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Functional calculus for semigroup generators via transference

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Abstract

In this article we apply a recently established transference principle in order to obtain the boundedness of certain functional calculi for semigroup generators. In particular, it is proved that if $-A$ generates a C_0-semigroup on a Hilbert space, then for each $\tau > 0$ the operator A has a bounded calculus for the closed ideal of bounded holomorphic functions on a (sufficiently large) right half-plane that satisfy $f(z) = O(e^{-\tau \operatorname{Re}(z)})$ as $|z| \to \infty$. The bound of this calculus grows at most logarithmically as $\tau \searrow 0$. As a consequence, $f(A)$ is a bounded operator for each holomorphic function f (on a right half-plane) with polynomial decay at ∞. Then we show that each semigroup generator has a so-called (strong) m-bounded calculus for all $m \in \mathbb{N}$, and that this property characterizes semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called γ-bounded semigroups, the Hilbert space results actually hold in general Banach spaces.

Keywords: Functional calculus; Transference; Operator semigroup; Fourier multiplier; γ-Boundedness

1. Introduction

Roughly speaking, a functional calculus for a (possibly unbounded) operator A on a Banach space X is a "method" of associating a closed operator $f(A)$ to each $f = f(z)$ taken from a set of functions (defined on some subset of the complex plane) in such a way that formulae valid for

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the functions turn into valid formulae for the operators upon replacing the independent variable z by A. A common way to establish such a calculus is to start with an algebra of "good" functions f where a definition of $f(A)$ as a bounded operator is more or less straightforward, and then extend this "primary" or "elementary calculus" by means of multiplicative "regularization" (see [7, Chapter 1] and [3]). It is then natural to ask which of the so constructed closed operators $f(A)$ are actually *bounded*, a question particularly relevant in applications, e.g., to evolution equations, see for instance [1,11].

The latter question links functional calculus theory to the theory of vector-valued singular integrals, best seen in the theory of sectorial (or strip-type) operators with a bounded H^∞-calculus, see for instance [13]. It appears there that in order to obtain nontrivial results the underlying Banach space must allow for singular integrals to converge, i.e., be a UMD space (or better, a Hilbert space). Furthermore, even if the Banach space is a Hilbert space, it turns out that simple resolvent estimates are not enough for the boundedness of an H^∞-calculus [7, Section 9.1].

However, some of the central positive results in that theory — McIntosh's theorem [15], the Boyadzhiev–deLaubenfels theorem [4] and the Hieber–Prüss theorem [10] — show that the presence of a C_0-group of operators does warrant the boundedness of certain H^∞-calculi. In [8] the underlying structure of these results was brought to light, namely a *transference principle*, a factorization of the operators $f(A)$ in terms of vector-valued Fourier multiplier operators. Finally, in [9] it was shown that C_0-*semi*groups also allow for such transference principles.

In the present paper, we develop this approach further. We apply the general form of the transference principle for semigroups given in [9] in order to obtain bounded functional calculi for generators of C_0-semigroups. These results, in particular Theorems 3.3, 3.7, and 4.3, are proved for general Banach spaces. However, they make use of (subalgebras of) the analytic $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multiplier algebra (see (2.1) below for a definition), and hence are useful only if the underlying Banach space has a geometry that allows for nontrivial Fourier multiplier operators. In case $X = H$ is a Hilbert space one obtains particularly nice results, which we want to summarize here. (See Section 4 for the definition of a strong m-bounded calculus.)

Theorem 1.1. *Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a Hilbert space H with $M := \sup_{t\in\mathbb{R}_+}\|T(t)\|$. Then the following assertions hold.*

(a) *For $\omega < 0$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ one has $f(A)T(\tau) \in \mathcal{L}(H)$ with*

$$\left\| f(A)T(\tau)\right\| \leqslant c(\tau)M^2\|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}, \tag{1.1}$$

where $c(\tau) = O(|\log(\tau)|)$ as $\tau \searrow 0$, and $c(\tau) = O(1)$ as $\tau \to \infty$.

(b) *For $\omega < 0 < \alpha$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ there is $C \geqslant 0$ such that*

$$\left\| f(A)(A-\lambda)^{-\alpha}\right\| \leqslant CM^2\|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} \tag{1.2}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. In particular, $\operatorname{dom}(A^\alpha) \subseteq \operatorname{dom}(f(A))$.

(c) *A has a strong m-bounded H^∞-calculus of type 0 for each $m \in \mathbb{N}$.*

(See Corollary 3.10 for (a) and (b) and Corollary 4.4 for (c).)

When X is a UMD space one can derive similar results, stated in Section 5. In Section 6 we extend the Hilbert space results to general Banach spaces by replacing the assumption of boundedness of the semigroup by its γ-*boundedness*, a concept strongly put forward by Kalton and Weis [12]. In particular, Theorem 1.1 holds true for γ-bounded semigroups on arbitrary Banach spaces with M being the γ-bound of the semigroup.

We stress the fact that in contrast to [7], where *sectorial* operators and, accordingly, functional calculi on sectors, were considered, the present article deals with general semigroup generators and with functional calculi on *half-planes*. (See Section 2.2 below.) The abstract theory of (holomorphic) functional calculi on half-planes can be found in [3] where the notion of an m-bounded calculus (for operators of half-plane type) has been introduced. Our Theorem 1.1(c) is basically contained in that paper (it follows directly from [3, Corollary 6.5 and (7.1)]).

The starting point of the present work was the article [19] by Hans Zwart, in particular [19, Theorem 2.5, 2.]. There it is shown that one has an estimate (1.1) with $c(\tau) = O(\tau^{-1/2})$ as $\tau \searrow 0$. (The case $\alpha > 1/2$ in (1.2) is an immediate consequence; however, that case is essentially trivial, see Lemma 2.4 below.)

In [19] and its sequel paper [17] the functional calculus for a semigroup generator is constructed in a rather unconventional way using ideas from systems theory. However, a closer inspection reveals that transference (i.e., the factorization over a Fourier multiplier) is present there as well, hidden in the very construction of the functional calculus.

1.1. Notation and terminology

We write $\mathbb{N} := \{1, 2, \ldots\}$ for the natural numbers and $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals. The letters X and Y are used to denote Banach spaces over the complex number field. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. For a closed operator A on X its *domain* is denoted by $\operatorname{dom}(A)$ and its *range* by $\operatorname{ran}(A)$. The *spectrum* of A is $\sigma(A)$ and the *resolvent set* $\rho(A) := \mathbb{C} \setminus \sigma(A)$. For all $z \in \rho(A)$ the operator $R(z, A) := (z - A)^{-1} \in \mathcal{L}(X)$ is the *resolvent* of A at z.

For $p \in [1, \infty]$, $\mathrm{L}^p(\mathbb{R}; X)$ is the Bochner space of equivalence classes of X-valued p-Lebesgue integrable functions on $\mathbb{R}$. The Hölder conjugate of p is p', defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The norm on $\mathrm{L}^p(\mathbb{R}; X)$ is usually denoted by $\|\cdot\|_p$.

For $\omega \in \mathbb{R}$ and $z \in \mathbb{C}$ we let $e_\omega(z) := e^{\omega z}$. By $\mathrm{M}(\mathbb{R})$ (resp. $\mathrm{M}(\mathbb{R}_+)$) we denote the space of complex-valued Borel measures on $\mathbb{R}$ (resp. $\mathbb{R}_+$) with the total variation norm, and we write $\mathrm{M}_\omega(\mathbb{R}_+)$ for the distributions μ on $\mathbb{R}_+$ of the form $\mu(\mathrm{d}s) = e^{\omega s}\nu(\mathrm{d}s)$ for some $\nu \in \mathrm{M}(\mathbb{R}_+)$. Then $\mathrm{M}_\omega(\mathbb{R}_+)$ is a Banach algebra under convolution with the norm

$$\|\mu\|_{\mathrm{M}_\omega(\mathbb{R}_+)} := \|e_{-\omega}\mu\|_{\mathrm{M}(\mathbb{R}_+)}.$$

For $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$ we let $\operatorname{supp}(\mu)$ be the topological support of $e_{-\omega}\mu$. A function g such that $e_{-\omega}g \in \mathrm{L}^1(\mathbb{R}_+)$ is usually identified with its associated measure $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$ given by $\mu(\mathrm{d}s) = g(s)\,\mathrm{d}s$. Functions and measures defined on $\mathbb{R}_+$ are identified with their extensions to $\mathbb{R}$ by setting them equal to zero outside $\mathbb{R}_+$.

For an open subset $\Omega \neq \emptyset$ of $\mathbb{C}$ we let $\mathrm{H}^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω, a unital Banach algebra with respect to the norm

$$\|f\|_\infty := \|f\|_{\mathrm{H}^\infty(\Omega)} := \sup_{z \in \Omega} \left|f(z)\right| \quad \bigl(f \in \mathrm{H}^\infty(\Omega)\bigr).$$

We shall mainly consider the case where Ω is equal to a right half-plane

$$\mathrm{R}_\omega := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > \omega\}$$

for some $\omega \in \mathbb{R}$ (we write $\mathbb{C}_+$ for R_0).

For convenience we abbreviate the coordinate function $z \mapsto z$ simply by the letter z. Under this convention, $f = f(z)$ for a function f defined on some domain $\Omega \subseteq \mathbb{C}$.

The *Fourier transform* of an X-valued tempered distribution Φ on $\mathbb{R}$ is denoted by $\mathcal{F}\Phi$. For instance, if $\mu \in \mathrm{M}(\mathbb{R})$ then $\mathcal{F}\mu \in \mathrm{L}^\infty(\mathbb{R})$ is given by

$$\mathcal{F}\mu(\xi) := \int_{\mathbb{R}} e^{-i\xi s}\,\mu(\mathrm{d}s) \quad (\xi \in \mathbb{R}).$$

For $\omega \in \mathbb{R}$ and $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$ we let $\hat{\mu} \in \mathrm{H}^\infty(\mathrm{R}_\omega) \cap \mathrm{C}(\overline{\mathrm{R}_\omega})$,

$$\hat{\mu}(z) := \int_0^\infty e^{-zs}\,\mu(\mathrm{d}s) \quad (z \in \mathrm{R}_\omega),$$

be the *Laplace–Stieltjes transform* of μ.

