

HOLOMORPHIC K - THEORY, ALGEBRAIC CO-CYCLES, AND LOOP GROUPS

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ABSTRACT. In this paper we study the “holomorphic K -theory” of a projective variety. This K - theory is defined in terms of the homotopy type of spaces of holomorphic maps from the variety to Grassmannians and loop groups. This theory has been introduced in various places such as [?], [?], and a related theory was considered in [?]. This theory is built out of studying algebraic bundles over a variety up to “algebraic equivalence”. In this paper we will give calculations of this theory for “flag like varieties” which include projective spaces, Grassmannians, flag manifolds, and more general homogeneous spaces, and also give a complete calculation for symmetric products of projective spaces. Using the algebraic geometric definition of the Chern character studied by the authors in

[?], we will show that there is a rational isomorphism of graded rings between holomorphic K - theory and the appropriate “morphic cohomology” groups, defined in [?] in terms of algebraic co-cycles in the variety. In so doing we describe a geometric model for rational morphic cohomology groups in terms of the homotopy type of the space of algebraic maps from the variety to the “symmetrized loop group” $\Omega U(n)/\Sigma_n$ where the symmetric group Σ_n acts on $U(n)$ via conjugation. This is equivalent to studying algebraic maps to the quotient of the infinite Grassmannians $BU(k)$ by a similar symmetric group action. We then use the Chern character isomorphism to prove a conjecture of Friedlander and Walker stating that if one localizes holomorphic K - theory by inverting the Bott class, then rationally this is isomorphic to topological K - theory. Finally this will allow us to produce explicit obstructions to periodicity in holomorphic K - theory, and show that these obstructions vanish for generalized flag manifolds.

INTRODUCTION

The study of the topology of holomorphic mapping spaces $Hol(X, Y)$, where X and Y are complex manifolds has been of interest to topologists and geometers for many years. In particular when Y is a Grassmannian or a loop group, the space of holomorphic maps

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yields parameter spaces for certain moduli spaces of holomorphic bundles (see [?], [?], [?]). In this paper we study the K -theoretic properties of such holomorphic mapping spaces.

More specifically, let X be any projective variety, let $Gr_n(\mathbb{C}^M)$ denote the Grassmannian of n -planes in \mathbb{C}^M (with its usual structure as a smooth projective variety), and let $\Omega U(n)$ denote the loop group of the unitary group $U(n)$, with its structure as an infinite dimensional smooth algebraic variety ([?]). We let

$$Hol(X; Gr_n(\mathbb{C}^M)) \quad \text{and} \quad Hol(X; \Omega U(n))$$

denote the spaces of algebraic maps between these varieties (topologized as subspaces of the corresponding spaces of continuous maps, with the compact open topologies). We use this notation because if X is smooth these spaces of algebraic maps are precisely the same as holomorphic maps between the underlying complex manifolds. The *holomorphic K -theory* space $K_{hol}(X)$ is defined to be the Quillen - Segal group completion of the union of these mapping spaces, which we write as

$$K_{hol}(X) = Hol(X; \mathbb{Z} \times BU)^+ = Hol(X; \Omega U)^+.$$

This group completion process will be described carefully below. The holomorphic K -groups will be defined to be the homotopy groups

$$K_{hol}^{-q}(X) = \pi_q(K_{hol}(X)).$$

A variant of this construction was first incidentally introduced in [?], and subsequently developed in [?] where one obtains various connective spectra associated to an algebraic variety X , using spaces of algebraic cycles. The case of Grassmannians is the one treated here. A theory related to holomorphic K -theory was also studied by Karoubi in [?] and the construction we use here coincides with the definition of “semi-topological K -theory” studied by Friedlander and Walker in [?]. Indeed their terminology reflects the fact that for a smooth projective variety X holomorphic K -theory sits between algebraic K -theory of the associated scheme and the topological K -theory of its underlying topological space. More precisely, using Morel and Voevodsky’s description algebraic K theory of a smooth variety X (via their work on \mathbb{A}^1 -homotopy theory [?]), Friedlander and Walker showed that there are natural transformations

$$K_{alg}(X) \xrightarrow{\alpha} K_{hol}(X) \xrightarrow{\beta} K_{top}(X)$$

so that the map $\beta : K_{hol}(X) \rightarrow K_{top}(X)$ is the map induced by including the holomorphic mapping space $Hol(X; \mathbb{Z} \times BU)$ in the topological mapping space $Map(X; \mathbb{Z} \times BU)$, and where the composition $\beta \circ \alpha : K_{alg}(X) \rightarrow K_{top}(X)$ is the usual transformation from algebraic

K -theory to topological K -theory induced by forgetting the algebraic structure of a vector bundle.

In this paper we calculate the holomorphic K -theory of a large class of varieties, including “flag - like varieties”, a class that includes Grassmannians, flag manifolds and more general homogeneous spaces. We also give a complete calculation of the holomorphic K -theory of arbitrary symmetric products of projective spaces. Since the algebraic K -theory of such symmetric product spaces is not in general known, these calculations should be of interest in their own right. We then study the Chern character for holomorphic K - theory, using the algebraic geometric description of the Chern character constructed by the authors in [?]. The target of the Chern character transformation is the “morphic cohomology” $L^*H^*(X) \otimes \mathbb{Q}$, defined in terms of algebraic co-cycles in X [?]. We then prove the following.

Theorem 1. *For any projective variety (or appropriate colimit of project varieties) X , the Chern character is a natural transformation*

$$ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

which is an isomorphism for every $q \geq 0$. Furthermore it preserves a natural multiplicative structure, so that it is an isomorphism of graded rings.

In the proof of this theorem we develop techniques which will yield the following interesting descriptions of morphic cohomology that don't involve the use of higher Chow varieties.

Consider the following quotient spaces by appropriate actions of the symmetric groups:

$$\begin{aligned} \Omega U / \Sigma &= \varinjlim_n \Omega U(n) / \Sigma_n, \\ BU / \Sigma &= \varinjlim_{n,m} Gr_m(\mathbb{C}^{nm}) / \Sigma_m \\ SP^\infty(\mathbb{C}\mathbb{P}^\infty) &= \varinjlim_n \left(\prod_n \mathbb{C}\mathbb{P}^\infty \right) / \Sigma_n \end{aligned}$$

Theorem 2. *If X is any projective variety, then the Quillen - Segal group completion of the following spaces of algebraic maps*

$$Mor(X, \Omega U / \Sigma)^+ \quad Mor(X, BU / \Sigma)^+, \quad \text{and} \quad Mor(X, SP^\infty(\mathbb{C}\mathbb{P}^\infty))^+$$

are all rationally homotopy equivalent. Moreover their k^{th} - rational homotopy groups (which we call π_k) are isomorphic to the rational morphic cohomology groups

$$\pi_k \cong \bigoplus_{p=1}^{\infty} L^p H^{2p-k}(X) \otimes \mathbb{Q}.$$

Among other things, the relation between morphic cohomology and the morphism space into the “symmetrized” loop group allows, using loop group machinery, a geometric description of these cohomology groups in terms of a certain moduli space of algebraic bundles with symmetric group action.

We then use Theorem ?? to prove the following result about “Bott periodic holomorphic K - theory”. $K_{hol}^*(X)$ is a module over $K_{hol}^*(point)$ in the usual way, and since $K_{hol}^*(point) = K_{top}^*(point)$, we have a “Bott class” $b \in K_{hol}^{-2}(point)$. The module structure then defines a transformation

$$b_* : K_{hol}^{-q}(X) \rightarrow K_{hol}^{-q-2}(X).$$

If $K_{hol}^*(X)[\frac{1}{b}]$ denotes the localization of $K_{hol}^*(X)$ obtained by inverting this operator, we will then prove the following rational version of a conjecture of Friedlander and Walker [?].

Theorem 3. *The map $\beta : K_{hol}^*(X)[\frac{1}{b}] \otimes \mathbb{Q} \rightarrow K_{top}^*(X) \otimes \mathbb{Q}$ is an isomorphism.*

Finally we describe a necessary conditions for the holomorphic K - theory of a smooth variety to be Bott periodic (i.e $K_{hol}(X) \cong K_{hol}(X)[\frac{1}{b}]$) in terms of the Hodge filtration of its cohomology. We will show that generalized flag varieties satisfy this condition and their holomorphic K - theory is Bott periodic. We also give examples of varieties for which these conditions fail and hence whose holomorphic K - theory is not Bott periodic.

This paper is organized as follows. In section 1 we give the definition of holomorphic K - theory in terms of loop groups and Grassmannians, and prove that the holomorphic K - theory space, $K_{hol}(X)$ is an infinite loop space. In section 2 we give a proof of a result of Friedlander and Walker that $K_{hol}^0(X)$ is the Grothendieck group of the monoid of algebraic bundles over X modulo a notion of “algebraic equivalence”. We prove this theorem here for the sake of completeness, and also because our proof allows us to compute the holomorphic K - theory of flag - like varieties, which we also do in section 2. In section 3 we identify the equivariant homotopy type of the holomorphic K - theory space $K_{hol}(\prod_n \mathbb{P}^1)$, where the group action is induced by the permutation action of the symmetric group Σ_n . This will allow us to compute the holomorphic K - theory of symmetric products of projective spaces, $K_{hol}(SP^m(\mathbb{P}^n))$. In section 4 we recall the Chern character defined in [?] we prove that is an isomorphism of rational graded rings(Theorem ??). In section 5 we prove Theorem ??

giving alternative descriptions of morphic cohomology. Finally in section 6 rational maps in the holomorphic K - theory spaces $K_{hol}(X)$ are studied, and they are used, together with the Chern character isomorphism, to prove Theorem ?? regarding Bott periodic holomorphic K - theory.

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1. THE HOLOMORPHIC K -THEORY SPACE

In this section we define the holomorphic K -theory space $K_{hol}(X)$ for a projective variety X and show that it is an infinite loop space.

For the purposes of this paper we let $\Omega U(n)$ denote the group of based *algebraic* loops in the unitary group $U(n)$. That is, an element of $\Omega U(n)$ is a map $\gamma : S^1 \rightarrow U(n)$ such that $\gamma(1) = 1$ and γ has finite Fourier series expansion. Namely, γ can be written in the form

$$\gamma(z) = \sum_{k=-N}^{k=N} A_k z^k$$

for some N , where the A_k 's are $n \times n$ matrices. It is well known that the inclusion of the group of algebraic loops into the space of all smooth (or continuous) loops is a homotopy equivalence of infinite dimensional complex manifolds [?].

Let X be a projective variety. It was shown by Valli in [?] that the holomorphic mapping space $Hol(X, \Omega U(n))$ has a \mathcal{C}_2 - operad structure in the sense of May [?]. Here \mathcal{C}_2 is the little 2-dimensional cube operad. This in particular implies that the Quillen - Segal group completion, which we denote with the superscript $+$ (after Quillen's $+$ - construction), $Hol(X, \Omega U(n))^+$ has the structure of a two - fold loop space. (Recall that up to homotopy, the Quillen - Segal group completion of a topological monoid A is the loop space of the classifying space, $\Omega B A$.) By taking the limit over n , we define the holomorphic K -theory space to be the group completion of the holomorphic mapping space.

