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Global inversion theorems via coercive functionals on metric spaces

Olivia Gutú*

Centro de Investigación en Matemáticas, Universidad Autónoma del Estado de Hidalgo, 42184 Pachuca, Mexico

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Abstract

Let X and Y be metric spaces. We give sufficient metric conditions for a local homeomorphism $f : X \to Y$ to be a global one. We achieve this by means of auxiliary coercive functionals; several expected global inversion theorems are obtained by choosing different functionals.

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Let X be a metric space. We shall say that a functional $k : X \to \mathbb{R}$ is *coercive* if for some $x_0 \in X$, $|k(x)| \to \infty$ when $d(x, x_0) \to \infty$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map such that $f'(w)$ is invertible at every $w \in \mathbb{R}^n$. It is known that f is a diffeomorphism onto $\mathbb{R}^n$ if and only if there exists a coercive C^1 map $k : \mathbb{R}^n \to \mathbb{R}$ such that $\|k'(w) \circ f'(w)^{-1}\|$ is bounded, for $w \in \mathbb{R}^n$ (see e.g. [16]).

Using ODE techniques, Zampieri [16] proved that the second implication is also true for a class of local homeomorphisms between infinite dimensional Banach spaces. Zampieri derived classical global inversion theorems, such as the Hadamard global inversion theorem [8,14], by choosing a special coercive functional k. In this paper we present an analogue of this result for local homeomorphisms $f : X \to Y$, with X a complete metric space and Y a strong contractible metric space (Definition 1.1).

In the first section, we give a topological lemma (Lemma 1.6) inspired in some ideas included in a remarkable paper by Fritz John [10]. In the second section, we consider the upper and lower

* Tel.: +52 771 7 17 21 58; fax: +52 771 7 17 20 91.
E-mail address: olivia@uaeh.edu.mx.

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scalar derivatives of the local homeomorphism f at x, denoted respectively by $D_x^+ f$ and $D_x^- f$ (for f a smooth map between Banach spaces those are $\|f'(x)\|$ and $\|f'(x)^{-1}\|^{-1}$). We use scalar derivatives to give metric conditions in order to get a technical result (Lemma 2.4) which will be needed to obtain our main result.

In the last section we present our main result. Under some technical assumptions, we prove that if there exist both $\alpha > 0$ and a coercive functional $k : X \to \mathbb{R}$ such that

$$\limsup_{v \to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} \leq \alpha, \quad \forall w \in X,$$

then f is a global homeomorphism.

With a natural choice of different coercive functionals, we obtain global inversion theorems in the metric space context (Corollaries 3.6–3.8), in the line of previous work by John [10], Katriel [11], Ioffe [9], Gutú and Jaramillo [7]. On the other hand, we obtain an analogue of Zampieri's theorem for Riemannian manifolds (Corollary 3.4).

1. Strong contractibility and maximal inverse

Definition 1.1. Let (Y, d) be a metric space. We shall say that Y is *strongly contractible* if there exists $y_0 \in Y$ and a homotopy $H : Y \times [0, 1] \to Y$ such that:

(1) $H(y, 0) = y_0$ and $H(y, 1) = y$ for all $y \in Y$,
(2) $H(y_0, t) = y_0$ for all $t \in [0, 1]$, and
(3) for every y fixed, the path $H(y, \cdot)$ has finite length.

Example 1.2. Every *convex subset of a normed vector space* Y is strongly contractible. Given $y_0 \in Y$, the natural homotopy is given by the lines joining every point y with y_0.

Example 1.3. Let Y be a *finite or infinite dimensional, geodesically complete and simply connected Riemannian (or more generally, Finslerian) manifold with seminegative curvature*. By the Cartan–Hadamard theorem, for every $y_0 \in Y$ the exponential map $\exp_{y_0} : T_{y_0}Y \to Y$ is a diffeomorphism. See [12, IX, Section 3, Theorem 3.8] and [13, Theorem 1.10]. Then, the homotopy $(y, t) \to \exp_{y_0}(ty)$ defines a strong contraction for Y at y_0.