2. Fourier multipliers and functional calculus

We briefly discuss some of the concepts that will be used in what follows.

2.1. Fourier multipliers

We shall need results from Fourier analysis as collected in [7, Appendix E]. Fix a Banach space X and let $m \in \mathrm{L}^\infty(\mathbb{R}; \mathcal{L}(X))$ and $p \in [1, \infty]$. Then m is a *bounded* $\mathrm{L}^p(\mathbb{R}; X)$*-Fourier multiplier* if there exists $C \geqslant 0$ such that

$$T_m(\varphi) := \mathcal{F}^{-1}(m \cdot \mathcal{F}\varphi) \in \mathrm{L}^p(\mathbb{R}; X) \quad \text{and} \quad \|T_m(\varphi)\|_p \leqslant C\|\varphi\|_p$$

for each X-valued Schwartz function φ. In this case the mapping T_m extends uniquely to a bounded operator on $\mathrm{L}^p(\mathbb{R}; X)$ if $p < \infty$ and on $\mathrm{C}_0(\mathbb{R}; X)$ if $p = \infty$. We let $\|m\|_{\mathcal{M}_p(X)}$ be the norm of the operator T_m and let $\mathcal{M}_p(X)$ be the unital Banach algebra of all bounded $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multipliers, endowed with the norm $\|\cdot\|_{\mathcal{M}_p(X)}$.

For $\omega \in \mathbb{R}$ and $p \in [1, \infty]$ we let

$$\mathrm{AM}_p^X(\mathrm{R}_\omega) := \{f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \mid f(\omega + i\cdot) \in \mathcal{M}_p(X)\} \tag{2.1}$$

be the *analytic* $\mathrm{L}^p(\mathbb{R}; X)$*-Fourier multiplier algebra* on R_ω, endowed the norm

$$\|f\|_{\mathrm{AM}_p^X} := \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} := \|f(\omega + i\cdot)\|_{\mathcal{M}_p(X)}.$$

Here $f(\omega + i\cdot) \in \mathrm{L}^\infty(\mathbb{R})$ denotes the *trace* of the holomorphic function f on the boundary $\partial\mathrm{R}_\omega = \omega + i\mathbb{R}$. By classical Hardy space theory,

$$f(\omega + is) := \lim_{\omega' \searrow \omega} f\big(\omega' + is\big) \tag{2.2}$$

exists for almost all $s \in \mathbb{R}$, with $\|f(\omega + i\cdot)\|_{\mathrm{L}^\infty(\mathbb{R})} = \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}$.

Remark 2.1 *(Important!).* To simplify notation we sometimes omit the reference to the Banach space X and write $\mathrm{AM}_p(\mathrm{R}_\omega)$ instead of $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ whenever it is convenient.

The space $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ is a unital Banach algebra, contractively embedded in $\mathrm{H}^\infty(\mathrm{R}_\omega)$, and $\mathrm{AM}_1^X(\mathrm{R}_\omega) = \mathrm{AM}_\infty^X(\mathrm{R}_\omega)$ is contractively embedded in $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ for all $p \in (1, \infty)$, cf. [7, p. 347].

For our main results we need two lemmas about the analytic multiplier algebra.

Lemma 2.2. *For every Banach space X, all $\omega \in \mathbb{R}$ and $p \in [1, \infty]$,*

$$\mathrm{AM}_p^X(\mathrm{R}_\omega) = \Big\{ f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \ \Big|\ \sup_{\omega' > \omega} \big\| f\big(\omega' + i\cdot\big)\big\|_{\mathcal{M}_p(X)} < \infty \Big\}$$

with $\|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} = \sup_{\omega' > \omega} \|f(\omega' + i\cdot)\|_{\mathcal{M}_p(X)}$ *for all* $f \in \mathrm{AM}_p^X(\mathrm{R}_\omega)$.

Proof. Let $\omega \in \mathbb{R}$, $p \in [1, \infty]$ and $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$. For all $\omega' > \omega$ and $s \in \mathbb{R}$,

$$f\big(\omega' + is\big) = \frac{\omega' - \omega}{\pi} \int_{\mathbb{R}} \frac{f(\omega - ir)}{(s - r)^2 + (\omega' - \omega)^2}\, \mathrm{d}r$$

by [16, Theorem 5.18]. The right-hand side is the convolution of $f(\omega - i\cdot)$ and the Poisson kernel $P_{\omega' - \omega}(r) := \frac{\omega' - \omega}{\pi(r^2 + (\omega' - \omega)^2)}$. Since $\|P_{\omega' - \omega}\|_{\mathrm{L}^1(\mathbb{R})} = 1$,

$$\big\|f\big(\omega' + i\cdot\big)\big\|_{\mathcal{M}_p(X)} \leqslant \big\|f(\omega - i\cdot)\big\|_{\mathcal{M}_p(X)} = \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)}.$$

The converse follows from (2.2) and [7, Lemma E.4.1]. □

For $\mu \in \mathrm{M}(\mathbb{R})$ and $p \in [1, \infty]$ we let $L_\mu \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}; X))$,

$$L_\mu(f) := \mu * f \quad \big(f \in \mathrm{L}^p(\mathbb{R}; X)\big), \tag{2.3}$$

be the convolution operator associated to μ.

Lemma 2.3. *For each $\omega \in \mathbb{R}$ the Laplace transform induces an isometric algebra isomorphism from $\mathrm{M}_\omega(\mathbb{R}_+)$ onto $\mathrm{AM}_1^{\mathbb{C}}(\mathrm{R}_\omega) = \mathrm{AM}_1^X(\mathrm{R}_\omega)$. Moreover,*

$$\|\hat{\mu}\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} = \|L_{e_{-\omega}\mu}\|_{\mathcal{L}(\mathrm{L}^p(X))}$$

for all $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$, $p \in [1, \infty]$.

Proof. The mappings $\mu \mapsto e_{-\omega}\mu$ and $f \mapsto f(\cdot + \omega)$ are isometric algebra isomorphisms $\mathrm{M}_\omega(\mathbb{R}_+) \to \mathrm{M}(\mathbb{R}_+)$ and $\mathrm{AM}_p(\mathrm{R}_\omega) \to \mathrm{AM}_p(\mathbb{C}_+)$, respectively. Hence it suffices to let $\omega = 0$. The Fourier transform induces an isometric isomorphism from $\mathrm{M}(\mathbb{R})$ onto $\mathcal{M}_1(X)$ [7, p. 347, 8.]. If $\mu \in \mathrm{M}(\mathbb{R}_+)$ and $f = \hat{\mu} \in \mathrm{H}^\infty(\mathbb{C}_+)$ then $f(i\cdot) = \mathcal{F}\mu \in \mathcal{M}_1(X)$ with $\|f(i\cdot)\|_{\mathcal{M}_1(X)} = \|\mu\|_{\mathrm{M}(\mathbb{R}_+)}$. Moreover, for $p \in [1, \infty]$,

$$\left\|f(i\cdot)\right\|_{\mathcal{M}_p(X)} = \sup_{\|g\|_p \leqslant 1} \left\|\mathcal{F}^{-1}\big(f(i\cdot)\mathcal{F}g\big)\right\|_p = \sup_{\|g\|_p \leqslant 1} \|\mu * g\|_p = \|L_\mu\|_{\mathcal{L}(\mathrm{L}^p(X))}.$$

If $f \in \mathrm{AM}_1(\mathbb{C}_+)$ then $f(i\cdot) = \mathcal{F}\mu$ for some $\mu \in \mathrm{M}(\mathbb{R})$. An application of Liouville's theorem shows that $\operatorname{supp}(\mu) \subseteq \mathbb{R}_+$, hence $f = \hat{\mu}$. □

2.2. *Functional calculus*

We assume that the reader is familiar with the basic notions and results of the theory of C_0-semigroups as developed, e.g., in [5], and just recall some facts which will be needed in this article.

Each C_0-semigroup $T = (T(t))_{t \in \mathbb{R}_+}$ on a Banach space X has *type* (M, ω) for some $M \geqslant 1$ and $\omega \in \mathbb{R}$, which means that $\|T(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. The *generator* of T is the unique closed operator $-A$ such that

$$(\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t \quad (x \in X)$$

for $\mathrm{Re}(\lambda)$ large. The *Hille–Phillips* (*functional*) *calculus* for A is defined as follows. Fix $M \geqslant 1$ and $\omega_0 \in \mathbb{R}$ such that T has type $(M, -\omega_0)$. For $\mu \in \mathrm{M}_{\omega_0}(\mathbb{R}_+)$ define $T_\mu \in \mathcal{L}(X)$ by

$$T_\mu x := \int_0^\infty T(t)x \,\mu(\mathrm{d}t) \quad (x \in X). \tag{2.4}$$

For $f = \hat{\mu} \in \mathrm{AM}_1(\mathrm{R}_{\omega_0})$ set $f(A) := T_\mu$. (This is allowed by the injectivity of the Laplace transform, see Lemma 2.3.) The mapping $f \mapsto f(A)$ is an algebra homomorphism. In a second step the definition of $f(A)$ is extended to a larger class of functions via *regularization*, i.e.,

$$f(A) := e(A)^{-1}(ef)(A)$$

if there exists $e \in \mathrm{AM}_1(\mathrm{R}_{\omega_0})$ such that $e(A)$ is injective and $ef \in \mathrm{AM}_1(\mathrm{R}_{\omega_0})$. Then $f(A)$ is a closed and (in general) unbounded operator on X and the definition of $f(A)$ is independent of the choice of regularizer e. The following lemma shows in particular that for $\omega < \omega_0$ the operator $f(A)$ is defined for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ by virtue of the regularizer $e(z) = (z - \lambda)^{-1}$, where $\mathrm{Re}(\lambda) < \omega$.