Definition 1.

$$K_{hol}(X) = Hol(X, \Omega U)^+.$$

If $A \subset X$ is a subvariety, we then define the *relative* holomorphic K -theory

$$K_{hol}(X, A)$$

to be the homotopy fiber of the natural restriction map, $K_{hol}(X) \rightarrow K_{hol}(A)$.

Before we go on we point out certain basic properties of $K_{hol}(X)$.

1. By the geometry of loop groups studied in [?] (more specifically the ‘‘Grassmannian model of a loop group’’) one knows that every element of the algebraic loop group $\Omega U(n)$ lies in a finite dimensional Grassmannian. When one takes the limit over n , it was observed in [?] that one has the holomorphic diffeomorphism $\mathbb{Z} \times BU \cong \Omega U$, where here BU is given the complex structure as a limit of Grassmannians, and ΩU denotes the limit of the algebraic loop groups $\Omega U(n)$. Thus we could have replaced ΩU by $\mathbb{Z} \times BU$ in the definition of $K_{hol}(X)$. That is, we have an equivalent definition:

Definition 2.

$$K_{hol}(X) = Hol(X, \mathbb{Z} \times BU)^+.$$

This definition has the conceptual advantage that

$$\pi_0(Hol(X, BU(n))) = \lim_{m \rightarrow \infty} \pi_0(Hol(X, Gr_n(\mathbb{C}^m)))$$

where $Gr_n(\mathbb{C}^m)$ is the Grassmannian of dimension n linear subspaces of \mathbb{C}^m . Moreover this set corresponds to equivalence classes of rank n holomorphic bundles over X that are embedded (holomorphically) in an m - dimensional trivial bundle.

2. It is necessary to take the group completion in our definition of $K_{hol}(X)$. For example, the results of [?] imply that

$$Hol_*(\mathbb{P}^1, \Omega U) \cong \prod_{k=0}^{\infty} BU(k)$$

where Hol_* denotes basepoint preserving holomorphic maps. Thus this holomorphic mapping space is not an infinite loop space without group completing. In fact after we group complete we obtain

$$K_{hol}(\mathbb{P}^1, *) \cong \mathbb{Z} \times BU$$

and so we have the ‘‘periodicity’’ result

$$K_{hol}(\mathbb{P}^1, *) \cong K_{hol}(*).$$

A more general form of ‘‘holomorphic Bott periodicity’’ is contained in D. Rowland’s Ph.D thesis [?] where it is shown that

$$K_{hol}(X \times \mathbb{P}^1, X) \cong K_{hol}(X)$$

for any smooth projective variety X . A more general projective bundle theorem was proved in [?].

We now observe that the two fold loop space mentioned above for holomorphic K -theory can actually be extended to an infinite loop structure.

Proposition 4. *The space $K_{hol}(X) = Hol(X, \mathbb{Z} \times BU)^+$ is an infinite loop space.*

Proof. Let \mathcal{L}_* be the linear isometries operad. That is, \mathcal{L}_m is the space of linear (complex) isometric embeddings of $\oplus_m \mathbb{C}^\infty$ into \mathbb{C}^∞ . These spaces are contractible, and the usual operad action

$$\mathcal{L}_m \times_{\Sigma_m} (Gr_n(\mathbb{C}^\infty))^m \longrightarrow Gr_{nm}(\mathbb{C}^\infty)$$

give holomorphic maps for each $\alpha \in \mathcal{L}_m$. It is then simple to verify that this endows the holomorphic mapping space

$$\Pi_n Hol(X, Gr_n(\mathbb{C}^\infty))$$

with the structure of a \mathcal{L}_* - operad space. Since this is an E_∞ operad in the sense of May [?], this implies that the group completion, $K_{hol}(X) = (\Pi_n Hol(X, Gr_n(\mathbb{C}^\infty)))^+$ has the structure of an infinite loop space. \square

As is usual, we define the (negative) *holomorphic K - groups* to be the homotopy groups of this infinite loop space:

Definition 3. For $q \geq 0$,

$$K_{hol}^{-q}(X) = \pi_q(K_{hol}(X)) = \pi_q(Hol(X, \Omega U)^+).$$

Remarks.

a. Notice that as usual, the holomorphic K -theory is a ring. Namely, the spectrum (in the sense of stable homotopy theory) corresponding to the infinite loop space $K_{hol}(X)$, is in fact a ring spectrum. The ring structure is induced by tensor product operation on Grassmannians,

$$Gr_n(\mathbb{C}^m)^{\otimes k} \rightarrow Gr_{nk}(\mathbb{C}^{mk}).$$

We leave it to the reader to check the details that this structure does indeed induce a ring structure on the holomorphic K -theory. Indeed, this parallels the well-known fact that whenever \mathbf{E} is a ring spectrum and X is an arbitrary space, then $\text{Map}(X, \mathbf{E})$ has a natural

structure of ring spectrum, where $\text{Map}(-, -)$ denotes the space of continuous maps, with the appropriate compact-open, compactly generated topology; cf. [?].

b. A variant of this construction was first incidentally introduced in [?], and subsequently developed in [?] where one obtains various connective spectra associated to an algebraic variety X , using spaces of algebraic cycles. The case of Grassmannians is, up to π_0 considerations, the one treated here. A theory related to holomorphic K - theory was also studied by Karoubi in [?], and the definition given here coincides with the notion of “semi - topological K - theory” introduced and studied by Friedlander and Walker in [?].

2. THE HOLOMORPHIC K - THEORY OF FLAG VARIETIES AND A GENERAL DESCRIPTION OF $K_{hol}^0(X)$.

The main goal of this section is to prove the following theorem which yields an effective calculation of $K_{hol}^0(X)$, when X is a flag variety.

Theorem 5. *Let X be a generalized flag variety. That is, X is a homogeneous space of the form $X = G/P$ where G is a complex algebraic group and $P < G$ is a parabolic subgroup. Then the natural map from holomorphic K - theory to topological K - theory,*

$$\beta : K_{hol}^0(X) \longrightarrow K_{top}^0(X)$$

is an isomorphism

The proof of this theorem involves a comparison of holomorphic K - theory with algebraic K - theory. As a consequence of this comparison we will recover Friedlander and Walker’s description of $K_{hol}^0(X)$ for any smooth projective variety X in terms of “algebraic equivalence classes” of algebraic bundles [?]. We begin by defining this notion of algebraic equivalence.

Definition 4. Let X be a projective variety (not necessarily smooth), and $E_0 \rightarrow X$ and $E_1 \rightarrow X$ algebraic bundles. We say that E_0 and E_1 are *algebraically equivalent* if there exists a constructible, connected algebraic curve T and an algebraic bundle E over $X \times T$, so that the restrictions of E to $X \times \{t_0\}$ and $X \times \{t_1\}$ are E_0 and E_1 respectively, for some $t_0, t_1 \in T$. Here a *constructible* curve means a finite union of irreducible algebraic (not necessarily complete) curves in some projective space.

Notice that two algebraically equivalent bundles are *topologically* isomorphic, but *not* necessarily isomorphic as algebraic bundles.

Theorem 6. *For any smooth projective algebraic variety X , the group $K_{hol}^0(X)$ is isomorphic to the Grothendieck group completion of the monoid of algebraic equivalence classes of algebraic bundles over X .*

The description of $Mor(X, BU(n))$ given in [?] provides our first step in understanding $K_{hol}^0(X)$.

As in [?], if X is a projective variety then we call an algebraic bundle $E \rightarrow X$ *embeddable*, if there exists an algebraic embedding of E into a trivial bundle: $E \hookrightarrow X \times \mathbb{C}^N$ for some large N . Let $\phi : E \hookrightarrow X \times \mathbb{C}^N$ be such an embedding. We identify an embedding ϕ with the composition $\phi : E \hookrightarrow X \times \mathbb{C}^N \hookrightarrow X \times \mathbb{C}^{N+M}$, where \mathbb{C}^N is included in \mathbb{C}^{N+M} as the first N coordinates. We think of such an equivalence class of embeddings as an embedding $E \hookrightarrow X \times \mathbb{C}^\infty$. We refer to the pair (E, ϕ) as an *embedded* algebraic bundle.

Let X be any projective variety, and let E be a rank k holomorphic bundle over X that is holomorphically embeddable in a trivial bundle, define $Hol_E(X, BU(k))$ to be the space of holomorphic maps $\gamma : X \rightarrow BU(k)$ such that $\gamma^*(\xi_k) \cong E$, where $\xi_k \rightarrow BU(k)$ is the universal holomorphic bundle. This is topologized as a subspace of the continuous mapping space, which is endowed with the compact - open topology.

Let $Aut(E)$ be the gauge group of holomorphic bundle automorphisms of E . The following lemma identifies the homotopy type of $Hol_E(X, BU(k))$ in terms of $Aut(E)$.

Lemma 7. *$Hol_E(X, BU(k))$ is naturally homotopy equivalent to the classifying space*

$$Hol_E(X, BU(k)) \simeq B(Aut(E)).$$

Proof. As was described in [?], elements in $Hol_E(X, BU(k))$ are in bijective correspondence to isomorphism classes of rank k embedded holomorphic bundles, (ζ, ϕ) . By modifying the embedding ϕ via an isomorphism between ζ and E , we see that $Hol_E(X, BU(k))$ is homeomorphic to the space of holomorphic embeddings $\psi : E \hookrightarrow X \times \mathbb{C}^\infty$, modulo the action of the holomorphic automorphism group, $Aut(E)$. The space of holomorphic embeddings of E in an infinite dimensional trivial bundle is easily seen to be contractible [?], and the action of $Aut(E)$ is clearly free, with local sections. Again, see [?] for details. The lemma follows. \square

Corollary 8. *The space $Hol_E(X; BU(k))$ is connected.*

Now as above, we say that two embedded algebraic bundles (E_0, ϕ_0) and (E_1, ϕ_1) , are *path equivalent* if there is topologically embedded bundle (E, ϕ) , over $X \times I$, which gives a path equivalence between E_0 and E_1 , and over each slice $X \times \{t\}$ is an embedded algebraic bundle. Finally, notice that the set of (algebraic) isomorphism classes of embedded algebraic bundles forms an abelian monoid.

Lemma 9. *For any projective algebraic variety X , the group $K_{hol}^0(X)$ is isomorphic to the Grothendieck group completion of the monoid of path equivalence classes of embedded algebraic bundles over X .*

Proof. Recall that

$$K_{hol}^0(X) = \pi_0 \left(\coprod_n Mor(X, BU(n)) \right)^+.$$

But the set of path components of the Quillen - Segal group completion of a topological E_∞ space is the Grothendieck group completion of the discrete monoid of path components of the original E_∞ - space. Now as observed above the morphism space $Hol(X, BU(n))$ is given by configurations of isomorphism classes of embedded algebraic bundles, (E, ϕ) . Thus $\pi_0(Hol(X, BU(n)))$ is the set of path equivalence classes of such pairs; i.e the set of path equivalence classes of embedded algebraic bundles of rank n . We may therefore conclude that $K_{hol}^0(X)$ is the Grothendieck group completion of the monoid of path equivalence classes of embeddable algebraic bundles. \square

We now strengthen this result as follows.