The assumption of seminegative curvature is not necessary for a manifold to be strongly contractible, as the next example shows:

Example 1.4. Every *infinite dimensional sphere* is strongly contractible: Let E be a Banach space, let $S_E = \{y \in E : \|y\| = 1\}$ and $B_E = \{y \in E : \|y\| \leq 1\}$. It was proved in [3] that there exists a Lipschitz map $r : B_E \to S_E$ such that $r(y) = y$, for every $y \in S_E$. Let $y_0 \in S_E$; then the map $H(y, t) = r(ty + (1 - t)y_0)$ defines a strong contraction for S_E at y_0. Note that the length of the path $H(y, \cdot)$ is finite because H is Lipschitz. On the other hand, the homotopy H can be taken to be of class C^p whenever S_E is C^p smooth; see [2].

Example 1.5. A path $p : [0, 1] \to Y$ in a metric space Y is said to be a *geodesic* if there exists $\alpha > 0$ such that $d(p(t), p(t')) = \alpha \cdot |t - t'|$, for all $t, t' \in [0, 1]$. A metric space in which the distance between two points is given by the infimum of the lengths of the curves that join them is called a *length space*. If Y is a complete simply connected length space with curvature ≤ 0 (in the sense of Alexandrov [1]) then:

(1) The space Y is uniquely geodesic (i.e. every two points can be joined by a unique geodesic).

(2) If p_1 and p_2 are two geodesics in Y beginning at the same point, then for all $t \in [0, 1]$, $d(p_1(t), p_2(t)) \leq t\, d(p_1(1), p_2(1))$.

See [4, II.1.2, II.1.3, II.4]. Hence, the map $(y, t) \to H(y, t)$ defined by the unique geodesic joining y to y_0 at time t is continuous, by property (2). These kinds of spaces include the CAT(0) spaces and a large class of polyhedral complexes; see [4, Chapter II.5].

If p is a path, we shall denote its image by $\operatorname{Im} p$. Let $f : X \to Y$ be a local homeomorphism between metric spaces. Suppose that Y is strongly contractible at $y_0 \in f(x)$ for some $x \in X$. If $y \in Y$, we shall denote by p_y the path $t \mapsto H(y, t)$. We define $\mathscr{S}_{y_0}$ as the set of all $y \in Y$ such that there exists a continuous path $q_y : [0, 1] \to X$ satisfying $f \circ q_y = p_y$ and $q_y(0) = x$. Let $f_x^{-1} : \mathscr{S}_{y_0} \to X$ be the map $y \mapsto q_y(1)$.

As a first step to getting our main result and in analogy with [10, Lemma 3], we have the following lemma.

Lemma 1.6. *The set $\mathscr{S}_{y_0}$ is open, and f_x^{-1} is a continuous inverse of f defined on that set.*

Proof. Let $y \in \mathscr{S}_{y_0}$. Since f is a local homeomorphism and $\operatorname{Im} q_y$ and $\operatorname{Im} p_y$ are compact, we can consider a finite cover $V_1 \cup V_2 \cup \cdots \cup V_m$ of $\operatorname{Im} q_y$ and a finite cover $B_1 \cup B_2 \cup \cdots \cup B_m$ of $\operatorname{Im} p_y$ such that B_k is a ball and $f|_{V_k} : V_k \to B_k$ is a homeomorphism, for $k = 1, \ldots, m$. Note that $\operatorname{Im} p_y \subset \bigcup_{k=1}^{m} B_k$. Let s_k be the inverse of $f|_{V_k}$.