Lemma 2.4. *Let* $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{C}$ *and* $\omega, \omega_0 \in \mathbb{R}$ *with* $\mathrm{Re}(\lambda) < \omega < \omega_0$. *Then*

$$f(z)(z - \lambda)^{-\alpha} \in \mathrm{AM}_1(\mathrm{R}_{\omega_0}) \quad \textit{for all } f \in \mathrm{H}^\infty(\mathrm{R}_\omega).$$

Proof. After shifting we may suppose that $\omega = 0$. Set $h(z) := f(z)(z-\lambda)^{-\alpha}$ for $z \in \mathbb{C}_+$. Then $h(i\cdot) \in \mathrm{L}^2(\mathbb{R})$ with

$$\|h(i\cdot)\|^2_{\mathrm{L}^2(\mathbb{R})} \leqslant \int_{\mathbb{R}} \frac{|f(is)|^2}{|is-\lambda|^{2\alpha}}\,\mathrm{d}s \leqslant \|f\|^2_{\mathrm{H}^\infty(\mathbb{C}_+)} \int_{\mathbb{R}} \frac{1}{|is-\lambda|^{2\alpha}}\,\mathrm{d}s,$$

hence the Paley–Wiener Theorem [16, Theorem 5.28] implies that $h = \hat{g}$ for some $g \in \mathrm{L}^2(\mathbb{R}_+)$. Then $e_{-\omega_0} g \in \mathrm{L}^1(\mathbb{R}_+)$ and $\widehat{e_{-\omega_0} g}(z) = h(z+\omega_0)$ for $z \in \mathbb{C}_+$. Lemma 2.3 yields $h \in \mathrm{AM}_1(\mathrm{R}_{\omega_0})$ with

$$\|h\|_{\mathrm{AM}_1(\mathrm{R}_{\omega_0})} = \|h(\cdot+\omega_0)\|_{\mathrm{AM}_1(\mathbb{C}_+)} = \|e_{-\omega_0} g\|_{\mathrm{L}^1(\mathbb{R}_+)}. \qquad \square$$

The Hille–Phillips calculus is an extension of the holomorphic functional calculus for the operators of half-plane type discussed in [3]. An operator A is *of half-plane type* $\omega_0 \in \mathbb{R}$ if $\sigma(A) \subseteq \overline{\mathrm{R}_{\omega_0}}$ with

$$\sup_{\lambda \in \mathbb{C}\setminus \mathrm{R}_\omega} \|R(\lambda, A)\| < \infty \quad \text{for all } \omega < \omega_0.$$

One can associate operators $f(A) \in \mathcal{L}(X)$ to certain elementary functions via Cauchy integrals and regularize as above to extend the definition to all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. If $-A$ generates a C_0-semigroup of type $(M, -\omega_0)$ then A is of half-plane type ω_0, and by combining [3, Proposition 2.8] and [7, Proposition 3.3.2] one sees that for $\omega < \omega_0$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ the definitions of $f(A)$ via the Hille–Phillips calculus and the half-plane calculus coincide.

For a proof of the next, fundamental, lemma see [3, Theorem 3.1].

Lemma 2.5 *(Convergence Lemma). Let A be a densely defined operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space X. Let $\omega < \omega_0$ and $(f_j)_{j\in J} \subseteq \mathrm{H}^\infty(\mathrm{R}_\omega)$ be a net satisfying the following conditions*:

(1) $\sup\{|f_j(z)| \mid z \in \mathrm{R}_\omega, j \in J\} < \infty$;
(2) $f_j(A) \in \mathcal{L}(X)$ *for all* $j \in J$ *and* $\sup_{j\in J} \|f_j(A)\| < \infty$;
(3) $f(z) := \lim_{j\in J} f_j(z)$ *exists for all* $z \in \mathrm{R}_\omega$.

Then $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$, $f(A) \in \mathcal{L}(X)$, $f_j(A) \to f(A)$ *strongly and*

$$\|f(A)\| \leqslant \limsup_{j\in J} \|f_j(A)\|.$$

Let A be an operator of half-plane type ω_0 and $\omega < \omega_0$. For a Banach algebra F of functions continuously embedded in $\mathrm{H}^\infty(\mathrm{R}_\omega)$, we say that A has a *bounded F-calculus* if there exists a constant $C \geqslant 0$ such that $f(A) \in \mathcal{L}(X)$ with

$$\|f(A)\|_{\mathcal{L}(X)} \leqslant C\|f\|_F \quad \text{for all } f \in F. \tag{2.5}$$

The operator $-A$ generates a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type (M, ω) if and only if $-(A+\omega)$ generates the semigroup $(e^{-\omega t}T(t))_{t\in\mathbb{R}_+}$ of type $(M, 0)$. The functional calculi for

A and $A+\omega$ are linked by the simple composition rule "$f(A+\omega) = f(\omega+z)(A)$" [7, Theorem 2.4.1]. Henceforth we shall mainly consider bounded semigroups; all results carry over to general semigroups by shifting.

3. Functional calculus for semigroup generators

Define the function $\eta : (0,\infty) \times (0,\infty) \times [1,\infty] \to \mathbb{R}_+$ by

$$\eta(\alpha, t, q) := \inf\bigl\{\|\psi\|_q \|\varphi\|_{q'} \bigm| \psi * \varphi \equiv e_{-\alpha} \text{ on } [t,\infty)\bigr\}. \tag{3.1}$$

The set on the right-hand side is not empty: choose for instance $\psi := \mathbf{1}_{[0,t]} e_{-\alpha}$ and $\varphi := \frac{1}{t} e_{-\alpha}$. By Lemma A.1,

$$\eta(\alpha, t, q) = O\bigl(\bigl|\log(\alpha t)\bigr|\bigr) \quad \text{as } \alpha t \to 0,$$

for $q \in (1,\infty)$.

For the following result recall the definitions of the operators L_μ from (2.3) and T_μ from (2.4).

Proposition 3.1. *Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semigroup of type $(M,0)$ on a Banach space X. Let $p \in [1,\infty]$, $\tau, \omega > 0$ and $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\mu) \subseteq [\tau,\infty)$. Then*

$$\|T_\mu\|_{\mathcal{L}(X)} \leqslant M^2 \eta(\omega, \tau, p) \|L_{e_\omega \mu}\|_{\mathcal{L}(\mathrm{L}^p(X))}. \tag{3.2}$$

Proof. We can factorize T_μ as $T_\mu = P \circ L_{e_\omega \mu} \circ \iota$, where

- $\iota : X \to \mathrm{L}^p(\mathbb{R}; X)$ is given by

$$\iota(x)(s) := \begin{cases} \psi(-s)T(-s)x & \text{if } s \leqslant 0, \\ 0 & \text{if } s > 0 \end{cases} \qquad (x \in X).$$

- $P : \mathrm{L}^p(\mathbb{R}; X) \to X$ is given by

$$Pf := \int_0^\infty \varphi(t) T(t) f(t)\,\mathrm{d}t \quad \bigl(f \in \mathrm{L}^p(\mathbb{R}; X)\bigr).$$

- $\psi \in \mathrm{L}^p(\mathbb{R}_+)$ and $\varphi \in \mathrm{L}^{p'}(\mathbb{R}_+)$ are such that $\psi * \varphi \equiv e_{-\omega}$ on $[\tau,\infty)$.

This is deduced as in the transference principle from [9, Section 2], using that $\mu = (\psi * \varphi) e_\omega \mu$. Hölder's inequality then implies

$$\|T_\mu\| \leqslant M^2 \|\psi\|_p \|L_{e_\omega \mu}\|_{\mathcal{L}(\mathrm{L}^p(X))} \|\varphi\|_{p'},$$

and taking the infimum over all such ψ and φ yields (3.2). □

Now define, for a Banach space X, $\omega \in \mathbb{R}$, $p \in [1,\infty]$ and $\tau > 0$, the space

$$\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega) := \bigl\{ f \in \mathrm{AM}^X_p(\mathrm{R}_\omega) \bigm| f(z) = O\bigl(e^{-\tau\,\mathrm{Re}(z)}\bigr) \text{ as } |z| \to \infty \bigr\},$$

endowed with the norm of $\mathrm{AM}^X_p(\mathrm{R}_\omega)$.

Lemma 3.2. *For every Banach space X, $\omega \in \mathbb{R}$, $p \in [1,\infty]$ and $\tau > 0$*

$$\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega) = \mathrm{AM}^X_p(\mathrm{R}_\omega) \cap e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_\omega) = e_{-\tau}\mathrm{AM}^X_p(\mathrm{R}_\omega). \tag{3.3}$$

In particular, $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega)$ is a closed ideal in $\mathrm{AM}^X_p(\mathrm{R}_\omega)$.

Proof. The first equality in (3.3) is clear, and so is the inclusion $e_{-\tau}\mathrm{AM}_p(\mathrm{R}_\omega) \subseteq \mathrm{AM}_{p,\tau}(\mathrm{R}_\omega)$. Conversely, if $f \in \mathrm{AM}_p(\mathrm{R}_\omega) \cap e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_\omega)$ then $e_\tau f \in \mathrm{AM}_p(\mathrm{R}_\omega)$ since

$$\bigl\| e^{\tau(\omega+i\cdot)} f(\omega+i\cdot) \bigr\|_{\mathcal{M}_p(X)} = e^{\tau\omega} \bigl\| f(\omega+i\cdot) \bigr\|_{\mathcal{M}_p(X)}.$$

Now suppose that $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{AM}_{p,\tau}(\mathrm{R}_\omega)$ converges to $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$. The Maximum Principle implies $\|e_\tau f_n\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} = e^{\tau\omega}\|f_n\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}$, hence $(e_\tau f_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{H}^\infty(\mathrm{R}_\omega)$. Since it converges pointwise to $e_\tau f$, (3.3) implies $f \in \mathrm{AM}_{p,\tau}(\mathrm{R}_\omega)$. □

We are now ready to prove the main result of this section. Note that the union of the ideals $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega)$ for $\tau > 0$ is dense in $\mathrm{AM}^X_p(\mathrm{R}_\omega)$ with respect to pointwise and bounded convergence of sequences. If there were a single constant independent of τ bounding the $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega)$-calculus for all τ, the Convergence Lemma would imply that A has a bounded $\mathrm{AM}^X_p(\mathrm{R}_\omega)$-calculus, but this is known to be false in general [7, Corollary 9.1.8].

Theorem 3.3. *For each $p \in (1,\infty)$ there exists a constant $c_p \geqslant 0$ such that the following holds. Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a Banach space X and let $\tau, \omega > 0$. Then $f(A) \in \mathcal{L}(X)$ and*

$$\bigl\|f(A)\bigr\| \leqslant \begin{cases} c_p M^2 |\log(\omega\tau)| \, \|f\|_{\mathrm{AM}^X_p} & \text{if } \omega\tau \leqslant \min(\frac{1}{p}, \frac{1}{p'}), \\ 2M^2 e^{-\omega\tau} \|f\|_{\mathrm{AM}^X_p} & \text{if } \omega\tau > \min(\frac{1}{p}, \frac{1}{p'}) \end{cases}$$

for all $f \in \mathrm{AM}^X_{p,\tau}(\mathrm{R}_{-\omega})$. In particular, A has a bounded $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_{-\omega})$-calculus.

