *The existence of an extremally disconnected, ω_1-Specker space without isolated points is equivalent to that of a Souslin line.*

Proof. This follows from Remark 1.7 of [4], as an extremally disconnected, ω_1-Specker space necessarily has countable cellularity. □

These lemmas then produce the following curious outcome.

Proposition 6.5. *Suppose that no Souslin line exists. Then any extremally disconnected space which is ω_1-Specker is homeomorphic to $\beta\mathbb{N}$.*

Proof. First, as the cellularity of X is countable, X is strongly ω_1-Specker, owing to Proposition 5.7. There must be an isolated point in X, owing to Lemma 6.4, and the subset N of all isolated points is countable. Now the set $U = X \setminus \mathrm{cl}_X N$ is open and therefore extremally disconnected. $Y = \mathrm{cl}_X U$ is its Stone–Čech compactification and it is a regular closed subset of X. Thus, by Lemma 6.3, Y too is strongly ω_1-Specker. As Y is extremally disconnected and has no isolated points, this amounts to a contradiction, unless $U = \emptyset$, in which case $X = \mathrm{cl}_X N \cong \beta\mathbb{N}$, as promised. □

References

[1] R.N. Ball, A.W. Hager, Epicomplete Archimedean ℓ-groups and vector lattices, Trans. Amer. Math. Soc. 322 (1990) 459–478.
[2] R.N. Ball, A.W. Hager, Epicompletion of Archimedean ℓ-groups and vector lattices with unit, J. Austral. Math. Soc. 48 (1990) 25–367.
[3] R.N. Ball, A.W. Hager, C.W. Neville, The quasi F_κ-cover of a compact Hausdorff space and the κ-ideal completion of an Archimedean ℓ-group, in: R.M. Shortt (Ed.), General Topology and its Applications, in: Dekker Notes, Vol. 123, Dekker, New York, 1990, pp. 7–50.
[4] A. Bella, A.W. Hager, J. Martinez, S. Woodward, H. Zhou, Specker spaces and their absolutes, I, Topology Appl. 72 (1996) 259–271.
[5] A. Bella, J. Martinez, S. Woodward, Algebras and spaces of dense constancies, Czech. Math. J., to appear.
[6] P.F. Conrad, The essential closure of an Archimedean lattice-ordered group, Duke Math. J. 38 (1971) 151–160.
[7] F. Dashiell, A.W. Hager, M. Henriksen, Order-Cauchy completions of rings and vector lattices of continuous functions, Canad. J. Math. 32 (1980) 657–685.
[8] L. Gillman, M. Jerison, Rings of Continuous Functions, in: Graduate Texts in Math., Vol. 43, Springer-Verlag, Berlin, 1976.
[9] A.W. Hager, On inverse-closed subalgebras of $C(X)$, Proc. London Math. Soc. (3) 19 (1969) 233–257.
[10] A.W. Hager, Minimal covers of topological spaces, in: Papers on General Topology and Related Category Theory and Topological Algebra, in: Ann. New York Acad. Sci., Vol. 552, 1989, pp. 44–59.
[11] A.W. Hager, α-cut complete Boolean algebras, Algebra Univ. 39 (1998) 57–70.
[12] A.W. Hager, J. Martinez, α-projectable and laterally α-complete Archimedean lattice-ordered groups, in: S. Berhanu (Ed.), Proc. Conf. in Memory of T. Retta, Temple, PA/Addis-Ababa 1995, Ethiopian J. Sci. (1996) 73–84.
[13] A.W. Hager, J. Martinez, Singular Archimedean lattice-ordered groups, Algebra Univ. 40 (1998) 119–147.
[14] A.W. Hager, J. Martinez, More on the laterally σ-complete reflection of an Archimedean lattice-ordered group, Order 15 (1999) 247–260.
[15] A.W. Hager, J. Martinez, Hulls for various kinds of α-completeness in Archimedean lattice-ordered groups, Order 16 (1999) 89–103.
[16] A.W. Hager, J. Martinez, Functorial rings of quotients, III: The maximum in Archimedean f-rings, J. Pure Appl. Algebra, to appear.
[17] A.W. Hager, J. Martinez, The ring of α-quotients, Algebra Universalis, to appear.
[18] A.W. Hager, L.C. Robertson, Representing and ringifying a Riesz space, Sympos. Math. 21 (1977) 411–431.
[19] M. Henriksen, J. Vermeer, R.G. Woods, The quasi F-cover of Tychonoff spaces, Trans. Amer. Math. Soc. 303 (1987) 779–803.
[20] M. Henriksen, J. Vermeer, R.G. Woods, Wallman covers of compact spaces, Diss. Math. 280 (1989).
[21] C.B. Huijsmans, B. de Pagter, Maximal d-ideals in a Riesz space, Canad. J. Math. 35 (1983) 1010–1029.

[22] R.B. Jensen, The fine structure of the constructible hierarchy, Ann. Math. Logic 4 (1972) 229–308 (with a section by J. Silver);
R.B. Jensen, Ann. Math. Logic 4 (1972) 443, Erratum.
[23] J. Martinez, W.Wm. McGovern, When the maximum ring of quotients of $C(X)$ is uniformly complete, Topology Appl. 116 (2001) 185–198.
[24] J. Martinez, S. Woodward, Specker spaces and their absolutes, II, Algebra Universalis 35 (1996) 333–341.
[25] R.S. Pierce, Distributivity in Boolean algebras, Pacific J. Math. 7 (1957) 983–992.
[26] W. Rudin, Continuous functions on compact spaces without perfect subsets, Proc. Amer. Math. Soc. 8 (1957) 39–42.
[27] A.I. Singh, M.A. Swardson, Levels of quotient rings of continuous functions, in: Algebra and its Applications (Athens, OH, 1999), in: Contemp. Math., Vol. 259, 2000, pp. 515–527.
[28] R.M. Solovey, S. Tennenbaum, Iterated Cohen extensions and Souslin's problem, Ann. Math. 94 (1971) 201–245.
[29] J. Vermeer, On perfect irreducible preimages, Topology Proc. 9 (1984) 173–189.
[30] R.C. Walker, The Stone–Čech Compactification, in: Ergebnisse der Math., Vol. 83, Springer-Verlag, Berlin, 1974.
[31] E.C. Weinberg, Higher degrees of distributivity in lattices of continuous functions, Purdue University Dissertation, January, 1961.