Let $\rho > 0$ be the Lebesgue number of $[0, 1]$ for the finite cover $\{q_y^{-1}(V_k \cap \operatorname{Im} p_q)\}_{k=1}^{m}$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0, 1]$ such that $t_j - t_{j-1} < \rho$. For every $j = 1, \ldots, n$, there exists $k_j \in \{1, 2, \ldots, m\}$ such that $q_y[t_{j-1}, t_j] \subset V_{k_j}$. Let $\tilde{V}_j = V_{k_j} \cap V_{k_{j+1}}$. Note that $q_y(t_j) \in \tilde{V}_j$, $\tilde{V}_j$ is open, $f|_{\tilde{V}_j}$ is a homeomorphism and $f(\tilde{V}_j)$ is open and path-connected. If $z \in f(\tilde{V}_j)$, there exists a path $r : [t_j, 1] \to f(\tilde{V}_j)$ joining $p_y(t_j)$ and z. Let q_1 and q_2 be the continuous paths defined both by q_y on $[0, t_j]$ and by $s_{k_j} \circ r$ and $s_{k_{j+1}} \circ r$ on $[t_j, 1]$ respectively. Then, since f is a local homeomorphism, by the unique path lifting property (see e.g. [15, Chapter 2] for the definition of this) we have that $q_1 \equiv q_2$; in particular, $s_{k_j}(z) = s_{k_{j+1}}(z)$, in other words,

$$s_{k_j} \equiv s_{k_{j+1}} \quad \text{on } f(\tilde{V}_j).$$

Hence, we can define a map F on the union $\sigma = \bigcup_{j=1}^{n} f(\tilde{V}_j)$ as s_{k_j} on each $f(\tilde{V}_j)$. Note that for $z \in \sigma$, $f(F(z)) = z$ and for $z \in \operatorname{Im} p_y$, $F(z) = f_x^{-1}(z)$. The set σ is open and contains the path p_y. Furthermore, F is continuous on σ since in a neighborhood of any point $z \in \sigma$ the mapping F coincides with one of the maps s_{k_j} which are continuous.

Since H is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that, if $z \in B_\delta(y)$, then $d(p_z(t), p_y(t)) < \varepsilon$, for all $t \in [0, 1]$. Then, we can choose ε small enough such that $\operatorname{Im} p_z \in \sigma$, if $z \in B_\delta(y)$. The restriction of F to $\operatorname{Im} p_z$ defines an inverse of f such that $F(p_z(0)) = F(y_0) = x$. Therefore, by the definition of $\mathscr{S}_{y_0}$, if $z \in B_\delta(y)$, then $z \in \mathscr{S}_{y_0}$ and $F(z) = f_x^{-1}(z)$.

Thus, $\mathscr{S}_{y_0}$ contains an open neighborhood of y and in that neighborhood f_x^{-1} coincides with the function F which is known to be continuous. Therefore, $\mathscr{S}_{y_0}$ is open and f_x^{-1} is continuous in $\mathscr{S}_{y_0}$. □

Remark 1.7. Let $x \in X$ be fixed, and $y_0 = f(x)$. Let $f : X \to Y$ be a local homeomorphism between metric spaces, Y strongly contractible at y_0. If f is a homeomorphism onto Y, then every path can be lifted. In particular f lifts the paths p_y, for all $y \in Y$. Therefore $\mathscr{S}_{y_0} = Y$. Conversely, if $\mathscr{S}_{y_0} = Y$, by Lemma 1.6, f is a global homeomorphism. Therefore, the following statements are equivalent:

(1) f is a global homeomorphism onto Y.
(2) $\mathscr{S}_{y_0} = Y$.

2. Scalar derivatives and global inversion via coercive functionals

In this section we are going to introduce the scalar derivatives for a continuous map, in order to establish metric conditions on a local homeomorphism $f : X \to Y$ to ensure that $\mathscr{S}_{y_0} = Y$, for some $y_0 \in f(X)$.

For a continuous map $f : X \to Y$ the *upper and lower scalar derivatives* at $x \in X$ are defined by

$$D_x^- f = \liminf_{v \to x} \frac{d(f(v), f(x))}{d(v, x)}, \qquad D_x^+ f = \limsup_{v \to x} \frac{d(f(v), f(x))}{d(v, x)},$$

where v is restricted to points of X different from x. We shall be dealing only with applications for which

$$0 < D_x^- f \le D_x^+ f < \infty.$$

Let V and W be open sets in the metric spaces X and Y respectively. It is easy to see that if $g : V \to W$ is a homeomorphism then, for every $v \in V$ and $w = g(v) \in W$,

$$D_w^+(g^{-1}) = (D_v^- g)^{-1} \quad \text{and} \quad D_w^-(g^{-1}) = (D_v^+ g)^{-1}.$$

Note that if $f : X \to Y$ is locally a bi-Lipschitz homeomorphism then, for every $x \in X$, there exist strictly positive numbers α and β such that $\alpha \le D_x^- f \le D_x^+ f \le \beta$ in a neighborhood of x.