Proof. First consider $f \in \mathrm{AM}_{1,\tau}(\mathrm{R}_{-\omega})$. Let $\delta_\tau \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ be the unit point mass at τ. By Lemmas 3.2 and 2.3 there exists $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ such that $f = e_{-\tau}\hat{\mu} = \widehat{\delta_\tau * \mu}$. Since $\delta_\tau * \mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\delta_\tau * \mu) \subseteq [\tau,\infty)$, Proposition 3.1 and Lemma 2.3 yield

$$\bigl\|f(A)\bigr\| \leqslant M^2 \eta(\omega,\tau,p)\|f\|_{\mathrm{AM}^X_p}. \tag{3.4}$$

Now suppose $f \in \mathrm{AM}_{p,\tau}(\mathrm{R}_{-\omega})$ is arbitrary. For $\epsilon > 0$, $k \in \mathbb{N}$ and $z \in \mathrm{R}_{-\omega}$ set $g_k(z) := \frac{k}{z-\omega+k}$ and $f_{k,\epsilon}(z) := f(z+\epsilon)g_k(z+\epsilon)$. Lemma 2.4 yields $f_{k,\epsilon} \in \mathrm{AM}_{1,\tau}(\mathrm{R}_{-\omega})$, hence, by what we have already shown,

$$\bigl\|f_{k,\epsilon}(A)\bigr\| \leqslant M^2 \eta(\omega,\tau,p)\|f_{k,\epsilon}\|_{\mathrm{AM}^X_p}.$$

The inclusion $\mathrm{AM}_1(\mathrm{R}_{-\omega}) \subseteq \mathrm{AM}_p(\mathrm{R}_{-\omega})$ is contractive, so Lemma 2.3 implies that $g_k \in \mathrm{AM}_p(\mathrm{R}_{-\omega})$ with

$$\|g_k\|_{\mathrm{AM}_p^X} \leqslant \|g_k\|_{\mathrm{AM}_1} = k\|e_{-k}\|_{\mathrm{L}^1(\mathbb{R}_+)} = 1.$$

Combining this with Lemma 2.2 yields

$$\|f_{k,\epsilon}\|_{\mathrm{AM}_p^X} \leqslant \big\|f(\cdot+\epsilon)\big\|_{\mathrm{AM}_p^X}\big\|g_k(\cdot+\epsilon)\big\|_{\mathrm{AM}_p^X} \leqslant \|f\|_{\mathrm{AM}_p^X}.$$

In particular, $\sup_{k,\epsilon}\|f_{k,\epsilon}\|_\infty < \infty$ and $\sup_{k,\epsilon}\|f_{k,\epsilon}(A)\| < \infty$. The Convergence Lemma 2.5 implies that $f(A) \in \mathcal{L}(X)$ satisfies (3.4). Lemma A.1 concludes the proof. □

Remark 3.4. Because $\mathrm{AM}_1(\mathrm{R}_{-\omega}) = \mathrm{AM}_\infty(\mathrm{R}_{-\omega})$ is contractively embedded in $\mathrm{AM}_p(\mathrm{R}_{-\omega})$, Theorem 3.3 also holds for $p=1$ and $p=\infty$. However, A trivially has a bounded AM_1-calculus by Lemma 2.3 and the Hille–Phillips calculus.

Note that the exponential decay of $|f(z)|$ is only required as the real part of z tends to infinity. If $|f(z)|$ decays exponentially as $|z| \to \infty$ the result is not interesting, by Lemma 2.4.

We can equivalently formulate Theorem 3.3 as a statement about composition with semigroup operators.

Corollary 3.5. *Under the assumptions of Theorem 3.3, $f(A)T(\tau) \in \mathcal{L}(X)$ and*

$$\big\|f(A)T(\tau)\big\| \leqslant \begin{cases} c_p M^2|\log(\omega\tau)|e^{\omega\tau}\|f\|_{\mathrm{AM}_p^X} & \text{if } \omega\tau \leqslant \min(\frac{1}{p},\frac{1}{p'}),\\ 2M^2\|f\|_{\mathrm{AM}_p^X} & \text{if } \omega\tau > \min(\frac{1}{p},\frac{1}{p'}) \end{cases}$$

for all $f \in \mathrm{AM}_p^X(\mathrm{R}_{-\omega})$.

Proof. Note that $f(A)T(\tau) = (e_{-\tau}f)(A)$ and $\|e_{-\tau}f\|_{\mathrm{AM}_p^X} = e^{\omega\tau}\|f\|_{\mathrm{AM}_p^X}$. □

3.1. Additional results

As a first corollary of Theorem 3.3 we obtain a sufficient condition for a semigroup generator to have a bounded AM_p-calculus.

Corollary 3.6. *Let $-A$ generate a bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ with*

$$\bigcup_{\tau>0}\operatorname{ran}\big(T(\tau)\big) = X.$$

Then A has a bounded $\mathrm{AM}_p^X(\mathrm{R}_\omega)$-calculus for all $\omega < 0$, $p \in [1,\infty]$.

Proof. Using Corollary 3.5, note that $f(A)T(\tau) \in \mathcal{L}(X)$ implies $\operatorname{ran}(T(\tau)) \subseteq \operatorname{dom}(f(A))$. An application of the Closed Graph Theorem (using the Convergence Lemma) yields (2.5). □

Theorem 3.7. *Let* $p \in (1, \infty)$, $\omega > 0$ *and* $\alpha, \lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) < 0 < \mathrm{Re}(\alpha)$. *There exists a constant* $C = C(p, \alpha, \lambda, \omega) \geqslant 0$ *such that the following holds. Let* $-A$ *generate a* C_0*-semigroup* $(T(t))_{t \in \mathbb{R}_+}$ *of type* $(M, 0)$ *on a Banach space* X. *Then* $\mathrm{dom}((A-\lambda)^\alpha) \subseteq \mathrm{dom}(f(A))$ *and*

$$\|f(A)(A-\lambda)^{-\alpha}\| \leqslant CM^2 \|f\|_{\mathrm{AM}_p^X}$$

for all $f \in \mathrm{AM}_p^X(\mathrm{R}_{-\omega})$.

Proof. First note that $-(A-\lambda)$ generates the exponentially stable semigroup $(e^{\lambda t} T(t))_{t \in \mathbb{R}_+}$. Hence Corollary 3.3.6 in [7] allows us to write

$$(A-\lambda)^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{\lambda t} T(t) x \,\mathrm{d}t \quad (x \in X).$$

Fix $f \in \mathrm{AM}_p(\mathrm{R}_{-\omega})$ and set $a := \frac{1}{\omega} \min\{\frac{1}{p}, \frac{1}{p'}\}$. By Corollary 3.5,

$$\int_0^\infty t^{\mathrm{Re}(\alpha)-1} e^{\mathrm{Re}(\lambda) t} \|f(A)T(t)x\| \,\mathrm{d}t \leqslant CM^2 \|f\|_{\mathrm{AM}_p^X} \|x\| < \infty$$

for all $x \in X$, where

$$C = c_p \int_0^a t^{\mathrm{Re}(\alpha)-1} \bigl|\log(\omega t)\bigr| e^{(\mathrm{Re}(\lambda)+\omega)t} \,\mathrm{d}t + 2 \int_a^\infty t^{\mathrm{Re}(\alpha)-1} e^{\mathrm{Re}(\lambda)t} \,\mathrm{d}t$$

is independent of f, M, and x. Since $f(A)$ is a closed operator, this implies that $(A-\lambda)^{-\alpha}$ maps into $\mathrm{dom}(f(A))$ with

$$f(A)(A-\lambda)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{\lambda t} f(A) T(t) \,\mathrm{d}t$$

as a strong integral. □

Remark 3.8. Theorem 3.7 shows that for each analytic multiplier function f the domain $\mathrm{dom}(f(A))$ is relatively large, it contains the real interpolation spaces $(X, \mathrm{dom}(A))_{\theta,q}$ and the complex interpolation spaces $[X, \mathrm{dom}(A)]_\theta$ for all $\theta \in (0, 1)$ and $q \in [1, \infty]$. This follows from [14, Proposition 1.1.4] and [7, Corollary 6.6.3] for real interpolation spaces and then from [14, Proposition 2.1.10] for the complex interpolation spaces.

Remark 3.9. We can describe the range of $f(A)(A-\lambda)^{-\alpha}$ in Theorem 3.7 more explicitly. In fact,

$$\mathrm{ran}\bigl(f(A)(A-\lambda)^{-\alpha}\bigr) \subseteq \mathrm{dom}\bigl((A-\lambda)^\beta\bigr)$$

for all $\mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$. Indeed, this follows if we show that $\mathrm{ran}(A-\lambda)^{-\alpha} \subseteq \mathrm{dom}((A-\lambda)^{\beta} f(A))$, and [7, Theorem 1.3.2] implies

$$\mathrm{dom}\big((A-\lambda)^{\beta} f(A)\big) = \mathrm{dom}\big(f(A)\big) \cap \mathrm{dom}\big(\big[(z-\lambda)^{\beta} f(z)\big](A)\big).$$

The inclusion $\mathrm{ran}((A-\lambda)^{-\alpha}) \subseteq \mathrm{dom}(f(A))$ follows from Theorem 3.7. Since

$$\big[(z-\lambda)^{\beta} f(z)\big](A)(A-\lambda)^{-\alpha} = \big[(z-\lambda)^{\beta-\alpha} f(z)\big](A) = f(A)(A-\lambda)^{\beta-\alpha},$$

the same holds for the inclusion $\mathrm{ran}((A-\lambda)^{-\alpha}) \subseteq \mathrm{dom}([(z-\lambda)^{\beta} f(z)](A))$.

3.2. Semigroups on Hilbert spaces

If $X = H$ is a Hilbert space, Plancherel's Theorem implies $\mathrm{AM}_2^H = \mathrm{H}^\infty$ with equality of norms. Hence the theory above specializes to the following result, implying (a) and (b) of Theorem 1.1.