Lemma 10. *Two embedded bundles (E_0, ϕ_0) and (E_1, ϕ_1) are path equivalent if and only if they are algebraically equivalent.*

Proof. Let $f : X \times I \rightarrow BU(n)$ be the (continuous) classifying map for the topological bundle E over $X \times I$, which gives the path equivalence between E_0 and E_1 , and denote f_0 and f_1 the restrictions of f to $X \times \{0\}$ and $X \times \{1\}$. Since $X \times I$ is compact, the image of f is contained in some Grassmannian $Gr_n(\mathbb{C}^m) \subset BU(n)$. It follows that f_0 and f_1 lie in the same path component of $Hol(X, Gr_n(\mathbb{C}^m))$. Since $Hol(X, Gr_n(\mathbb{C}^m))$ is a disjoint union of constructible subsets of the Chow monoid $\mathcal{C}_{dim X}(X \times Gr_n(\mathbb{C}^m))$, then f_0 and f_1 lie in

the same connected component of a constructible subset in some projective space. Using the fact that any two points in an irreducible algebraic variety Y lie in some irreducible algebraic curve $C \subset Y$ (see [?, p. 56]), one concludes that any two points in a connected constructible subset of projective space lie in a connected constructible curve. Let T be a connected constructible curve contained in $Hol(X, Gr_n(\mathbb{C}^m))$ and containing f_0 and f_1 . Under the canonical identification $Hol(T, Hol(X, Gr_n(\mathbb{C}^m))) \cong Hol(X \times T, Gr_n(\mathbb{C}^m))$, one identifies the inclusion

$$i : T \hookrightarrow Hol(X, Gr_n(\mathbb{C}^m))$$

with an algebraic morphism $\bar{i} : X \times T \rightarrow Gr_n(\mathbb{C}^m)$. This map classifies the desired embedded bundle E over $X \times T$.

The converse is clear. □

The above two lemmas imply the following.

Proposition 11. *For any projective algebraic variety X , (not necessarily smooth), the group $K_{hol}^0(X)$ is isomorphic to the Grothendieck group completion of the monoid of algebraic equivalence classes of embedded algebraic bundles over X .*

Notice that Theorem ?? implies that we can remove the “embedded” condition in the statement of this proposition. We will show how that can be done later in this section.

Recall from the last section that the forgetful map from the category of colimits of projective varieties to the category of topological spaces, induces a map of morphism spaces,

$$Hol(X; \mathbb{Z} \times BU) \rightarrow Map(X; \mathbb{Z} \times BU)$$

which induces a natural transformation

$$\beta : K_{hol}(X) \rightarrow K_{top}(X).$$

Corollary 12. *Let X be a colimit of projective varieties. Then the induced map $\beta : K_{hol}^0(X) \rightarrow K_{top}^0(X)$ is induced by sending the class of an embedded bundle to its underlying topological isomorphism type:*

$$\begin{aligned} \beta : K_{hol}^0(X) &\rightarrow K_{top}^0(X) \\ [E, \phi] &\rightarrow [E] \end{aligned}$$

In order to approach Theorem 1 we need to understand the relationship between algebraic K - theory, $K_{alg}^0(X)$, and holomorphic K - theory, $K_{hol}^0(X)$ for X a smooth projective variety. For such a variety $K_{alg}^0(X)$ is the Grothendieck group of the exact category of algebraic bundles over X . Roughly speaking the relationship between algebraic and holomorphic K -theories for a smooth variety is the passage from isomorphism classes of holomorphic bundles to algebraic equivalence classes of holomorphic bundles. This relationship was made precise in [?] using the Morel - Voevodsky description of algebraic K - theory of a smooth scheme in terms of an appropriate morphism space. In particular, recall that

$$K_{alg}^0(X) = Mor_{\mathcal{H}((Sm/\mathbb{C})_{Nis})}(X, R\Omega B(\sqcup_{n \geq 0} BGL_n(\mathbb{C}))).$$

where $\mathcal{H}((Sm/\mathbb{C})_{Nis})$ is the homotopy category of smooth schemes over \mathbb{C} , using the Nisnevich topology. See [?] for details. In particular a morphism of projective varieties, $f : X \rightarrow Gr_n(\mathbb{C}^M)$ induces an element in the above morphism space and hence a class $\langle f \rangle \in K_{alg}^0(X)$. It also induces a class $[f] \in \pi_0(Hol(X; \mathbb{Z} \times BU)^+ = K_{hol}^0(X))$. As seen in [?] this correspondence extends to give a forgetful map from the morphisms in the homotopy category $\mathcal{H}((Sm/\mathbb{C})_{Nis})$ to homotopy classes of morphisms in the category of colimits of projective varieties. This defines a natural transformation

$$\alpha : K_{alg}^0(X) \rightarrow K_{hol}^0(X)$$

for X a colimit of smooth projective varieties.

Lemma 13. *For X a smooth projective variety the transformation*

$$\alpha : K_{alg}^0(X) \rightarrow K_{hol}^0(X)$$

is surjective.

Proof. As observed above, the set of path components of the Quillen - Segal group completion of a topological monoid Y is the Grothendieck - group completion of the discrete monoid of path components:

$$\pi_0(Y^+) = (\pi_0(Y))^+.$$

Therefore we have that $K_{hol}^0(X)$ is the Grothendieck group completion of $\pi_0(Hol(X, \mathbb{Z} \times BU))$. Thus every element $\gamma \in K_{hol}^0(X)$ can be written as

$$\gamma = [f] - [g]$$

where f and g are holomorphic maps from X to some Grassmannian. By the above observations $\gamma = \alpha(\langle f \rangle - \langle g \rangle)$. \square

This lemma and Proposition ?? allow us to prove the following interesting splitting property of $K_{hol}^0(X)$ which is not immediate from its definition.

Theorem 14. *Let*

$$0 \rightarrow [F, \phi_F] \rightarrow [E, \phi_E] \rightarrow [G, \phi_G] \rightarrow 0$$

be a short exact sequence of embedded holomorphic bundles over a smooth projective variety X . Then in $K_{hol}^0(X)$ we have the relation

$$[E, \phi_E] = [F, \phi_F] + [G, \phi_G].$$

Proof. This follows from Lemma ?? and the fact that short exact sequences split in $K_{alg}^0(X)$. \square

Lemma ?? will also allow us to prove Theorem ?? which we now proceed to do. We begin with a definition.

Definition 5. We say that a smooth projective variety X is *flag - like* if the following properties hold on its K - theory:

1. the usual forgetful map

$$\psi : K_{alg}^0(X) \rightarrow K_{top}^0(X)$$

is an isomorphism, and

2. $K_{alg}^0(X)$ is generated (as an abelian group) by embeddable holomorphic bundles.

Remark: We call such varieties “flag - like” because generalized flag varieties (homogeneous spaces G/P as in the statement of Theorem ??) satisfy these conditions. We now state a strengthening of Theorem ?? which we prove.

Theorem 15. *Suppose X is a flag - like smooth projective variety. Then the homomorphisms*

$$\alpha : K_{alg}^0(X) \rightarrow K_{hol}^0(X)$$

and

$$\beta : K_{hol}^0(X) \rightarrow K_{top}^0(X)$$

are isomorphisms of rings.

Proof. Let X be a flag - like smooth projective variety. Since every embeddable holomorphic bundle is represented by a holomorphic map $f : X \rightarrow Gr_n(\mathbb{C}^M)$, for some Grassmannian, then property (2) implies that $K_{alg}^0(X)$ is generated by classes $\langle f \rangle$, where f is such a holomorphic map. But then $\psi(\langle f \rangle) \in K_{top}^0(X)$ clearly is the class represented by f in $\pi_0(Map(X, \mathbb{Z} \times BU)) = K_{top}^0(X)$. But this means that the map $\psi : K_{alg}^0(X) \rightarrow K_{top}^0(X)$ is given by the composition

$$\beta \circ \alpha : K_{alg}^0(X) \rightarrow K_{hol}^0(X) \rightarrow K_{top}^0(X).$$

But since ψ is an isomorphism this means α is injective. But we already saw in corollary ?? that α is surjective. Thus α , and therefore β , are isomorphisms. Clearly from their descriptions, α and β preserve tensor products, and hence are ring isomorphisms. \square

We now use this result to prove Theorem ?. Let X be a smooth, projective variety and let $e : X \hookrightarrow \mathbb{C}P^n$ be a projective embedding. We begin by describing a construction which will allow *any* holomorphic bundle E over X to be viewed as representing an element of $K_{hol}^0(X)$ (i.e E does not necessarily have to be embeddable).

So let $E \rightarrow X$ be a holomorphic bundle over X . Recall that by tensoring E with a line bundle of sufficiently negative Chern class, it will become embeddable. (This is dual to the statement that tensoring a holomorphic bundle over a smooth projective variety with a line bundle with sufficiently large Chern class produces holomorphic bundle that is generated by global sections.) So for sufficiently large k , the bundle $E \otimes O(-k)$ is embeddable. Here $O(-k)$ is the k - fold tensor product of the canonical line bundle $O(-1)$ over $\mathbb{C}P^n$, which, by abuse of notation, we identify with its restriction to X . Now choose a holomorphic embedding

$$\phi : E \otimes O(-k) \hookrightarrow X \times \mathbb{C}^N.$$

Then the pair $(E \otimes O(-k), \phi)$ determines an element of $K_{hol}^0(X)$.

Now from Theorem ?? we know that $K_{alg}^0(\mathbb{C}P^n) \cong K_{hol}^0(\mathbb{C}P^n) \cong K_{top}^0(\mathbb{C}P^n)$ as rings. But since $O(-k) \otimes O(k) = 1 \in K_{top}^0(\mathbb{C}P^n)$, this means that if

$$\iota_k = \otimes_k \iota : O(-k) = \otimes_k O(-1) \hookrightarrow \otimes_k \mathbb{C}^{n+1}$$

is the canonical embedding, then the pair $(O(-k), \iota_k)$ represents an invertible class in $K_{hol}^0(\mathbb{C}P^n)$. We denote its inverse by $O(-k)^{-1} \in K_{hol}^0(\mathbb{C}P^n)$, and, as before, we use the same notation to denote its restriction to $K_{hol}^0(X)$.

Write $A(E) = (E \otimes O(-k), \phi) \otimes O(-k)^{-1} \in K_{hol}^0(X)$.

Proposition 16. *The assignment to the holomorphic bundle E the class*

$$A(E) = [E \otimes O(-k), \phi] \otimes O(-k)^{-1} \in K_{hol}^0(X)$$

is well defined, and only depends on the (holomorphic) isomorphism class of E .

Proof. We first verify that given any holomorphic bundle $E \rightarrow X$, that $A(E)$ is a well defined element of $K_{hol}^0(X)$. That is, we need to show that this class is independent of the choices made in its definition. More specifically, we need to show that

$$[E \otimes O(-k), \phi] \otimes O(-k)^{-1} = [E \otimes O(-q), \psi] \otimes O(-q)^{-1}$$

for any appropriate choices of k, q, ϕ , and ψ . We do this in two steps.

Case 1: $k = q$. In this case it suffices to show that $(E \otimes O(-k), \phi)$ and $(E \otimes O(-k), \psi)$ lie in the same path component of the morphism space $Hol(X, BU(n))$, where n is the rank of E . Now using the notation of Lemma ?? we see that these two elements both lie in $Hol_{E \otimes O(-k)}(X, BU(n))$, which, as proved in Corollary ?? there is path connected.