Example 2.1. If X and Y are Banach spaces and $f : X \to Y$ is a C^1 mapping such that $f'(x)$ is invertible for all $x \in X$, it is known that $D_x^+ f = \|f'(x)\|$ and $D_x^- f = \|f'(x)^{-1}\|^{-1}$ (see [10]). We have showed in [7, Example 3.2] that the same statement holds if X and Y are connected and complete Riemannian manifolds, even in the infinite dimensional case. For connected and complete Finsler manifolds we got the inequalities $D_x^+ f \le \|f'(x)\|$ and $D_x^- f \ge \|f'(x)^{-1}\|^{-1}$.

Example 2.2. Also, in [7, Remark 3.4] we proved that if f is a local homeomorphism then the lower scalar derivative coincides with the Ioffe–Katriel surjection constant (cf. [11,9]) i.e.

$$D_x^- f = \liminf_{t \to 0} t^{-1} \sup\{r \ge 0 : B_r(f(x)) \subset f(B_t(x))\}.$$

The proof of the following lemma can be found in [7, Theorems 3.8 and 3.9].

Lemma 2.3. *Let $f : X \to Y$ be a continuous map between metric spaces.*

(1) *If $q : [a, b] \to X$ has finite length and $p = f \circ q$, then*

$$\ell(p) \le \sup_{x \in \operatorname{Im} q} D_x^+ f \cdot \ell(q).$$

(2) *Let $q : [a, b] \to X$ be a path and suppose that $p = f \circ q$ has finite length; then*

$$\ell(p) \geq \inf_{x \in \operatorname{Im} q} D_x^- f \cdot \ell(q).$$

In what follows, we are going to work with local homeomorphisms that satisfy

$$\sup\{(D_u^- f)^{-1} : u \in U\} < \infty, \quad \text{for all bounded subset } U \text{ of } X. \tag{2.1}$$

Note that if f is locally bi-Lipschitz and the balls in X satisfy the Heine–Borel property, then condition (2.1) is fulfilled.

Recall that a function $h : X \to Y$ between metric spaces is said to be *coercive* if for some $y_0 \in Y$ and $x_0 \in X$

$$d(f(x), y_0) \to \infty \quad \text{when } d(x, x_0) \to \infty.$$

Lemma 2.4. *Let $f : X \to Y$ be a local homeomorphism, with X complete and Y strongly contractible at $y_0 = f(x)$ for some $x \in X$. Suppose that the condition (2.1) holds. If there exists a continuous coercive function $k : X \to \mathbb{R}$ such that*

$$\sup\{D_z^+(k \circ f_x^{-1}) : z \in \mathscr{S}_{y_0}\} < \infty \tag{2.2}$$

then $\mathscr{S}_{y_0} = Y$.

Proof. Let $y \in Y$; we shall prove that p_y has a lifting q_y. Since f is a local homeomorphism, there exists $b \in (0, 1)$ and $q_y : [0, b) \to X$ such that $f \circ q_y = p_y$ on $[0, b)$. Consider the continuous inverse $f_x^{-1} : \mathscr{S}_{y_0} \longrightarrow X$. Let $t \in [0, b)$ be fixed. By Lemma 2.3(1) to $k \circ f_x^{-1}$, we have

$$|k(x) - k(q_y(t))| \leq \ell(k \circ q_y|_{[0,t]}) \leq \sup\{D_z^+(k \circ f_x^{-1}) : z \in \operatorname{Im} p_y|_{[0,t)}\} \cdot \ell(p_y|_{[0,t]}).$$