Corollary 3.10. *Let $-A$ generate a bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a Hilbert space H. Then the following assertions hold.*

(a) *There exists a universal constant $c \geqslant 0$ such that the following holds. Let $\tau, \omega > 0$. Then $f(A) \in \mathcal{L}(H)$ and*

$$\|f(A)\| \leqslant \begin{cases} cM^2|\log(\omega\tau)|\,\|f\|_\infty & \text{if } \omega\tau \leqslant \frac{1}{2},\\ 2M^2 e^{-\omega\tau}\|f\|_\infty & \text{if } \omega\tau > \frac{1}{2}\end{cases}$$

for all $f \in e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$. Moreover, $f(A)T(\tau) \in \mathcal{L}(H)$ with

$$\|f(A)T(\tau)\| \leqslant \begin{cases} cM^2|\log(\omega\tau)|e^{\omega\tau}\|f\|_\infty & \text{if } \omega\tau \leqslant \frac{1}{2},\\ 2M^2\|f\|_\infty & \text{if } \omega\tau > \frac{1}{2}\end{cases}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_{-\omega})$.

(b) *If*

$$\bigcup\nolimits_{\tau>0} \mathrm{ran}\big(T(\tau)\big) = H,$$

then A has a bounded $\mathrm{H}^\infty(\mathrm{R}_\omega)$-calculus for all $\omega < 0$.

(c) *For $\omega < 0$ and $\alpha, \lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0 < \mathrm{Re}(\alpha)$ there is $C = C(\alpha, \lambda, \omega) \geqslant 0$ such that*

$$\|f(A)(A-\lambda)^{-\alpha}\| \leqslant CM^2\|f\|_\infty$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. In particular, $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(f(A))$.

Part (c) shows that, even though semigroup generators on Hilbert spaces do not have a bounded H^∞-calculus in general, each function f that decays with polynomial rate $\alpha > 0$ at infinity yields a bounded operator $f(A)$. For $\alpha > \frac{1}{2}$ this is already covered by Lemma 2.4, but for $\alpha \in (0, \frac{1}{2}]$ it appears to be new.

Remark 3.11. Part (c) of Corollary 3.10 yields a statement about stability of numerical methods. Let $-A$ generate an exponentially stable semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space, let $r \in \mathrm{H}^\infty(\mathbb{C}_+)$ be such that $\|r\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \leqslant 1$, and let $\alpha, h > 0$. Then

$$\sup\bigl\{\bigl\|r(hA)^n x\bigr\| \bigm| n \in \mathbb{N},\ x \in \operatorname{dom}\bigl(A^\alpha\bigr)\bigr\} < \infty \tag{3.5}$$

follows from (c) in Corollary 3.10 after shifting the generator. Elements of the form $r(hA)^n x$ are often used in numerical methods to approximate the solution of the abstract Cauchy problem associated to $-A$ with initial value x, and (3.5) shows that such approximations are stable whenever $x \in \operatorname{dom}(A^\alpha)$.

4. *m*-Bounded functional calculus

In this section we describe another transference principle for semigroups, one that provides estimates for the norms of operators of the form $f^{(m)}(A)$ for f an analytic multiplier function and $f^{(m)}$ its m-th derivative, $m \in \mathbb{N}$. We use terminology from Section 5 of [3]. Moreover, we recall our notational simplification $\mathrm{AM}_p(\mathrm{R}_\omega) := \mathrm{AM}_p^X(\mathrm{R}_\omega)$ (Remark 2.1).

Let $\omega < \omega_0$ be real numbers. An operator A of half-plane type ω_0 on a Banach space X has an *m-bounded $\mathrm{AM}_p^X(\mathrm{R}_\omega)$-calculus* if there exists $C \geqslant 0$ such that $f^{(m)}(A) \in \mathcal{L}(X)$ with

$$\bigl\|f^{(m)}(A)\bigr\| \leqslant C\|f\|_{\mathrm{AM}_p^X} \quad \text{for all } f \in \mathrm{AM}_p^X(\mathrm{R}_\omega).$$

This is well defined since the Cauchy integral formula implies that $f^{(m)}$ is bounded on every half-plane $\mathrm{R}_{\omega'}$ with $\omega' > \omega$.

We say that A has a *strong m-bounded AM_p^X-calculus of type ω_0* if A has an m-bounded $\mathrm{AM}_p^X(\mathrm{R}_\omega)$-calculus for *every* $\omega < \omega_0$ such that for some $C \geqslant 0$ one has

$$\bigl\|f^{(m)}(A)\bigr\| \leqslant \frac{C}{(\omega_0-\omega)^m}\|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} \tag{4.1}$$

for all $f \in \mathrm{AM}_p^X(\mathrm{R}_\omega)$ and $\omega < \omega_0$.

Lemma 4.1. *Let A be an operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space X, and let $p \in [1,\infty]$ and $m \in \mathbb{N}$. If A has a strong m-bounded AM_p^X-calculus of type ω_0, then A has a strong n-bounded AM_p^X-calculus of type ω_0 for all $n > m$.*

Proof. Let $\omega < \alpha < \beta < \omega_0$, $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} f^{(n)}(\beta+is) &= \frac{n!}{2\pi i}\int_{\mathbb{R}} \frac{f(\alpha+ir)}{(\alpha+ir-(\beta+is))^{n+1}}\,\mathrm{d}r \\ &= \frac{n!}{2\pi i}\bigl(f(\alpha+i\cdot) * (\alpha-\beta-i\cdot)^{-n-1}\bigr)(s) \end{aligned}$$

for all $s \in \mathbb{R}$, by the Cauchy integral formula. Hence, using Lemma 2.2,

$$\begin{aligned}\|f^{(n)}(\beta+i\cdot)\|_{\mathcal{M}_p(X)} &\leqslant \frac{n!}{2\pi}\|(\alpha-\beta-i\cdot)^{-n-1}\|_{\mathrm{L}^1(\mathbb{R})}\|f(\alpha+i\cdot)\|_{\mathcal{M}_p(X)}\\ &\leqslant \frac{C}{(\beta-\alpha)^n}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}\end{aligned}$$

for some $C=C(n)\geqslant 0$ independent of f, β, α and ω. Letting α tend to ω yields

$$\|f^{(n)}\|_{\mathrm{AM}_p(\mathrm{R}_\beta)} = \|f^{(n)}(\beta+i\cdot)\|_{\mathcal{M}_p(X)} \leqslant \frac{C}{(\beta-\omega)^n}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}. \tag{4.2}$$

Now let $n>m$. Applying (4.2) with $n-m$ in place of n shows that $f^{(n-m)}\in \mathrm{AM}_p(\mathrm{R}_\beta)$ with

$$\|f^{(n)}(A)\| \leqslant \frac{C'}{(\omega_0-\beta)^m}\|f^{(n-m)}\|_{\mathrm{AM}_p(\mathrm{R}_\beta)} \leqslant \frac{CC'}{(\omega_0-\beta)^m(\beta-\omega)^{n-m}}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}.$$

Finally, letting $\beta=\frac{1}{2}(\omega+\omega_0)$,

$$\|f^{(n)}(A)\| \leqslant \frac{C''}{(\omega_0-\omega)^n}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}$$

for some $C''\geqslant 0$ independent of f and ω. □

For the transference principle in Proposition 3.1 it is essential that the support of $\mu\in\mathrm{M}_\omega(\mathbb{R}_+)$ is contained in some interval $[\tau,\infty)$ with $\tau>0$. In general one cannot expect to find such a transference principle for arbitrary μ, as this would allow one to prove that semigroup generators have a bounded analytic multiplier calculus. But this is known to be false in general, cf. [7, Corollary 9.1.8]. However, if we let $t\mu$ be given by $(t\mu)(\mathrm{d}t) := t\mu(\mathrm{d}t)$ then we can deduce the following transference principle. We use the conventions $1/\infty := 0$, $\infty^0 := 1$.

Proposition 4.2. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a Banach space X. Let $p\in[1,\infty]$, $\omega<0$ and $\mu\in\mathrm{M}_\omega(\mathbb{R}_+)$. Then*

$$\|T_{t\mu}\| \leqslant \frac{M^2}{|\omega|}p^{-1/p}\big(p'\big)^{-1/p'}\|L_{e_{-\omega}\mu}\|_{\mathcal{L}(\mathrm{L}^p(X))}.$$

Proof. We can factorize $T_{t\mu}$ as $T_{t\mu}=P\circ L_{e_{-\omega}\mu}\circ\iota$, where ι and P are as in the proof of Proposition 3.1 with $\psi,\varphi := \mathbf{1}_{\mathbb{R}_+}e_\omega$. This follows from the abstract transference principle in [9, Section 2], since $(\psi*\varphi)e_{-\omega}\mu = t\mu$. Then

$$\begin{aligned}\|T_{t\mu}\| &\leqslant M^2\|e_\omega\|_{p'}\|L_{e_{-\omega}\mu}\|_{\mathcal{L}(\mathrm{L}^p(X))}\|e_\omega\|_p\\ &= \frac{M^2}{|\omega|}p^{-1/p}\big(p'\big)^{-1/p'}\|L_{e_{-\omega}\mu}\|_{\mathcal{L}(\mathrm{L}^p(X))},\end{aligned}$$

by Hölder's inequality. □

We are now ready to prove our main result on m-bounded functional calculus, a generalization of Theorem 7.1 in [3] to arbitrary Banach spaces.