Case 2: General Case: Suppose without loss of generality that $q > k$. Then clearly the classes $(E \otimes O(-k), \phi) \otimes O(-k)^{-1}$ and $(E \otimes O(-k) \otimes O(-(q-k)), \phi \otimes \iota_{q-k}) \otimes O(-k)^{-1} \otimes O(-(q-k))^{-1}$ represent the same element of $K_{hol}^0(X)$. But this latter class is $(E \otimes O(-q), \phi \otimes \iota_{q-k}) \otimes O(-q)^{-1}$ which we know by case 1 represents the same K - theory class as $(E \otimes O(-q), \psi) \otimes O(-q)^{-1}$.

Thus $A(E)$ is a well defined class in $K_{hol}^0(X)$. Clearly the above arguments also verify that $A(E)$ only depends on the isomorphism type of E . \square

Notice that this argument implies that K_{hol}^0 encodes *all* holomorphic bundles (not just embeddable ones). We will use this to complete the proof of theorem ??.

Proof. Let X be a smooth projective variety and let \mathcal{H}_X denote the Grothendieck group of monoid of algebraic equivalence classes of holomorphic bundles over X . We show that the correspondence A described in the above theorem induces an isomorphism

$$A : \mathcal{H}_X \xrightarrow{\cong} K_{hol}^0(X).$$

We first show that A is well defined. That is, we need to know if E_0 and E_1 are algebraically equivalent, then $A(E_0) = A(E_1)$. So let $E \rightarrow X \times T$ be an algebraic equivalence. Since T is a curve in projective space, we can find a projective embedding of the product, $e : X \times T \hookrightarrow \mathbb{C}P^n$. Now for sufficiently large k , $E \otimes O(-k)$ is embeddable, and given an embedding ϕ_E ,

the pair $(E \otimes O(-k), \phi_E)$ defines an algebraic equivalence between the embedded bundles $(E_0 \otimes O(-k), \phi_0)$ and $(E_1 \otimes O(-k), \phi_1)$, where the ϕ_i are the appropriate restrictions of the embedding ϕ . Thus

$$[E_0 \otimes O(-k), \phi_0] = [E_1 \otimes O(-k), \phi_1] \in K_{hol}^0(X).$$

Thus

$$[E_0 \otimes O(-k), \phi_0] \otimes O(-k)^{-1} = [E_1 \otimes O(-k), \phi_1] \otimes O(-k)^{-1} \in K_{hol}^0(X).$$

But these classes are $A(E_0)$ and $A(E_1)$. Thus $A : \mathcal{H}_X \rightarrow K_{hol}^0(X)$ is well defined.

Notice also that A is surjective. This is because, as was seen in the proof of the last theorem, if (E, ϕ) is an embedded holomorphic bundle, then $A(E) = [E, \phi] \in K_{hol}^0(X)$. The essential point here being that the choice of the embedding ϕ does not affect the holomorphic K -theory class, since the space of such choices is connected.

Finally notice that A is injective. This follows from two facts:

1. The classes $[O(-k)^{-1}]$ are units in the ring structure of $K_{hol}^0(X)$, and
2. If bundles of the form $E_0 \otimes O(-k)$ and $E_1 \otimes O(-k)$ are algebraically equivalent then the bundles E_0 and E_1 are algebraically equivalent.

□

3. THE EQUIVARIANT HOMOTOPY TYPE OF $K_{hol}(\prod_n \mathbb{P}^1)$ AND THE HOLOMORPHIC K -THEORY OF SYMMETRIC PRODUCTS OF PROJECTIVE SPACES

The goal of this section is to completely identify the holomorphic K -theory of symmetric products of projective spaces, $K_{hol}(SP^n(\mathbb{P}^m))$. Since the algebraic K -theory of these spaces is not in general known, this will give us new information about algebraic bundles over these symmetric product spaces. These spaces are particularly important in this paper since, as we will point out below, symmetric products of projective spaces are representing spaces for morphic cohomology.

Our approach to this question is to study the equivariant homotopy type of $K_{hol}(\prod_n \mathbb{P}^1)$, where the symmetric group Σ_n acts on the holomorphic K -theory space $K_{hol}(\prod_n \mathbb{P}^1) = Hol(\prod_n \mathbb{P}^1; \mathbb{Z} \times BU)^+$ by permuting the coordinates of $\prod_n \mathbb{P}^1$. It acts on the topological K -theory space $K_{top}(\prod_n \mathbb{P}^1) = Map(\prod_n \mathbb{P}^1; \mathbb{Z} \times BU)$ in the same way. The main result of this section is the following.

Theorem 17. *The natural map $\beta : K_{hol}(\prod_n \mathbb{P}^1) \rightarrow K_{top}(\prod_n \mathbb{P}^1)$ is a Σ_n - equivariant homotopy equivalence.*

Before we begin the proof of this theorem we observe the following consequences:

Corollary 18. *Let $G < \Sigma_n$ be a subgroup. Then the induced map on the K - theories of the orbit spaces,*

$$\alpha : K_{hol}(\prod_n \mathbb{P}^1/G) \rightarrow K_{top}(\prod_n \mathbb{P}^1/G)$$

is a homotopy equivalence.

Proof. By Theorem ??, $\alpha : K_{hol}(\prod_n \mathbb{P}^1) \rightarrow K_{top}(\prod_n \mathbb{P}^1)$ is a Σ_n - equivariant homotopy equivalence. Therefore it induces a homotopy equivalence on the fixed point sets,

$$\alpha : K_{hol}(\prod_n \mathbb{P}^1)^G \xrightarrow{\cong} K_{top}(\prod_n \mathbb{P}^1)^G.$$

But these fixed point sets are $K_{hol}(\prod_n \mathbb{P}^1/G)$ and $K_{top}(\prod_n \mathbb{P}^1/G)$ respectively. \square

Corollary 19. *$\alpha : K_{hol}(\mathbb{C}\mathbb{P}^n) \rightarrow K_{top}(\mathbb{C}\mathbb{P}^n)$ is a homotopy equivalence.*

Proof. Let $G = \Sigma_n$ in the above corollary. $\prod_n (\mathbb{P}^1)/\Sigma_n = SP^n(\mathbb{P}^1) \cong \mathbb{C}\mathbb{P}^n$. \square

The following example will be important because as seen earlier, symmetric products of projective spaces form representing spaces for morphic cohomology.

Corollary 20. *Let r and k be any positive integers. Then*

$$\alpha : K_{hol}(SP^r(\mathbb{C}\mathbb{P}^k)) \rightarrow K_{top}(SP^r(\mathbb{C}\mathbb{P}^k))$$

is a homotopy equivalence.

Proof. Let G be the wreath product $G = \Sigma_r \wr \Sigma_k$ viewed as a subgroup of the symmetric group Σ_{rk} . The obtain an identification of orbit spaces

$$\left(\prod_{rk} \mathbb{P}^1 \right) / \left(\Sigma_r \wr \Sigma_k \right) = SP^r(SP^k(\mathbb{P}^1)) \cong SP^r(\mathbb{C}\mathbb{P}^k).$$

Finally, apply the above corollary when $n = rk$. \square

Observe that this corollary gives a complete calculation of the holomorphic K - theory of symmetric products of projective spaces, since their topological K - theory is known.

In order to begin the proof of Theorem ?? we need to expand our notion of holomorphic K - theory to include unions of varieties. So let A and B be subvarieties of $\mathbb{C}\mathbb{P}^n$, then define $Hol(A \cup B, \mathbb{Z} \times BU)$ to be the space of those continuous maps on $A \cup B$ that are holomorphic when restricted to A and B . This space still has the action of the little isometry operad and so we can take a group completion and define $K_{hol}(A \cup B) = Hol(A \cup B, \mathbb{Z} \times BU)^+$. If $A \cup B$ is connected, then we can define the reduced holomorphic K - theory as before, $\tilde{K}_{hol}(A \cup B) =$ the homotopy fiber of the restriction map $K_{hol}(A \cup B) \rightarrow K_{hol}(x_0)$, where $x_0 \in A \cap B$. With this we can now define the holomorphic K - theory of a smash product of varieties.

Definition 6. Let X and Y be connected projective varieties with basepoints x_0 and y_0 respectively. We define $\tilde{K}_{hol}(X \wedge Y)$ to be the homotopy fiber of the restriction map

$$\rho : \tilde{K}_{hol}(X \times Y) \rightarrow \tilde{K}_{hol}(X \vee Y)$$

where $X \vee Y = \{(x, y_0)\} \cup \{(x_0, y)\} \subset X \times Y$.

Recall that in topological K -theory, the Bott periodicity theorem can be viewed as saying the Bott map $\beta : \tilde{K}_{top}(X) \rightarrow \tilde{K}_{top}(X \wedge S^2)$ is a homotopy equivalence for any space X . In [?] Rowland studies the holomorphic analogue of this result. She studies the Bott map $\beta : \tilde{K}_{hol}(X) \rightarrow \tilde{K}_{hol}(X \wedge \mathbb{P}^1)$ and, using the index of a family of $\bar{\partial}$ operators, defines a map $\bar{\partial} : \tilde{K}_{hol}(X \wedge \mathbb{P}^1) \rightarrow \tilde{K}_{hol}(X)$. Using a refinement of Atiyah's proof of Bott periodicity [?], she proves the following.

Theorem 21. *Given any smooth projective variety X , the Bott map*

$$\beta : \tilde{K}_{hol}(X) \rightarrow \tilde{K}_{hol}(X \wedge \mathbb{P}^1)$$

is a homotopy equivalence of infinite loop spaces. Moreover its homotopy inverse is given by the map

$$\bar{\partial} : \tilde{K}_{hol}(X \wedge \mathbb{P}^1) \rightarrow \tilde{K}_{hol}(X).$$

The fact that the Bott map β is an isomorphism also follows from the “projective bundle theorem” of Friedlander and Walker [?] which was proven independently, using different

techniques. This result in the case when $X = S^0$ was proved in [?]. The statement in this case is

$$\tilde{K}_{hol}(\mathbb{P}^1) \simeq \tilde{K}_{hol}(S^0) = \mathbb{Z} \times BU = \tilde{K}_{top}(S^0) \simeq \tilde{K}_{top}(S^2).$$

Combining this with Theorem ?? (iterated several times) we get the following:

Corollary 22. *For a positive integer k , let $\bigwedge_k \mathbb{P}^1 = (\mathbb{P}^1)^{(k)}$ be the k -fold smash product of \mathbb{P}^1 . Then we have homotopy equivalences*

$$\tilde{K}_{hol}((\mathbb{P}^1)^{(k)}) \simeq \mathbb{Z} \times BU \simeq \tilde{K}_{top}(S^{2k}),$$

We will use this result to prove Theorem ?. We actually will prove a splitting result for $K_{hol}(\prod_n \mathbb{P}^1)$ which we now state.

Let \mathcal{S}_n denote the category whose objects are (unordered) subsets of $\{1, \dots, n\}$. Morphisms are inclusions. Notice that the cardinality of the set of objects,

$$|Ob(\mathcal{S}_n)| = 2^n.$$

Notice also that the set of objects $Ob(\mathcal{S}_n)$ has an action of the symmetric group Σ_n induced by the permutation action of Σ_n on $\{1, \dots, n\}$.