By hypothesis, there exists $\alpha > 0$ such that $|k(x) - k(q_y(t))| \leq \alpha$, for all $t \in [0, b)$. Since k is coercive, then $\operatorname{Im} q_y$ is bounded. Let $s < t \in [0, b)$; applying Lemma 2.3(2), we have

$$d(q_y(s), q_y(t)) \leq \ell(p_y|_{[s,t]}) \cdot \sup\{(D_u^- f)^{-1} : u \in \operatorname{Im} q_y|_{[s,t]}\},$$

but

$$\sup\{(D_u^- f)^{-1} : u \in \operatorname{Im} q_y|_{[s,t]}\} \leq \sup\{(D_u^- f)^{-1} : u \in \operatorname{Im} q_y\} \leq \beta,$$

for some β. Hence $d(q_y(s), q_y(t)) \leq \beta \ell(p_y|_{[s,t]})$. Since p_y is continuous and is defined on $[0, 1]$, this shows that if $t_n \to b$, then $\{q_y(t_n)\}$ is a Cauchy sequence; therefore $\lim_{t_n \to b} q_y(t)$ exists and lies in X by the completeness of X. Since the local homeomorphisms have the unique path lifting property, q_y can be extended, by a standard argument, to $[0, 1]$ and it satisfies $f \circ q_y = p_y$ (see e.g. [7]). Therefore, $\mathscr{S}_{y_0} = Y$. □

3. Main result and special cases of coercive functional

Now, we are ready to present our main result:

Theorem 3.1. *Let $f : X \to Y$ be a local homeomorphism, X complete and Y strongly contractible at a point in $f(X)$. Suppose that the condition (2.1) is satisfied. If there exist both a coercive continuous function $k : X \to \mathbb{R}$ and $\alpha > 0$ such that*

$$\limsup_{v \to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} \leq \alpha, \quad \forall w \in X \tag{3.1}$$

then f is a global homeomorphism.

Proof. Let $y_0 = f(x)$, for some $x \in X$ such that Y is strongly contractible at y_0. Let $z \in \mathscr{S}_{y_0}$ be fixed. Since f is a continuous injection on $f_x^{-1}(\mathscr{S}_{y_0})$, then we can set $w = f_x^{-1}(z)$ and applying the change of variable $u = f(v)$, for every $u \in \mathscr{S}_{y_0}$, we have

$$\limsup_{u\to z} \frac{|k \circ f_x^{-1}(u) - k \circ f_x^{-1}(z)|}{d(u,z)} = \limsup_{v\to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))}.$$

By Lemma 2.4 and the inequality (3.1) we get that f is a global homeomorphism. □

Example 3.2. Let X and Y be Banach spaces (or complete connected Riemannian manifolds). Suppose that the local homeomorphism $f : X \to Y$ and the coercive functional $k : X \to \mathbb{R}$ are C^1 maps and, for all $w \in X$, $f'(w)$ is invertible. Then,

$$\limsup_{v\to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} = \|k'(w) \circ f'(w)^{-1}\|, \quad \forall w \in X.$$

To see this, let $w \in X$, $z \in f(w)$ and W be an open neighborhood of w such that $f|_W$ is a homeomorphism. Let $g = (f|_W)^{-1}$; then

$$\limsup_{v\to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} = \limsup_{y\to z} \frac{|(k \circ g)(y) - (k \circ g)(z)|}{d(y,z)} = D_z^+(k \circ g).$$

On the other hand, by Example 2.1, we have $D_z^+(k \circ g) = \|(k \circ g)'(z)\| = \|k'(g(z))g'(z)\|$. Since $g'(z) = f'(g(z))^{-1} = f'(w)^{-1}$, then $D_z^+(k \circ g) = \|k'(w) \circ f'(w)^{-1}\|$.

In the smooth Banach space setting, Theorem 3.1 is equivalent to the next corollary.

Corollary 3.3. *Let $f : X \to Y$ be a C^1 map between Banach spaces such that $f'(w) \in$ Isom (X, Y), for all $w \in X$. Suppose that*

$$\sup\{\|f'(w)^{-1}\| : \|w\| \le r\} < \infty, \quad \forall r : 0 < r < \infty. \tag{3.2}$$

If there exists a C^1 coercive function $k : X \to \mathbb{R}$ such that

$$\sup_{w\in X} \|k'(w) \circ f'(w)^{-1}\| < \infty \tag{3.3}$$

then f is a global homeomorphism.

The result in Corollary 3.3 was obtained by Zampieri [16, Theorem 3.1] from a different approach.