Theorem 4.3. *Let A be a densely defined operator of half-plane type 0 on a Banach space X. Then the following assertions are equivalent*:

(i) *$-A$ is the generator of a bounded C_0-semigroup on X.*
(ii) *A has a strong m-bounded AM_p^X-calculus of type 0 for some/all $p \in [1,\infty]$ and some/all $m \in \mathbb{N}$.*

In particular, if $-A$ generates a bounded C_0-semigroup then A has an m-bounded $\mathrm{AM}_p^X(\mathrm{R}_\omega)$-calculus for all $\omega < 0$, $p \in [1,\infty]$ and $m \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) By Lemma 4.1 it suffices to let $m = 1$. We proceed along the same lines as the proof of Theorem 3.3. Let $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ be the semigroup generated by $-A$ and fix $\omega < 0$, $p \in [1,\infty]$ and $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$. Define the functions $f_{k,\epsilon} := f(\cdot + \epsilon)g_k(\cdot + \epsilon)$ for $k \in \mathbb{N}$ and $\epsilon > 0$, where $g_k(z) := \frac{k}{z-\omega+k}$ for $z \in \mathrm{R}_\omega$. Then $f_{k,\epsilon} \in \mathrm{AM}_1(\mathrm{R}_\omega)$ by Lemma 2.4, and Lemma 2.3 yields $\mu_{k,\epsilon} \in \mathrm{M}_\omega(\mathbb{R}_+)$ with $f_{k,\epsilon} = \widehat{\mu_{k,\epsilon}}$. Now

$$f'_{k,\epsilon}(z) = \lim_{h\to 0} \frac{f_{k,\epsilon}(z+h) - f_{k,\epsilon}(z)}{h} = \lim_{h\to 0} \int_0^\infty \frac{e^{-(z+h)t} - e^{-zt}}{h} \mu_{k,\epsilon}(\mathrm{d}t)$$
$$= -\int_0^\infty t e^{-zt} \mu_{k,\epsilon}(\mathrm{d}t) = -\widehat{t\mu_{k,\epsilon}}(z)$$

for $z \in \mathrm{R}_\omega$, by the Dominated Convergence Theorem. Hence $f'_{k,\epsilon}(A) = -T_{t\mu_{k,\epsilon}}$, and Proposition 4.2 and Lemma 2.3 imply

$$\left\| f'_{k,\epsilon}(A) \right\| \leqslant \frac{M^2}{|\omega|} p^{-1/p} \left(p'\right)^{-1/p'} \| f_{k,\epsilon} \|_{\mathrm{AM}_p^X}.$$

Furthermore, $\sup_{k,\epsilon} \| f_{k,\epsilon} \|_{\mathrm{AM}_p^X} \leqslant \| f \|_{\mathrm{AM}_p^X}$. In particular, the $f_{k,\epsilon}$ are uniformly bounded. By the Cauchy integral formula, so are the derivatives $f'_{k,\epsilon}$ on every smaller half-plane. Since $f'_{k,\epsilon}(z) \to f'(z)$ for all $z \in \mathrm{R}_\omega$ as $k \to \infty$, $\epsilon \to 0$, the Convergence Lemma yields $f'(A) \in \mathcal{L}(X)$ with

$$\left\| f'(A) \right\| \leqslant \frac{M^2}{|\omega|} p^{-1/p} \left(p'\right)^{-1/p'} \| f \|_{\mathrm{AM}_p^X},$$

which is (4.1) for $m = 1$.

For (ii) $\Rightarrow$ (i) assume that A has a strong m-bounded AM_p-calculus of type 0 for some $p \in [1,\infty]$ and some $m \in \mathbb{N}$. Then

$$e_{-t} \in \mathrm{AM}_1(\mathrm{R}_\omega) \subseteq \mathrm{AM}_p(\mathrm{R}_\omega)$$

for all $t > 0$ and $\omega < 0$, with

$$\|e_{-t}\|_{\mathrm{AM}_p(\mathrm{R}_\omega)} \leqslant \|e_{-t}\|_{\mathrm{AM}_1(\mathrm{R}_\omega)} = e^{-t\omega}.$$

Now $(e_{-t})^{(m)} = (-t)^m e_{-t}$ implies

$$t^m \|e^{-tA}\| \leqslant \frac{C}{|\omega|^m} e^{-t\omega}.$$

Letting $\omega := -\frac{1}{t}$ and using Lemma 2.5 in [3] yields the required statement. □

If $X = H$ is a Hilbert space then Plancherel's theorem yields the following result, which is a generalization of Theorem 7.1 in [3]. It contains part (c) of Theorem 1.1.

Corollary 4.4. *Let A be a densely defined operator of half-plane type 0 on a Hilbert space H. Then the following assertions are equivalent:*

(i) *$-A$ is the generator of a bounded C_0-semigroup on H.*
(ii) *A has a strong m-bounded H^∞-calculus of type 0 for some/all $m \in \mathbb{N}$.*

In particular, if $-A$ generates a bounded C_0-semigroup then A has an m-bounded $\mathrm{H}^\infty(\mathrm{R}_\omega)$-calculus for all $\omega < 0$ and $m \in \mathbb{N}$.

5. Semigroups on UMD spaces

A Banach space X is a *UMD space* if the function $t \mapsto \operatorname{sgn}(t)$ is a bounded $\mathrm{L}^2(X)$-Fourier multiplier. For $\omega \in \mathbb{R}$ we let

$$\mathrm{H}_1^\infty(\mathrm{R}_\omega) := \left\{ f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \;\middle|\; (z-\omega) f'(z) \in \mathrm{H}^\infty(\mathrm{R}_\omega) \right\}$$

be the *analytic Mikhlin algebra* on R_ω, a Banach algebra endowed with the norm

$$\|f\|_{\mathrm{H}_1^\infty} = \|f\|_{\mathrm{H}_1^\infty(\mathrm{R}_\omega)} := \sup_{z \in \mathrm{R}_\omega} \left|f(z)\right| + \left|(z-\omega) f'(z)\right| \quad \left(f \in \mathrm{H}_1^\infty(\mathrm{R}_\omega)\right).$$

The vector-valued Mikhlin multiplier theorem [7, Theorem E.6.2] and Lemma 2.2 yield the continuous inclusion

$$\mathrm{H}_1^\infty(\mathrm{R}_\omega) \hookrightarrow \mathrm{AM}_p^X(\mathrm{R}_\omega)$$

for each $p \in (1, \infty)$, if X is a UMD space. Combining this with Theorems 3.3 and 4.3 and Corollaries 3.5 and 3.6 proves the following theorem.

Theorem 5.1. *Let $-A$ generate a C_0-semigroup $(T(t))_{t \in \mathbb{R}_+}$ of type $(M, 0)$ on a UMD space X. Then the following assertions hold.*

(a) *For each $p \in (1, \infty)$ there exists a constant $c_p = c(p, X) \geqslant 0$ such that the following holds. Let $\tau, \omega > 0$. Then $f(A) \in \mathcal{L}(X)$ with*

$$\|f(A)\| \leqslant \begin{cases} c_p M^2 |\log(\omega\tau)| \, \|f\|_{\mathrm{H}_1^\infty} & \text{if } \omega\tau \leqslant \min\{\frac{1}{p}, \frac{1}{p'}\}, \\ 2 c_p M^2 e^{-\omega\tau} \|f\|_{\mathrm{H}_1^\infty} & \text{if } \omega\tau > \min\{\frac{1}{p}, \frac{1}{p'}\} \end{cases}$$

for all $f \in \mathrm{H}^\infty_1(\mathrm{R}_{-\omega}) \cap e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$, *and* $f(A)T(\tau) \in \mathcal{L}(X)$ *with*

$$\|f(A)T(\tau)\| \leqslant \begin{cases} c_p M^2 |\log(\omega\tau)| e^{\omega\tau} \|f\|_{\mathrm{H}^\infty_1} & \text{if } \omega\tau \leqslant \min\{\frac{1}{p}, \frac{1}{p'}\}, \\ 2c_p M^2 \|f\|_{\mathrm{H}^\infty_1} & \text{if } \omega\tau > \min\{\frac{1}{p}, \frac{1}{p'}\} \end{cases}$$

for all $f \in \mathrm{H}^\infty_1(\mathrm{R}_{-\omega})$.

(b) *If*

$$\bigcup_{\tau>0} \operatorname{ran}\big(T(\tau)\big) = X,$$

then A *has a bounded* $\mathrm{H}^\infty_1(\mathrm{R}_\omega)$*-calculus for all* $\omega < 0$.

(c) A *has a strong* m*-bounded* H^∞_1*-calculus of type* 0 *for all* $m \in \mathbb{N}$.

Remark 5.2. Theorem 3.7 yields the domain inclusion $\operatorname{dom}(A^\alpha) \subseteq \operatorname{dom}(f(A))$ for all $\alpha \in \mathbb{C}_+$, $\omega < 0$ and $f \in \mathrm{H}^\infty_1(\mathrm{R}_\omega)$, on a UMD space X. However, this inclusion in fact holds true on a general Banach space X. Indeed, for $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) < 0$, Bernstein's Lemma [2, Proposition 8.2.3] implies $\frac{f(z)}{(\lambda-z)^\alpha} \in \mathrm{AM}_1(\mathbb{C}_+)$, hence $f(A)(\lambda - A)^{-\alpha} \in \mathcal{L}(X)$ and $\operatorname{dom}(A^\alpha) \subseteq \operatorname{dom}(f(A))$. An estimate

$$\|f(A)(\lambda - A)^{-\alpha}\| \leqslant C\|f\|_{\mathrm{H}^\infty_1(\mathrm{R}_\omega)}$$

then follows from an application of the Closed Graph Theorem and the Convergence Lemma.

Remark 5.3. To apply Theorem 5.1 one can use the continuous inclusion

$$\mathrm{H}^\infty\big(\mathrm{R}_\omega \cup (S_\varphi + a)\big) \subseteq \mathrm{H}^\infty_1(\mathrm{R}_{\omega'}) \tag{5.1}$$

for $\omega' > \omega$, $a \in \mathbb{R}$ and $\varphi \in (\pi/2, \pi]$. Here $\mathrm{R}_\omega \cup (S_\varphi + a)$ is the union of R_ω and the translated sector $S_\varphi + a$, where

$$S_\varphi := \big\{z \in \mathbb{C} \bigm| |\arg(z)| < \varphi\big\}.$$

Indeed, to derive (5.1) it suffices to let $a = 0$, and using Cauchy's integral formula as in [7, Lemma 8.2.6] yields the desired result.

6. γ-Bounded semigroups

The geometry of the underlying Banach space X played an essential role in the results of Sections 3 and 4 in the form of properties of the analytic multiplier algebra AM^X_p. To wit, in order to identify nontrivial functions in AM^X_p one needs a geometric assumption on X, for instance that it is a Hilbert or a UMD space. In this section we take a different approach and make additional assumptions on the semigroup instead of the underlying space. We show that if the semigroup in question is *γ-bounded* then one can recover the Hilbert space results on an arbitrary Banach space X.

For this section we assume the reader to be familiar with the basics of the theory of γ-radonifying operators and γ-boundedness as collected in the survey article by van Neerven [18]. We use terminology and results from [9].