Let X be a space with a basepoint $x_0 \in X$. For $\theta \in Ob(\mathcal{S}_n)$, define

$$\prod_{\theta} X = X^{\theta} \subset X^n$$

by $X^{\theta} = \{(x_1, \dots, x_n) \text{ such that if } j \text{ is not an element of } \theta, \text{ then } x_j = x_0 \in X\}$. Notice that if θ is a subset of $\{1, \dots, n\}$ of cardinality k , then $X^{\theta} \cong X^k$. The smash product $\bigwedge_{\theta} X = X^{(\theta)}$ is defined similarly. The following is the splitting theorem that will allow us to prove Theorem ?.

Theorem 23. *Let X be a smooth projective variety (or a union of smooth projective varieties). Then there is a natural Σ_n -equivariant homotopy equivalence*

$$J : \tilde{K}_{hol}(X^n) \longrightarrow \prod_{\theta \in Ob(\mathcal{S}_n)} \tilde{K}_{hol}(X^{(\theta)}).$$

where the action of Σ_n on the right hand side is induced by the permutation action of Σ_n on the objects $Ob(\mathcal{S}_n)$.

Proof. In order to prove this theorem we begin by recalling the equivariant stable splitting theorem of a product proved in [?]. An alternate proof of this can be found in [?].

Given a space X with a basepoint $x_0 \in X$, let $\Sigma^\infty(X)$ denote the suspension spectrum of X . We refer the reader to [?] for a discussion of the appropriate category of equivariant spectra.

Theorem 24. *There is a natural Σ_n equivariant homotopy equivalence of suspension spectra*

$$J : \Sigma^\infty(X^n) \xrightarrow{\cong} \Sigma^\infty(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)}(X^{(\theta)})).$$

As a corollary of this splitting theorem we get the following splitting of topological K -theory spaces.

Corollary 25. *There is a Σ_n -equivariant homotopy equivalence of topological K -theory spaces,*

$$J^* : \prod_{\theta \in \text{Ob}\mathcal{S}_n} \tilde{K}_{top}(X^{(\theta)}) \rightarrow \tilde{K}_{top}(X^n).$$

Proof. Given to spectra E and F , let $sMap(E, F)$ be the spectrum consisting of spectrum maps from E to F . We again refer the reader to [?] for a discussion of the appropriate category of spectra. If Ω^∞ is the zero space functor from spectra to infinite loop spaces, then $\Omega^\infty(sMap(E, F)) = Map_\infty(\Omega^\infty(E), \Omega^\infty(F))$, where Map_∞ refers to the space of infinite loop maps.

Let bu denote the connective topological K -theory spectrum, whose zero space is $\mathbb{Z} \times BU$. Now Theorem ?? yields a Σ_n equivariant homotopy equivalence of the mapping spectra,

$$J^* : sMap(\Sigma^\infty(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)}(X^{(\theta)}), bu) \xrightarrow{\cong} sMap(\Sigma^\infty(X^n), bu),$$

and therefore of infinite loop mapping spaces,

$$J^* : Map_\infty(\Omega^\infty \Sigma^\infty(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)}(X^{(\theta)}), \mathbb{Z} \times BU) \xrightarrow{\cong} Map_\infty(\Omega^\infty \Sigma^\infty(X^n), \mathbb{Z} \times BU).$$

But since $\Omega^\infty \Sigma^\infty(Y)$ is, in an appropriate sense, the free infinite loop space generated by a space Y , then given any other infinite loop space W , the space of infinite loop maps, $Map_\infty(\Omega^\infty \Sigma^\infty(Y), W)$ is equal to the space of (ordinary) maps $Map(Y, W)$. Thus we have a Σ_n -equivariant homotopy equivalence of mapping spaces,

$$J^* : Map((\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)}(X^{(\theta)}), \mathbb{Z} \times BU) \xrightarrow{\cong} Map(X^n, \mathbb{Z} \times BU).$$

□

Notice that Theorem ?? is the holomorphic version of Corollary ?? . In order to prove this result, we need to develop a holomorphic version of the arguments used in proving Theorem ?. For this we consider the notion of ‘‘holomorphic stable homotopy equivalence’’, as follows. Suppose that X is a smooth projective variety (or union of varieties) and E is a spectrum whose zero space is a smooth projective variety (or a union of such), define $sHol(\Sigma^\infty(X), E)$ to be the subspace of $Map_\infty(\Omega^\infty \Sigma^\infty(X), \Omega^\infty(E))$ consisting of those infinite loop maps $\phi : \Omega^\infty \Sigma^\infty(X) \rightarrow \Omega^\infty(E)$ so that the composition

$$X \hookrightarrow \Omega^\infty \Sigma^\infty(X) \xrightarrow{\phi} \Omega^\infty(E)$$

is holomorphic. Notice, for example, that $sHol(\Sigma^\infty(X); bu) = Hol(X, \mathbb{Z} \times BU)$.

Now suppose X and Y are both smooth projective varieties, (or unions of such).

Definition 7. A map of suspension spectra, $\psi : \Sigma^\infty(X) \rightarrow \Sigma^\infty(Y)$ is called a *holomorphic stable homotopy equivalence*, if the following two conditions are satisfied.

1. ψ is a homotopy equivalence of spectra.
2. If E is any spectrum whose zero space is a smooth projective variety (or a union of such), then the induced map on mapping spectra, $\psi^* : sMap(\Sigma^\infty(Y), E) \rightarrow sMap(\Sigma^\infty(X), E)$ restricts to a map

$$\psi^* s : Hol(\Sigma^\infty(Y), E) \rightarrow sHol(\Sigma^\infty(X), E)$$

which is a homotopy equivalence.

With this notion we can complete the proof of Theorem ?. This requires a proof of Theorem ? that will respect holomorphic stable homotopy equivalences. The version of this theorem given in [?] will do this. We now recall that proof and refer to [?] for details.

Let X be a connected space with basepoint $x_0 \in X$. Let X_+ denote X with a disjoint basepoint, and let $X \vee S^0$ denote the wedge of X with the two point space S^0 . Topologically X_+ and $X \vee S^0$ are the same spaces, but their basepoints are in different connected components. However their suspension spectra $\Sigma^\infty(X_+)$ and $\Sigma^\infty(X \vee S^0)$ are stably homotopy equivalent spectra with units (i.e via a stable homotopy equivalence $j : \Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$ that respects the obvious unit maps $\Sigma^\infty(S^0) \rightarrow \Sigma^\infty(X_+)$ and $\Sigma^\infty(S^0) \rightarrow \Sigma^\infty(X \vee S^0)$.) Moreover it is clear that if X is a smooth projective variety then $\Sigma^\infty(X_+)$ and $\Sigma^\infty(X \vee S^0)$ are holomorphically stably homotopy equivalent in the above sense. Now by taking smash products n -times of this equivalence, we get a Σ_n -equivariant holomorphic stable homotopy equivalence,

$$J_n : \Sigma^\infty((X_+)^{(n)}) = (\Sigma^\infty((X_+))^{(n)}) \xrightarrow{j^{(n)}} (\Sigma^\infty(X \vee S^0))^{(n)} = \Sigma^\infty((X \vee S^0)^{(n)}).$$

Now notice that the n - fold smash product $(X_+)^{(n)}$ is naturally (and Σ_n equivariantly) homeomorphic to the cartesian product $(X^n)_+$. Notice also that the n fold iterated smash product of $X \vee S^0$ is Σ_n - equivariantly homeomorphic to the wedge of the smash products,

$$(X \vee S^0)^{(n)} = \left(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)} X^{(\theta)} \right) \vee S^0.$$

Thus J_n gives a Σ_n - equivariant stable homotopy equivalence,

$$J_n : \Sigma^\infty(((X^n)_+)) \xrightarrow{\simeq} \Sigma^\infty\left(\left(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)} X^{(\theta)}\right) \vee S^0\right).$$

which gives a proof of Theorem ???. Moreover when X is a smooth projective variety (or a union of such) this equivariant stable homotopy equivalence is a holomorphic one. In particular, given any such X , this implies there is a Σ_n equivariant homotopy equivalence

$$J_n^* : sHol_*(\Sigma^\infty((X^n)_+); bu) \xrightarrow{\simeq} sHol_*(\Sigma^\infty\left(\left(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)} X^{(\theta)}\right) \vee S^0\right); bu).$$

where $sHol_*$ refers to those maps of spectra that preserve the units. If we remove the units from each of these mapping spectra we conclude that we have a Σ_n equivariant homotopy equivalence

$$J_n^* : sHol(\Sigma^\infty(X^n); bu) \xrightarrow{\simeq} sHol(\Sigma^\infty\left(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)} X^{(\theta)}\right); bu).$$

But these spaces are precisely $Hol(X^n, \mathbb{Z} \times BU)$ and $Hol(\bigvee_{\theta \in \text{Ob}(\mathcal{S}_n)} X^{(\theta)}; \mathbb{Z} \times BU) = \prod_{\theta \in \text{Ob}(\mathcal{S}_n)} Hol(X^{(\theta)}; \mathbb{Z} \times BU)$ respectively. Theorem ??? now follows. \square

We are now in a position to prove Theorem ???.

Proof. By theorems ?? and ?? we have the following homotopy commutative diagram:

$$\begin{array}{ccc} \tilde{K}_{hol}((\mathbb{P}^1)^n) & \xrightarrow[\simeq]{J_n^*} & \prod_{\theta \in \text{Ob}(\mathcal{S}_n)} \tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)}) \\ \beta \downarrow & & \downarrow \beta \\ \tilde{K}_{top}((\mathbb{P}^1)^n) & \xrightarrow[\simeq]{J_n^*} & \prod_{\theta \in \text{Ob}(\mathcal{S}_n)} \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)}). \end{array}$$

Notice that all the maps in this diagram are Σ_n equivariant, and by the results of theorems ?? and ?? the horizontal maps are Σ_n -equivariant homotopy equivalences. Furthermore, by Corollary ?? the maps $\tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)}) \rightarrow \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)})$ are homotopy equivalences. Now since the Σ_n action on $\prod_{\theta \in \text{Ob}(\mathcal{S}_n)} \tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)})$ and on $\prod_{\theta \in \text{Ob}(\mathcal{S}_n)} \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)})$ is given by permuting the factors according to the action of Σ_n on $\text{Ob}(\mathcal{S}_n)$, this implies that the right

hand vertical map in this diagram, $\beta : \prod_{\theta \in \text{Ob}(\mathcal{S}_n)} \tilde{K}_{hol}((\mathbb{P}^1)^{(\theta)}) \rightarrow \prod_{\theta \in \text{Ob}(\mathcal{S}_n)} \tilde{K}_{top}((\mathbb{P}^1)^{(\theta)})$ is a Σ_n -equivariant homotopy equivalence. Hence the left hand vertical map

$$\beta : \tilde{K}_{hol}((\mathbb{P}^1)^n) \rightarrow \tilde{K}_{top}((\mathbb{P}^1)^n)$$

is also a Σ_n - equivariant homotopy equivalence. This is the statement of Theorem ???. \square

4. THE CHERN CHARACTER FOR HOLOMORPHIC K - THEORY

In this section we study the Chern character for holomorphic K - theory that was defined by the authors in [?]. The values of this Chern character are in the rational Friedlander - Lawson “morphic cohomology groups”, $L^*H^*(X) \otimes \mathbb{Q}$. Our goal is to show that the Chern character is gives an isomorphism

$$ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \cong \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}.$$

Recall the following basic results about the Chern character proved in [?].