Let H be a Hilbert space and X a Banach space. A linear operator $T : H \to X$ is *γ-summing* if

$$\|T\|_\gamma := \sup_F \left(\mathbb{E} \left\| \sum_{h \in F} \gamma_h T h \right\|_X^2 \right)^{1/2} < \infty,$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $(\gamma_h)_{h\in F}$ is an independent collection of complex-valued standard Gaussian random variables on some probability space. Endow

$$\gamma_\infty(H; X) := \{T : H \to X \mid T \text{ is } \gamma\text{-summing}\}$$

with the norm $\|\cdot\|_\gamma$ and let the space $\gamma(H; X)$ of all *γ-radonifying operators* be the closure in $\gamma_\infty(H; X)$ of the finite-rank operators $H \otimes X$.

For a measure space (Ω, μ) let $\gamma(\Omega; X)$ (resp. $\gamma_\infty(\Omega; X)$) be the space of all weakly L^2-functions $f : \Omega \to X$ for which the integration operator $J_f : \mathrm{L}^2(\Omega) \to X$,

$$J_f(g) := \int_\Omega g \cdot f \, \mathrm{d}\mu \quad \big(g \in \mathrm{L}^2(\Omega)\big),$$

is γ-radonifying (γ-summing), and endow it with the norm $\|f\|_\gamma := \|J_f\|_\gamma$.

A collection $\mathcal{T} \subseteq \mathcal{L}(X)$ is *γ-bounded* if there exists a constant $C \geqslant 0$ such that

$$\left(\mathbb{E} \left\| \sum_{T \in \mathcal{T}'} \gamma_T T x_T \right\|^2 \right)^{1/2} \leqslant C \left(\mathbb{E} \left\| \sum_{T \in \mathcal{T}'} \gamma_T x_T \right\|^2 \right)^{1/2}$$

for all finite subsets $\mathcal{T}' \subseteq \mathcal{T}$, sequences $(x_T)_{T\in\mathcal{T}'} \subseteq X$ and independent complex-valued standard Gaussian random variables $(\gamma_T)_{T\in\mathcal{T}'}$. The smallest such C is the *γ-bound* of $\mathcal{T}$ and is denoted by $[\![\mathcal{T}]\!]^\gamma$. Every γ-bounded collection is uniformly bounded with supremum bound less than or equal to the γ-bound, and the converse holds if X is a Hilbert space.

An important result involving γ-boundedness is the *multiplier theorem*. We state a version that is tailored to our purposes. Given a Banach space Y, a function $g : \mathbb{R} \to Y$ is *piecewise* $W^{1,\infty}$ if $g \in W^{1,\infty}(\mathbb{R} \setminus \{a_1, \dots, a_n\}; Y)$ for some finite set $\{a_1, \dots, a_n\} \subseteq \mathbb{R}$.

Theorem 6.1 *(Multiplier Theorem). Let X and Y be Banach spaces and $T : \mathbb{R} \to \mathcal{L}(X, Y)$ a strongly measurable mapping such that*

$$\mathcal{T} := \big\{ T(s) \bigm| s \in \mathbb{R} \big\}$$

is γ-bounded. Suppose furthermore that there exists a dense subset $D \subseteq X$ such that $s \mapsto T(s)x$ is piecewise $W^{1,\infty}$ for all $x \in D$. Then the multiplication operator

$$\mathcal{M}_T : \mathrm{L}^2(\mathbb{R}) \otimes X \to \mathrm{L}^2(\mathbb{R}; Y), \qquad \mathcal{M}_T(f \otimes x) = f(\cdot) T(\cdot) x$$

extends uniquely to a bounded operator

$$\mathcal{M}_T : \gamma\big(\mathrm{L}^2(\mathbb{R}); X\big) \to \gamma\big(\mathrm{L}^2(\mathbb{R}); Y\big)$$

with $\|\mathcal{M}_T\| \leqslant [\![\mathcal{T}]\!]^\gamma$.

Proof. That $\mathcal{M}_T$ extends uniquely to a bounded operator into $\gamma_\infty(\mathrm{L}^2(\mathbb{R}); Y)$ with $\|\mathcal{M}_T\| \leqslant [\![\mathcal{T}]\!]^\gamma$ is the content of Theorem 5.2 in [18]. To see that in fact $\operatorname{ran}(\mathcal{M}_T) \subseteq \gamma(\mathbb{R}; Y)$ we employ a density argument. For $x \in D$ let $a_1, \dots, a_n \in \mathbb{R}$ be such that $s \mapsto T(s)x \in W^{1,\infty}(\mathbb{R} \setminus \{a_1, \dots, a_n\}; Y)$, and set $a_0 := -\infty$, $a_{n+1} := \infty$. Let $f \in \mathrm{C}_c(\mathbb{R})$ be given and note that

$$\int_{a_j}^{a_{j+1}} \|f\|_{\mathrm{L}^2(s,a_{j+1})} \big\|T(s)'x\big\|\,\mathrm{d}s < \infty$$

for all $j \in \{1, \dots, n\}$. Furthermore,

$$\int_{-\infty}^{a_1} \|f\|_{\mathrm{L}^2(-\infty,s)} \big\|T(s)'x\big\|\,\mathrm{d}s < \infty.$$

Corollary 6.3 in [9] yields $(\mathbf{1}_{(a_j,a_{j+1})} f)(\cdot)T(\cdot)x \in \gamma(\mathbb{R}; Y)$ for all $0 \leqslant j \leqslant n$, hence $f(\cdot)T(\cdot)x \in \gamma(\mathbb{R}; Y)$. Since $\mathrm{C}_c(\mathbb{R}) \otimes D$ is dense in $\mathrm{L}^2(\mathbb{R}) \otimes X$, which in turn is dense in $\gamma(\mathrm{L}^2(\mathbb{R}); X)$, the result follows. □

We are now ready to prove a generalization of part (a) of Corollary 3.10. Recall that

$$e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_\omega) = \big\{f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \;\big|\; f(z) = O\big(e^{-\tau \operatorname{Re}(z)}\big) \text{ as } |z| \to \infty\big\}$$

for $\tau > 0$, $\omega \in \mathbb{R}$.

Theorem 6.2. *There exists a universal constant* $c \geqslant 0$ *such that the following holds. Let* $-A$ *generate a* γ*-bounded* C_0*-semigroup* $(T(t))_{t\in\mathbb{R}_+}$ *with* $M := [\![T]\!]^\gamma$ *on a Banach space* X*, and let* $\tau, \omega > 0$*. Then* $f(A) \in \mathcal{L}(X)$ *with*

$$\big\|f(A)\big\| \leqslant \begin{cases} c\,M^2|\log(\omega\tau)|\,\|f\|_\infty & \text{if } \omega\tau \leqslant \frac{1}{2}, \\ 2M^2 e^{-\omega\tau}\|f\|_\infty & \text{if } \omega\tau > \frac{1}{2} \end{cases}$$

for all $f \in e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$.

In particular, A *has a bounded* $e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$*-calculus.*

Proof. We only need to show that the estimate (3.2) in Proposition 3.1 can be refined to

$$\|T_\mu\| \leqslant M^2\eta(\omega,\tau,2)\|L_{e_\omega\mu}\|_{\mathcal{L}(\gamma(\mathbb{R};X))} \tag{6.1}$$

for $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\mu) \subseteq [\tau, \infty)$. Then one uses that

$$\|L_{e_\omega\mu}\|_{\mathcal{L}(\gamma(\mathbb{R};X))} \leqslant \|\widehat{e_\omega\mu}\|_{\mathrm{H}^\infty(\mathbb{C}_+)} = \|\hat{\mu}\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})},$$

by the ideal property of $\gamma(\mathrm{L}^2(\mathbb{R}); X)$ [18, Theorem 6.2], and proceeds as in the proof of Theorem 3.3 to deduce the desired result.

To obtain (6.1) we factorize T_μ as $T_\mu = P \circ L_{e_\omega\mu} \circ \iota$, where $\iota : X \to \gamma(\mathbb{R}; X)$ and $P : \gamma(\mathbb{R}; X) \to X$ are given by

$$\iota x(s) := \psi(-s)T(-s)x \quad (x \in X, s \in \mathbb{R}),$$

$$Pg := \int_0^\infty \varphi(t)T(t)g(t)\,\mathrm{d}t \quad \big(g \in \gamma(\mathbb{R}; X)\big),$$

for $\psi, \varphi \in \mathrm{L}^2(\mathbb{R}_+)$ such that $\psi * \varphi \equiv e_{-\omega}$ on $[\tau, \infty)$. This factorization follows as in Section 2 of [9] once we show that the maps ι and P are well-defined and bounded. To this end, first note that $s \mapsto T(-s)x$ is piecewise $W^{1,\infty}$ for all x in the dense subset $\mathrm{dom}(A) \subseteq X$ and that

$$\psi(-\cdot) \otimes x \in \mathrm{L}^2(-\infty, 0) \otimes X \subseteq \gamma\big(\mathrm{L}^2(\mathbb{R}); X\big).$$

Therefore Theorem 6.1 yields $\iota x \in \gamma(\mathbb{R}, X)$ with

$$\|\iota x\|_\gamma = \|J_{\iota x}\|_\gamma \leqslant M\big\|\psi(-\cdot) \otimes x\big\|_\gamma = M\|\psi\|_{\mathrm{L}^2(\mathbb{R}_+)}\|x\|_X.$$

As for P, write

$$Pg = \int_0^\infty \varphi(t)T(t)g(t)\,\mathrm{d}t = J_{Tg}(\varphi)$$

and use Theorem 6.1 once again to see that $Tg \in \gamma(\mathbb{R}; X)$. Hence

$$\|Pg\|_X \leqslant \|J_{Tg}\|_\gamma \|\varphi\|_{\mathrm{L}^2(\mathbb{R}_+)} \leqslant M\|\varphi\|_{\mathrm{L}^2(\mathbb{R}_+)}\|g\|_\gamma.$$

Finally, estimating the norm of T_μ through this factorization and taking the infimum over all ψ and φ yields (6.1). □

Corollary 6.3. *Corollary 3.10 generalizes to γ-bounded semigroups on arbitrary Banach spaces upon replacing the uniform bound M of T by $[\![T]\!]^\gamma$.*

Theorem 4.3 can be extended in an almost identical manner to a γ-version.

Theorem 6.4. *Let $-A$ generate a γ-bounded C_0-semigroup on a Banach space X. Then A has a strong m-bounded H^∞-calculus of type 0 for all $m \in \mathbb{N}$.*

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Appendix A. Growth estimates

In this appendix we examine the function $\eta : (0,\infty) \times (0,\infty) \times [1,\infty] \to \mathbb{R}_+$ from (3.1):

$$\eta(\alpha,t,q) := \inf\bigl\{\|\psi\|_q\|\varphi\|_{q'} \bigm| \psi * \varphi \equiv e_{-\alpha} \text{ on } [t,\infty)\bigr\}.$$

We will use the notation $f \lesssim g$ for real-valued functions $f, g : Z \to \mathbb{R}$ on some set Z to indicate that there exists a constant $c \geqslant 0$ such that $f(z) \leqslant cg(z)$ for all $z \in Z$.

Lemma A.1. *For each $q \in (1,\infty)$ there exist constants $c_q, d_q \geqslant 0$ such that*

$$d_q\bigl|\log(\alpha t)\bigr| \leqslant \eta(\alpha,t,q) \leqslant c_q\bigl|\log(\alpha t)\bigr| \tag{A.1}$$

if $\alpha t \leqslant \min\{\frac{1}{q}, \frac{1}{q'}\}$. If $\alpha t > \min\{\frac{1}{q}, \frac{1}{q'}\}$ then

$$e^{-\alpha t} \leqslant \eta(\alpha,t,q) \leqslant 2e^{-\alpha t}. \tag{A.2}$$

Proof. First note that $\eta(\alpha,t,q) = \eta(\alpha t,1,q) = \eta(1,\alpha t,q)$ for all α, t and q. Indeed, for $\psi \in \mathrm{L}^q(\mathbb{R}_+)$, $\varphi \in \mathrm{L}^{q'}(\mathbb{R}_+)$ with $\psi * \varphi \equiv e_{-\alpha}$ on $[1,\infty)$ define $\psi_t(s) := \frac{1}{t^{1/q}}\psi(s/t)$ and $\varphi_t(s) := \frac{1}{t^{1/q'}}\varphi(s/t)$ for $s \geqslant 0$. Then

$$\psi_t * \varphi_t(r) = \int_0^\infty \psi\left(\frac{r-s}{t}\right)\varphi\left(\frac{s}{t}\right)\frac{\mathrm{d}s}{t} = \psi * \varphi\left(\frac{r}{t}\right)$$

for all $r \geqslant 0$, so $\psi_t * \varphi_t \equiv e_{-\alpha}$ on $[t,\infty)$. Moreover,

$$\|\psi_t\|_q^q = \int_0^\infty \left|\psi\left(\frac{s}{t}\right)\right|^q \frac{\mathrm{d}s}{t} = \int_0^\infty \bigl|\psi(s)\bigr|^q\,\mathrm{d}s = \|\psi\|_q^q,$$

and similarly $\|\varphi_t\|_{q'} = \|\varphi\|_{q'}$. Hence $\eta(\alpha,t,q) \leqslant \eta(\alpha t,1,q)$. Considering $\psi_{1/t}$ and $\varphi_{1/t}$ yields $\eta(\alpha,t,q) = \eta(\alpha t,1,q)$. The other equality follows immediately. Hence, to prove any of the inequalities in (A.1) or (A.2), we can assume either that $\alpha = 1$ or that $t = 1$ (but not both).

For the left-hand inequalities, we assume that $\alpha = 1$ and we first consider the left-hand inequality of (A.1). Let $t < 1$ and $\psi \in \mathrm{L}^q(\mathbb{R}_+)$, $\varphi \in \mathrm{L}^{q'}(\mathbb{R}_+)$ such that $\psi * \varphi \equiv e_{-1}$ on $[t,\infty)$. Then

$$\begin{aligned}
\left|\log(t)\right| = -\log(t) &= \int_t^1 \frac{\mathrm{d}s}{s} \leqslant e\int_t^1 e^{-s}\frac{\mathrm{d}s}{s} = e\int_t^1 \left|\psi * \varphi(s)\right|\frac{\mathrm{d}s}{s} \\
&\leqslant e\int_t^1\int_0^s \left|\psi(s-r)\right|\cdot\left|\varphi(r)\right|\mathrm{d}r\,\frac{\mathrm{d}s}{s} \\
&\leqslant e\int_0^\infty\int_r^\infty \frac{|\varphi(s-r)|}{s}\,\mathrm{d}s\,\left|\psi(r)\right|\mathrm{d}r \\
&= e\int_0^\infty\int_0^\infty \frac{|\psi(r)||\varphi(s)|}{s+r}\,\mathrm{d}s\,\mathrm{d}r \leqslant \frac{e\pi}{\sin(\pi/q)}\|\psi\|_q\|\varphi\|_{q'},
\end{aligned}$$

where we used Hilbert's absolute inequality [6, Theorem 5.10.1]. It follows that

$$\eta(1,t,q) \geqslant \frac{\sin(\pi/q)}{e\pi}\left|\log(t)\right|.$$

For the left-hand inequality of (A.2), we assume that $\alpha = 1$ and let $t > 0$ be arbitrary. Then

$$e^{-t} = (\psi * \varphi)(t) \leqslant \int_0^t \left|\psi(t-s)\right|\left|\varphi(s)\right|\mathrm{d}s \leqslant \|\psi\|_q\|\varphi\|_{q'}$$

by Hölder's inequality, hence $e^{-t} \leqslant \eta(1,t,q)$.

For the right-hand inequalities in (A.1) and (A.2), we assume that $t = 1$ and first consider the right-hand inequality in (A.1) for $\alpha \leqslant \min\{\frac{1}{q}, \frac{1}{q'}\}$. In the proof of Lemma A.1 in [9] it is shown that

$$(\psi_0 * \varphi_0)(s) = \begin{cases} s, & s \in [0,1), \\ 1, & s \geqslant 1, \end{cases}$$

for

$$\psi_0 := \sum_{j=0}^\infty \beta_j \mathbf{1}_{(j,j+1)} \quad \text{and} \quad \varphi_0 := \sum_{j=0}^\infty \beta'_j \mathbf{1}_{(j,j+1)},$$

where $(\beta_j)_j$ and $(\beta'_j)_j$ are sequences of positive scalars such that $\beta_j = O((1+j)^{-1/q})$ and $\beta'_j = O((1+j)^{-1/q'})$ as $j \to \infty$. Let $\psi := e_{-\alpha}\psi_0$ and $\varphi := e_{-\alpha}\varphi_0$. Then $\psi * \varphi \equiv e_{-\alpha}$ on $[1,\infty)$ and

$$\|\psi\|_q^q = \|e_{-\alpha}\psi_0\|_q^q = \sum_{j=0}^\infty \beta_j^q \int_j^{j+1} e^{-\alpha q s}\,\mathrm{d}s \lesssim \sum_{j=0}^\infty \frac{e^{-\alpha q j}}{1+j}$$

$$\leqslant 1 + \int_0^\infty \frac{e^{-\alpha q s}}{1+s}\,\mathrm{d}s = 1 + e^{\alpha q}\int_{\alpha q}^\infty \frac{e^{-s}}{s}\,\mathrm{d}s.$$

The constant in the first inequality depends only on q. Since $\alpha q \leqslant 1$,

$$\|\psi\|_q^q \lesssim 1 + e^{\alpha q}\left(\int_{\alpha q}^1 \frac{e^{-s}}{s}\,\mathrm{d}s + \int_1^\infty \frac{e^{-s}}{s}\,\mathrm{d}s\right) \leqslant 1 + \int_{\alpha q}^1 \frac{1}{s}\,\mathrm{d}s + e^{\alpha q}\int_1^\infty e^{-s}\,\mathrm{d}s$$
$$= 1 - \log(\alpha q) + e^{\alpha q - 1} \leqslant \log\left(\frac{1}{\alpha}\right) + 2.$$

Moreover, $\frac{1}{\alpha} \geqslant q > 1$ hence $\log(\frac{1}{\alpha}) \geqslant \log(q) > 0$ and

$$\log\left(\frac{1}{\alpha}\right) + 2 \leqslant \left(1 + \frac{2}{\log(q)}\right)\log\left(\frac{1}{\alpha}\right).$$

Therefore

$$\|\psi\|_q \lesssim \log\left(\frac{1}{\alpha}\right)^{1/q} = \left|\log(\alpha)\right|^{1/q},$$

for a constant depending only on q. In a similar manner we deduce

$$\|\varphi\|_{q'} \lesssim \left|\log(\alpha)\right|^{1/q'}$$

for a constant depending only on q' (and thus on q). This yields (A.1).

For the right-hand side of (A.2) we assume that $t = 1$ and, without loss of generality (since $\eta(\alpha, t, q) = \eta(\alpha, t, q')$), that $\alpha > \frac{1}{q}$. Let $\varphi := \mathbf{1}_{[0,1]} e_{\alpha(q-1)}$ and $\psi := \frac{\alpha q}{e^{\alpha q}-1}\mathbf{1}_{\mathbb{R}_+} e_{-\alpha}$. Then

$$\psi * \varphi(r) = \frac{\alpha q}{e^{\alpha q} - 1}\int_0^1 e^{\alpha(q-1)s} e^{-\alpha(r-s)}\,\mathrm{d}s = e^{-\alpha r}$$

for $r \geqslant 1$. Hence

$$\eta(\alpha, 1, q) \leqslant \|\psi\|_q \|\varphi\|_{q'} = \frac{\alpha q}{e^{\alpha q} - 1}\left(\int_0^\infty e^{-\alpha q s}\,\mathrm{d}s\right)^{1/q}\left(\int_0^1 e^{\alpha(q-1)q' s}\,\mathrm{d}s\right)^{1/q'}$$
$$= \frac{(\alpha q)^{(q-1)/q}}{e^{\alpha q} - 1}\left(\int_0^1 e^{\alpha q s}\,\mathrm{d}s\right)^{\frac{q-1}{q}} = \left(e^{\alpha q} - 1\right)^{-1/q} \leqslant 2^{1/q} e^{-\alpha} \leqslant 2e^{-\alpha},$$

where we have used the assumption $\alpha > \frac{1}{q}$ in the penultimate inequality. □

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