Theorem 26. *There is a natural transformation of functors from the category of colimits of projective varieties to algebras over the rational numbers,*

$$ch : K_{hol}^{-*}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-*}(X) \otimes \mathbb{Q}$$

that satisfies the following properties.

1. *The Chern character is compatible with the Chern character for topological K - theory. That is, the following diagram commutes:*

$$\begin{array}{ccc} K_{hol}^{-q}(X) \otimes \mathbb{Q} & \xrightarrow{\beta_*} & K_{top}^{-q}(X) \otimes \mathbb{Q} \\ ch \downarrow & & \downarrow ch \\ \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} & \xrightarrow{\phi_*} & \bigoplus_{k=0}^{\infty} H^{2k-q}(X; \mathbb{Q}) \end{array}$$

where ϕ_* is the natural transformation from morphic cohomology to singular cohomology as defined in [?]

2. *Let $ch_k : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow L^k H^{2k-q}(X) \otimes \mathbb{Q}$ be the projection of ch onto the k^{th} factor. Also let $c_k : K_{hol}^{-q}(X) \rightarrow L^k H^{2k-q}(X)$ be the k^{th} Chern class defined in [?, §6] (see [?, §4] for details). Then there is a polynomial relation between natural transformations*

$$c_k = k! ch_k + p(ch_1, \dots, ch_{k-1})$$

where $p(ch_1, \dots, ch_{k-1})$ is some polynomial in the first $k - 1$ Chern characters.

As mentioned above the goal of this section is to prove the following theorem regarding the Chern character.

Theorem 27. *For every $q \geq 0$, the Chern character for holomorphic K - theory*

$$ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism.

Proof. Recall from [?] that the suspension theorem in morphic cohomology implies that morphic cohomology can be represented by morphisms into spaces of zero cycles in projective spaces. Since zero cycles are given by points in symmetric products this can be interpreted in the following way. Let $SP^\infty(\mathbb{P}^\infty)$ be the infinite symmetric product of the infinite projective space. Given a projective variety X , let $Mor(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$ denote the colimit of the algebraic morphism spaces $Mor(X; SP^n(\mathbb{P}^m))$.

Lemma 28. *Let X be a colimit of projective varieties. Then*

$$\pi_q(Mor(X; (\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))^+) \cong \bigoplus_{k \geq 0} L^k H^{2k-q}(X).$$

Similarly, $\mathbb{Z} \times BU$ represents holomorphic K - theory in the sense that

$$(4.1) \quad \pi_q(Mor(X; \mathbb{Z} \times BU)^+) \cong K_{hol}^{-q}(X).$$

Thus to prove Theorem ?? we will describe a relationship between the representing spaces $\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$ and $\mathbb{Z} \times BU$.

Using the identification in Lemma ??, let

$$\iota \in \bigoplus_{k=1}^{\infty} L^k H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$$

correspond to the class in $\pi_0((Mor(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))^+)$ represented by the identity map $id : \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty) \rightarrow \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$.

Lemma 29. *There exists a unique class $\tau \in K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ with Chern character $ch(\tau) = \iota \in \bigoplus_{k=0}^{\infty} L^k H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$.*

Proof. By Corollary ?? in section 3, we know that for every k and n , $K_{hol}(SP^k(\mathbb{P}^n)) \rightarrow K_{top}(SP^k(\mathbb{P}^n))$ is a homotopy equivalence. It follows that by taking limits we have that $K_{hol}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \rightarrow K_{top}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$ is a homotopy equivalence. But we also know from [?] that the natural map

$$\phi : \bigoplus_{k=0}^{\infty} L^k H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q} \rightarrow \bigoplus_{k=0}^{\infty} H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Q})$$

is an isomorphism. This is true because for the following reasons.

1. Since products $\prod_n(\mathbb{P}^1)$ have ‘‘algebraic cell decompositions’’ in the sense of [?], its morphic cohomology and singular cohomology coincide,

$$\phi : L^k H^p(\prod_n(\mathbb{P}^1)) \xrightarrow{\cong} H^p(\prod_n \mathbb{P}^1).$$

2. Since both morphic cohomology and singular cohomology admit transfer maps ([?]) there is a natural identification of $L^k H^p(SP^r(\mathbb{P}^m)) \otimes \mathbb{Q}$ and $H^p(SP^r(\mathbb{P}^m); \mathbb{Q})$ with the $\Sigma_r \int \Sigma_m$ invariants in $L^k H^p(\prod_{rm} \mathbb{P}^1) \otimes \mathbb{Q}$ and $H^p(\prod_{rm} \mathbb{P}^1; \mathbb{Q})$ respectively. Since the natural transformation $\phi : L^k H^p(\prod_{rm} \mathbb{P}^1) \rightarrow H^p(\prod_{rm} \mathbb{P}^1)$ is equivariant, then we get an induced isomorphism on the invariants,

$$\phi : L^k H^p(SP^r(\mathbb{P}^m)) \otimes \mathbb{Q} \xrightarrow{\cong} H^p(SP^r(\mathbb{P}^m); \mathbb{Q}).$$

3. By taking limits over r and m we conclude that

$$\phi : L^k H^p(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q} \longrightarrow H^p(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Q})$$

is an isomorphism.

Using this isomorphism and the compatibility of the Chern character maps in holomorphic and topological K - theories, to prove this theorem it is sufficient to prove that there exists a unique class $\tau \in K_{top}^0((\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q})$ with (topological) Chern character

$$ch(\tau) = \iota \in [\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)] \otimes \mathbb{Q} \cong \bigoplus_{k=0}^{\infty} H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Q})$$

where $\iota \in [\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)]$ is the class represented by the identity map. But this follows because the Chern character in topological K - theory, $ch : \tilde{K}_{top}^0(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k=1}^{\infty} H^{2k}(X; \mathbb{Q})$ is an isomorphism. \square

We now show how the element $\tau \in K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ defined in the above lemma will yield an inverse to the Chern character transformation.

Theorem 30. *The element $\tau \in K_{top}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ defines natural transformations*

$$\tau_* : \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q} \longrightarrow K_{hol}^{-q}(X) \otimes \mathbb{Q}$$

such that the composition

$$ch \circ \tau_* : \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q} \rightarrow K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k \geq 0} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is equal to the identity.

Proof. The set of path components of the Quillen - Segal group completion of a topological monoid is the Grothendieck group completion of the discrete monoid of path components. If we use the notation $\hat{\mathcal{M}}$ to mean the Grothendieck group of a discrete monoid \mathcal{M} , this says that

$$K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) = \pi_0(Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)^+) \cong (\pi_0(Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)))^\wedge,$$

and hence

$$K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q} \cong (\pi_0(Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)_{\mathbb{Q}}))^\wedge,$$

where the subscript \mathbb{Q} denotes the holomorphic mapping space localized at the rationals. This means that τ can be represented as a difference of classes,

$$\tau = [\tau_1] - [\tau_2]$$

where $\tau_i \in Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)_{\mathbb{Q}}$.

Now consider the composition pairing

$$Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \times Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU) \rightarrow Hol(X; \mathbb{Z} \times BU) \rightarrow Hol(X; \mathbb{Z} \times BU)^+.$$

which localizes to a pairing

$$Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \times Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times BU)_{\mathbb{Q}} \rightarrow Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}} \rightarrow Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+.$$

Using this pairing, τ_1 and τ_2 each define transformations

$$\tau_i : Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \rightarrow Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+.$$

Using the fact that $Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+$ is an infinite loop space, then the subtraction map is well defined up to homotopy,

$$\tau_1 - \tau_2 : Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \rightarrow Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+.$$

We need the following intermediate result about this construction.

Lemma 31. *For any projective variety (or colimit of varieties) X , the map*

$$\tau_1 - \tau_2 : Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \rightarrow Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+$$

is a map of H - spaces.

Proof. Since the construction of these maps was done at the representing space level, it is sufficient to verify the claim in the case when X is a point. That is, we need to verify that the compositions

$$(4.2) \quad \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_{\mathbb{Q}} \times \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_{\mathbb{Q}} \xrightarrow{(\tau_1 - \tau_2) \times (\tau_1 - \tau_2)} (\mathbb{Z} \times BU)_{\mathbb{Q}} \times (\mathbb{Z} \times BU)_{\mathbb{Q}} \xrightarrow{\mu} (\mathbb{Z} \times BU)_{\mathbb{Q}}$$

and

$$(4.3) \quad \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_{\mathbb{Q}} \times \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_{\mathbb{Q}} \xrightarrow{\nu} \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)_{\mathbb{Q}} \xrightarrow{(\tau_1 - \tau_2)} (\mathbb{Z} \times BU)_{\mathbb{Q}}$$

represent the same elements of $K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty) \times \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$, where μ and ν are the monoid multiplications in $\mathbb{Z} \times BU$ and $\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$ respectively. But by Corollary ?? of the last section, this is the same as $K_{top}^0((\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \times (\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))) \otimes \mathbb{Q}$. Now in the topological category, we know that the class $\tau \in K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$ is the inverse to the Chern character and hence induces a rational equivalence of H - spaces

$$\tau : (\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \xrightarrow{\simeq} (\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

This implies that the compositions ?? and ?? represent the same elements of $K_{top}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$, and hence the same elements in $K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)) \otimes \mathbb{Q}$. \square

Thus the map

$$\tau_1 - \tau_2 : Hol(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}} \rightarrow Hol(X, \mathbb{Z} \times BU)_{\mathbb{Q}}^+$$

is an H - map from a \mathcal{C}_∞ operad spaces (as described in §1), to an infinite loop space. But any such rational H - map extends in a unique manner up to homotopy, to a map of H - spaces of their group completions

$$\tau_1 - \tau_2 : Hol(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+ \rightarrow Hol(X, \mathbb{Z} \times BU)_{\mathbb{Q}}^+.$$

This map is natural in the category of colimits of projective varieties X . Since any H - map between rational infinite loop spaces is homotopic to an infinite loop map, this then defines a natural transformation of rational infinite loop spaces,

$$(4.4) \quad \tau = \tau_1 - \tau_2 : Hol(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+ \rightarrow Hol(X, \mathbb{Z} \times BU)_{\mathbb{Q}}^+.$$

So when we apply homotopy groups τ defines natural transformations

$$(4.5) \quad \tau_* : \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \longrightarrow K_{hol}^{-q}(X) \otimes \mathbb{Q}.$$

Now notice that if we let $X = \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty)$ in (??), and $\iota \in Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty), \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+$ be the class represented by the identity map, then by definition, one has that

$$\tau(\iota) \in Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty), \mathbb{Z} \times BU)_{\mathbb{Q}}^+$$

represents the class $[\tau] \in K_{hol}^0(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$ described in Lemma ??. Moreover this lemma tells us that $ch([\tau]) = [\iota] \in \bigoplus_{k \geq 0}^{\infty} L^k H^{2k}(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))$. Now as in section 4, we view the Chern character as represented by an element $ch \in Hol(\mathbb{Z} \times BU; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+$ which is a map of rational infinite loop spaces, then this lemma tells us that the elements

$$ch \circ \tau(\iota) \in Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+$$

and

$$\iota \in Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+$$

are both maps of rational infinite loop spaces and lie in the same path component of $Hol(\mathbb{Z} \times SP^\infty(\mathbb{P}^\infty); \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+$. But this implies the $ch \circ \tau$ and ι define the homotopic natural transformations of rational infinite loop spaces,

$$ch \circ \tau \simeq \iota : Hol(X, \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+ \rightarrow Hol(X; \mathbb{Z} \times SP^\infty(\mathbb{P}^\infty))_{\mathbb{Q}}^+.$$

When we apply homotopy groups this means that

$$ch \circ \tau = id : \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

which was the claim in the statement of Theorem ??. □

We now can complete the proof of Theorem ??. That is we need to prove that

$$ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism. By Theorem ?? we know that ch is surjective. In order to show that it is injective, we prove the following:

Lemma 32. *The composition of natural transformations*

$$\tau_* \circ ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \rightarrow K_{hol}^{-q}(X) \otimes \mathbb{Q}$$

is the identity.

Proof. These transformations are induced on the representing level by maps of rational infinite loop spaces,

$$ch : (\mathbb{Z} \times BU)_{\mathbb{Q}} \rightarrow (\mathbb{Z} \times SP^{\infty}(\mathbb{P}^{\infty}))_{\mathbb{Q}}$$

and

$$\tau : (\mathbb{Z} \times SP^{\infty}(\mathbb{P}^{\infty}))_{\mathbb{Q}} \rightarrow (\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

The composition

$$\tau \circ ch : (\mathbb{Z} \times BU)_{\mathbb{Q}} \rightarrow (\mathbb{Z} \times BU)_{\mathbb{Q}}$$

represents an element of rational holomorphic K - theory,

$$[\tau \circ ch] \in K_{hol}^0(\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

Now the fact that τ is an inverse of the Chern character in topological K - theory tells us that

$$[\tau \circ ch] = j \in K_{top}^0(\mathbb{Z} \times BU)_{\mathbb{Q}},$$

where $j \in K_{top}^0(\mathbb{Z} \times BU) = \pi_0(\text{Map}(\mathbb{Z} \times BU; \mathbb{Z} \times BU))$ is the class represented by the identity map. But according to the results in §2, we know

$$K_{hol}^0(\mathbb{Z} \times BU)_{\mathbb{Q}} \cong K_{top}^0(\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

So by the compatibility of the Chern characters in holomorphic and topological K - theories, we conclude that

$$[\tau \circ ch] = j \in K_{hol}^0(\mathbb{Z} \times BU)_{\mathbb{Q}}.$$

This implies that $\tau \circ ch : (\mathbb{Z} \times BU)_{\mathbb{Q}} \rightarrow (\mathbb{Z} \times BU)_{\mathbb{Q}}$ and the identity map $id : (\mathbb{Z} \times BU)_{\mathbb{Q}} \rightarrow (\mathbb{Z} \times BU)_{\mathbb{Q}}$ induce the same natural transformations $Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+ \rightarrow Hol(X; \mathbb{Z} \times BU)_{\mathbb{Q}}^+$. Applying homotopy groups implies that

$$\tau_* \circ ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} \rightarrow K_{hol}^{-q}(X) \otimes \mathbb{Q}$$

is the identity as claimed. \square

This lemma implies that $ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k \geq 0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$ is injective. As remarked above this was the last remaining fact to be verified in the proof of Theorem ??.

□

We end this section with a proof that the total Chern class also gives a rational isomorphism in every dimension. Namely, recall the Chern classes

$$c_k : K_{hol}^{-q}(X) \rightarrow L^k H^{2k-q}(X)$$

defined originally in [?]. Taking the direct sum of these maps gives us the *total Chern class* map,

$$c : K_{hol}^{-q}(X) \rightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X).$$

We will prove the following result, which was conjectured by Friedlander and Walker in [?].

Theorem 33. *The total Chern class*

$$c : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is an isomorphism for all $q \geq 0$.

We note that in the case $q = 0$, this theorem was proved in [?]. The proof in general will follow quickly from our Theorem ?? stating that the total Chern character is a rational isomorphism.

Proof. We first prove that the total Chern class

$$c : K_{hol}^{-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$$

is injective. So suppose that for some $\alpha \in K_{hol}^{-q}(X) \otimes \mathbb{Q}$, we have that $c(\alpha) = 0$. So each Chern class $c_q(\alpha) = 0$ for $q \geq 0$. Now recall from section 4 that in the algebra of operations between $K_{hol}^{-q}(X) \otimes \mathbb{Q}$ and $\bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$, that the Chern classes and Chern character are related by a formula of the form

$$(4.6) \quad c_k = k! \, ch_k + p(ch_1, \dots, ch_{k-1})$$

where $p(ch_1, \dots, ch_{k-1})$ is some polynomial in the first $k - 1$ Chern classes. So since each $c_q(\alpha) = 0$ then an inductive argument using (??) implies that each $ch_q(\alpha) = 0$. Thus the total Chern character $ch(\alpha) = 0$. But since the total Chern character is an isomorphism

(Theorem ??), this implies that $\alpha = 0 \in K_{hol}^{-q}(X) \otimes \mathbb{Q}$. This proves that the total Chern class operation is injective.

We now prove that $c : K_{hol}^{-q}(X) \otimes \mathbb{Q} \longrightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$ is surjective. To do this we will prove that for every k and element $\gamma \in L^k H^{2k-q}(X) \otimes \mathbb{Q}$ there is a class $\alpha_k \in K_{hol}^{-q}(X)$ with $c_k(\alpha_k) = \gamma$ and $c_q(\alpha_k) = 0$ for $q \neq k$. We prove this by induction on k . So assume this statement is true for $k \leq m - 1$, and we now prove it for $k = m$. Let $\gamma_m \in L^m H^{2m-q}(X) \otimes \mathbb{Q}$. Since the total Chern character is an isomorphism, there is an element $\alpha_m \in K_{hol}^{-q}(X) \otimes \mathbb{Q}$ with $ch_m(\alpha_m) = \gamma_m$, and $ch_q(\alpha_m) = 0$ for $q \neq m$. But formula (??) implies that $c_q(\alpha_m) = 0$ for $q < m$, and $c_m(\alpha_m) = \frac{1}{m!}\gamma_m$. Thus the total Chern class has value $c(m!\alpha_m) = \gamma_m$. This proves that the total Chern class is surjective, and therefore that it is an isomorphism. \square

5. STABILITY OF RATIONAL MAPS AND BOTT PERIODIC HOLOMORPHIC K - THEORY

In this section we study the space of rational maps in the morphism spaces used to define holomorphic K - theory. We will show that the “stability property” for rational maps in the morphism space $Hol(X, \mathbb{Z} \times BU)$ amounts to the question of whether Bott periodicity holds in $K_{hol}^*(X)$. We then use the Chern character isomorphism proved in the last section to prove a conjecture of Friedlander and Walker [?] that rationally, Bott periodic holomorphic K - theory is isomorphic to topological K - theory. (Friedlander and Walker actually conjectured that this statement is true integrally.) Given a projective variety Y

with basepoint $y_0 \in Y$, let $Hol_{y_0}(\mathbb{P}^1, Y)$ denote the space of holomorphic (algebraic) maps $f : \mathbb{P}^1 \rightarrow Y$ satisfying the basepoint condition $f(\infty) = y_0$. We refer to this space as the space of based rational maps in Y . In [?] the “group completion” of this space of rational maps $Hol_{y_0}(\mathbb{P}^1, Y)^+$ was defined. This notion of group completion had the property that if $Hol_{y_0}(\mathbb{P}^1, Y)$ has the structure of a topological monoid, then $Hol_{y_0}(\mathbb{P}^1, Y)^+$ is the Quillen - Segal group completion. In general $Hol_{y_0}(\mathbb{P}^1, Y)^+$ was defined to be a space of limits of “chains” of rational maps, topologized using Morse theoretic considerations. We refer the reader to [?] for details. We recall also from that paper the following definition.

Definition 8. The space of rational maps in a projective variety Y is said to *stabilize*, if the group completion of the space of rational maps is homotopy equivalent to the space of continuous maps,

$$Hol_{y_0}(\mathbb{P}^1, Y)^+ \simeq \Omega^2 Y.$$

In [?] criteria for when the rational maps in a projective variety (or symplectic manifold) stabilize were discussed and analyzed. In this paper we study the implications in holomorphic K -theory of the stability of rational maps in the varieties $Hol(X, Gr_n(\mathbb{C}^M))$, where X is a smooth projective variety, and $Gr_n(\mathbb{C}^M)$ is the Grassmannian of n -dimensional subspaces of \mathbb{C}^M . (The fact that the space of morphisms from one projective variety to another is in turn algebraic is well known. See, for example [?, ?] for discussions about the algebraic structure of morphisms between varieties.) We actually study rational maps in $Hol(X; \mathbb{Z} \times BU)$, which is a colimit of projective varieties. In fact we will study rational maps in the group completion $Hol(X; \mathbb{Z} \times BU)^+$ by which we mean the group completion of the relative morphism space,

$$Hol_*(\mathbb{P}^1; Hol(X; \mathbb{Z} \times BU)^+) = Hol(\mathbb{P}^1 \times X, \infty \times X; \mathbb{Z} \times BU)^+.$$

Theorem 34. *Let X be a smooth projective variety. Then the space of rational maps in the group completed morphism space $Hol(X; \mathbb{Z} \times BU)^+$ stabilizes if and only if the holomorphic K -theory space $K_{hol}(X)$ satisfies Bott periodicity:*

$$K_{hol}(X) \simeq \Omega^2 K_{hol}(X).$$

Proof. The space of rational maps in the morphism space $Hol(X; \mathbb{Z} \times BU)^+$ stabilizes if and only if the group completion of its space of rational maps is the two fold loop space,

$$(5.1) \quad Hol_*(\mathbb{P}^1; Hol(X; \mathbb{Z} \times BU)^+) \simeq \Omega^2(Hol(X; \mathbb{Z} \times BU)^+).$$

But by definition, the left hand side is equal to $Hol(\mathbb{P}^1 \times X, \infty \times X; \mathbb{Z} \times BU)^+ = K_{hol}(\mathbb{P}^1 \times X; \infty \times X)$. But by Rowland's theorem [?] or by the more general "projective bundle theorem" proved in [?] we know that the Bott map

$$\beta : K_{hol}(X) \rightarrow K_{hol}(\mathbb{P}^1 \times X; \infty \times X)$$

is a homotopy equivalence. Combining this with property ??, we have that the space of rational maps in the morphism space $Hol(X; \mathbb{Z} \times BU)^+$ stabilizes if and only if the following composition is a homotopy equivalence

$$(5.2) \quad \begin{array}{ccc} B : K_{hol}(X) & \xrightarrow[\simeq]{\beta} & K_{hol}(\mathbb{P}^1 \times X; \infty \times X) \\ = Hol_*(\mathbb{P}^1; Hol(X; \mathbb{Z} \times BU)^+) & \longrightarrow & \Omega^2 Hol(X; \mathbb{Z} \times BU)^+ = \Omega^2 K_{hol}(X). \end{array}$$

□

By applying homotopy groups, the Bott map (??) $B : K_{hol}(X) \rightarrow \Omega^2 K_{hol}(X)$ defines a homomorphism

$$B_* : K_{hol}^{-q}(X) \rightarrow K_{hol}^{-q-2}(X)$$

Let $b \in K_{hol}^{-2}(point)$ be the image under B_* of the unit $1 \in K_{hol}^0(point)$. Clearly this class lifts the Bott class in topological K - theory, $b \in K_{top}^{-2}(point)$. Observe further that $B_* : K_{hol}^{-q}(X) \rightarrow K_{hol}^{-q-2}(X)$ is given by multiplication by the Bott class $b \in K_{hol}^{-2}(point)$, using the module structure of $K_{hol}^*(X)$ over the ring $K_{hol}^*(point)$. The homomorphism $B_* : K_{hol}^{-q}(X) \rightarrow K_{hol}^{-q-2}(X)$ was studied in [?] and it was conjectured there that if $K_{hol}^*(X)[\frac{1}{b}]$ denotes the localization of $K_{hol}^*(X)$ obtained by inverting the Bott class, then one obtains topological K - theory. We now prove the following rational version of this conjecture.

Theorem 35. *Let X be a smooth projective variety. Then the map from holomorphic K - theory to topological K - theory $\beta : K_{hol}(X) \rightarrow K_{top}(X)$ induces an isomorphism*

$$\beta_* : K_{hol}^*(X)[1/b] \otimes \mathbb{Q} \xrightarrow{\cong} K_{top}^*(X) \otimes \mathbb{Q}.$$

Proof. Consider the Chern character defined on the $K_{hol}^{-2}(point) \otimes \mathbb{Q}$

$$ch : K_{hol}^{-2}(point) \otimes \mathbb{Q} \rightarrow \bigoplus_k L^k H^{2k-2}(point) \otimes \mathbb{Q}.$$

Now the morphic cohomology of a point is equal to the usual cohomology of a point, $L^k H^{2k-2}(point) = H^{2k-2}(point)$, so this group is non zero if and only if $k = 1$. So the Chern character gives an isomorphism

$$ch : K_{hol}^{-2}(point) \otimes \mathbb{Q} \xrightarrow{\cong} L^1 H^0(point) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Let $s \in L^1 H^0(point) \otimes \mathbb{Q}$ be the Chern character of the Bott class, $s = ch(b)$. Since the Chern character is an isomorphism, $s \in L^1 H^0(point) \otimes \mathbb{Q} \cong \mathbb{Q}$ is a generator. We use this notation for the following reason.

Recall the operation in morphic cohomology $S : L^k H^q(X) \rightarrow L^{k+1} H^q(X)$ defined in [?]. Using the fact that $L^* H^*(X)$ is a module over $L^* H^*(point)$ (using the “join” multiplication in morphic cohomology), then this operation is given by multiplication by a generator of $L^1 H^0(point) = \mathbb{Z}$. Therefore up to a rational multiple, this operation on rational morphic cohomology, $S : L^k H^q(X) \otimes \mathbb{Q} \rightarrow L^{k+1} H^q(X) \otimes \mathbb{Q}$, is given by multiplication by the element $s = ch(b) \in L^1 H^0(point) \otimes \mathbb{Q}$.

In [?] it was shown that the natural map from morphic cohomology to singular cohomology $\phi : L^k H^q(X) \rightarrow H^q(X)$ makes the following diagram commute:

$$(5.3) \quad \begin{array}{ccc} L^k H^q(X) & \xrightarrow{S} & L^{k+1} H^q(X) \\ \phi \downarrow & & \downarrow \phi \\ H^q(X) & \xrightarrow{=} & H^q(X). \end{array}$$

It also follows from the ‘‘Poincare duality theorem’’ proved in [?] that if X is an n -dimensional smooth variety, then $L^s H^q(X) = H^q(X)$ for $s \geq n$. Furthermore for $k < n$ $\phi : L^k H^q(X) \rightarrow H^q(X)$ factors as the composition

$$(5.4) \quad \phi : L^k H^q(X) \xrightarrow{S} L^{k+1} H^q(X) \xrightarrow{S} \dots \xrightarrow{S} L^n H^q(X) = H^q(X)$$

Let $L^* H^q(X)[1/S]$ denote the localization of $L^* H^q(X)$ obtained by inverting the transformation $S : L^* H^q(X) \rightarrow L^{*+1} H^q(X)$. Specifically

$$L^* H^q(X)[1/S] = \varinjlim \{ L^* H^q(X) \xrightarrow{S} L^{*+1} H^q(X) \xrightarrow{S} \dots \}$$

Then (??) and (??) imply we have an isomorphism with singular cohomology,

$$(5.5) \quad \phi : L^* H^q(X)[1/S] \xrightarrow{\cong} H^q(X).$$

Again, since rationally the S operation is, up to multiplication by a nonzero rational number, given by multiplication by $s \in L^1 H^0(\text{point}) \otimes \mathbb{Q}$, we can all conclude that when rational morphic cohomology is localized by inverting s , we have an isomorphism with singular rational cohomology,

$$(5.6) \quad \phi : L^* H^q(X; \mathbb{Q})[1/s] \xrightarrow{\cong} H^q(X; \mathbb{Q}).$$

Now since the Chern character isomorphism $ch : K_{hol}^{-q}(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q}$ is an isomorphism of rings, then the following diagram commutes:

$$(5.7) \quad \begin{array}{ccc} K_{hol}^{-q}(X) \otimes \mathbb{Q} & \xrightarrow{\cdot b} & K_{hol}^{-q-2}(X) \otimes \mathbb{Q} \\ ch \downarrow \cong & & \cong \downarrow ch \\ \bigoplus_{k=0}^{\infty} L^k H^{2k-q}(X) \otimes \mathbb{Q} & \xrightarrow[\cdot s]{} & \bigoplus_{k=0}^{\infty} L^{k+1} H^{2k-q}(X) \otimes \mathbb{Q}. \end{array}$$

where the top horizontal map is multiplication by the Bott class $b \in K_{hol}^{-2}(\text{point})$, and the bottom horizontal map is multiplication by $s = ch(b) \in L^1 H^0(\text{point}) \otimes \mathbb{Q}$.

Moreover since the Chern character in holomorphic K - theory and and that for topological K - theory are compatible, this means we get a commutative diagram:

$$\begin{array}{ccc} K_{hol}^{-q}(X)[1/b] \otimes \mathbb{Q} & \xrightarrow{\beta} & K_{top}^{-q}(X) \otimes \mathbb{Q} \\ ch \downarrow & & \downarrow ch \\ \bigoplus_{k=0}^{\infty} L^* H^{2k-q}(X; \mathbb{Q})[1/s] & \xrightarrow{\phi} & \bigoplus_{k=0}^{\infty} H^{2k-q}(X; \mathbb{Q}). \end{array}$$

By (??) we know that the bottom horizontal map is an isomorphism. Moreover by Theorem ?? the left hand vertical map is an isomorphism. Of course the right hand vertical map is also a rational isomorphism. Hence the top horizontal map is a rational isomorphism,

$$\beta_* : K_{hol}^{-q}(X)[1/b] \otimes \mathbb{Q} \xrightarrow{\cong} K_{top}^{-q}(X) \otimes \mathbb{Q}.$$

□

In most of the calculations of $K_{hol}(X)$ done so far we have seen examples of when $K_{hol}(X) \cong K_{top}(X)$. In particular in these examples the holomorphic K - theory is periodic, $K_{hol}^*(X) \cong K_{hol}^*(X)[\frac{1}{b}]$. As we have seen from Theorem ??, these two conditions are rationally equivalent. We end by using the above results to give a necessary condition for the holomorphic K - theory to be Bott periodic, and use it to describe examples where periodicity fails, and therefore provide examples that have distinct holomorphic and topological K - theories.

Theorem 36. *Let X be a smooth projective variety. Then if $K_{hol}^*(X) \otimes \mathbb{Q} \cong K_{hol}^*(X)[\frac{1}{b}]$ (or equivalently $K_{hol}^*(X) \otimes \mathbb{Q} \cong K_{top}^*(X) \otimes \mathbb{Q}$), then in the Hodge filtration of its cohomology we have*

$$H^{k,k}(X; \mathbb{C}) \cong H^{2k}(X; \mathbb{C})$$

for every $k \geq 0$.

Proof. Consider the commutative diagram involving the total Chern character

$$(5.8) \quad \begin{array}{ccc} K_{hol}^0(X) \otimes \mathbb{C} & \xrightarrow{\beta_*} & K_{top}^0(X) \otimes \mathbb{C} \\ ch \downarrow & & \downarrow ch \\ \bigoplus_{k \geq 0} L^k H^{2k}(X) \otimes \mathbb{C} & \xrightarrow{\phi} & \bigoplus_{k \geq 0} H^{2k}(X; \mathbb{C}) \end{array}$$

By Theorem ??, if $K_{hol}^*(X)$ is Bott periodic, then the top horizontal map $\beta : K_{hol}^0(X) \otimes \mathbb{C} \rightarrow K_{top}^0(X) \otimes \mathbb{C}$ is an isomorphism. But by theorem ?? we know that the two vertical

maps in this diagram are isomorphisms. Thus if $K_{hol}^*(X)$ is Bott periodic, then the bottom horizontal map in this diagram is an isomorphism. That is,

$$\phi : L^k H^{2k}(X) \otimes \mathbb{C} \longrightarrow H^{2k}(X; \mathbb{C})$$

is an isomorphism, for every $k \geq 0$. But as is shown in [?], $L^k H^{2k}(X) \cong \mathcal{A}_k(X)$, where $\mathcal{A}_k(X)$ is the space of algebraic k -cycles in X up to algebraic (or homological) equivalence. Moreover the image of $\phi : L^k H^{2k}(X) \otimes \mathbb{C} \rightarrow H^{2k}(X; \mathbb{C})$ is the image of the natural map induced by including algebraic cycles in all cycles, $\mathcal{A}_k \otimes \mathbb{C} \rightarrow H^{2k}(X; \mathbb{C})$, which lies in Hodge filtration $H^{k,k}(X; \mathbb{C}) \subset H^{2k}(X; \mathbb{C})$. Thus $\phi : L^k H^{2k}(X) \otimes \mathbb{C} \rightarrow H^{2k}(X; \mathbb{C})$ is an isomorphism implies that the composition

$$\mathcal{A}_k(X) \otimes \mathbb{C} \rightarrow H^{k,k}(X; \mathbb{C}) \subset H^{2k}(X; \mathbb{C})$$

is an isomorphism. In particular this means that $H^{k,k}(X; \mathbb{C}) = H^{2k}(X; \mathbb{C})$. □

We end by noting that for a flag manifold X , we know by Theorem ?? that $K_{hol}^0(X) \cong K_{top}^0(X)$, and indeed $H^{p,p}(X; \mathbb{C}) \cong H^{2p}(X; \mathbb{C})$. However in general this theorem tells us that if we have a variety X having nonzero $H^{p,q}(X; \mathbb{C})$ for some $p \neq q$, then $K_{hol}^*(X)$ is *not* Bott periodic, and in particular is distinct from topological K -theory. Certainly abelian varieties of dimension ≥ 2 are examples of such varieties.

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