By Example 3.2, Corollary 3.3 is also true if we suppose that X and Y are complete and connected Riemannian manifolds, even infinite dimensional, and Y simply connected with seminegative curvature. For the finite dimensional case, we have the following result.

Corollary 3.4. *Let X and Y be complete and connected Riemannian n-manifolds, Y simply connected with seminegative curvature. Let $f : X \to Y$ be a C^1 map such that, for all $w \in X$, $f'(w)$ is invertible. The following statements are equivalent:*

(1) *There exists $k : X \to \mathbb{R}$ coercive with $\sup_{w\in X} \|k'(w) \circ f'(w)^{-1}\| < \infty$.*
(2) *f is a global diffeomorphism.*
(3) *f is coercive.*

Proof. Note that in this case, condition (2.1) is always satisfied, because X is complete and by the Hopf–Rinow Theorem, every bounded and closed set is compact. The implication (1) $\Rightarrow$ (2) follows from Theorem 3.1 and Example 3.2. To prove (3) $\Rightarrow$ (1), let $y_0 \in Y$ be fixed and consider the coercive functional $\delta(y) = d(y, y_0)$. It is clear that δ is not C^1 but satisfies $|\delta(y) - \delta(z)| \leq d(y, z)$, for all $y, z \in Y$. Therefore, for $\varepsilon > 0$, there exists a C^1 approximation $\tilde{\delta}$ of δ which satisfies, for all $y, z \in Y$, $|\tilde{\delta}(y) - \tilde{\delta}(z)| < (1 + \varepsilon)\, d(y, z)$ and $|\tilde{\delta}(y) - \delta(y)| < \varepsilon$ (so $\tilde{\delta}$ is also coercive); see [6]. Let $k = \tilde{\delta} \circ f$; since both $\tilde{\delta}$ and f are C^1 coercive functions, then k is a C^1 coercive functional, and for any $v, w \in X$:

$$\frac{|k(v) - k(w)|}{d(f(v), f(w))} = \frac{|\tilde{\delta}(f(v)) - \tilde{\delta}(f(w))|}{d(f(v), f(w))} < (1 + \varepsilon).$$

Then, by Example 3.2,

$$\limsup_{v \to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} = \|k'(w) \circ f'(w)^{-1}| \leq (1 + \varepsilon), \quad \forall w \in X.$$

We just have to check (2) $\Rightarrow$ (3): Since f is a global homeomorphism then it is proper, i.e. $f^{-1}(K)$ is compact whenever K is compact. Because Y is complete, again by the Hopf–Rinow Theorem, if C is bounded then $\bar{C}$ is compact, and then $f^{-1}(C) \subset f^{-1}(\bar{C})$ is bounded; therefore f is also a coercive function. $\square$

Remark 3.5. Note that in the previous corollary the assumption of seminegative curvature is used only to ensure that Y is strongly contractible, so that the same statement is true for all Y which are strongly contractible.

Note that the implication (2) $\Rightarrow$ (3) in Corollary 3.4 is not true for functions between infinite dimensional normed spaces. In order to see this, note the following: A topological space is said to be *invertible* if for every proper subset $U \subset X$ there exists a homeomorphism $f : X \to X$ sending $X \setminus U$ into U (observe that invertibility is preserved by homeomorphisms). In [5, Corollary 8] it is proved that every infinite dimensional normed space is invertible. Therefore, given an infinite dimensional normed space $(X, \|\cdot\|)$ it is enough to choose $U = \{x : \|x\| = 1\}$ and then find a non-coercive homeomorphism on X.

Returning to the metric space context, we can deduce easily the following corollaries.

Corollary 3.6. *Let $f : X \to Y$ be a local homeomorphism, X complete and Y strongly contractible at a point in $f(X)$. Suppose that the condition (2.1) holds. If f is coercive, then it is a global homeomorphism.*

Proof. Since f is coercive, the function $k : X \to \mathbb{R}$ defined by $k(x) = d(f(x), y_0)$, with $y_0 \in Y$ fixed, is coercive. Then, for all $w \in X$,

$$\limsup_{v \to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} = \limsup_{v \to w} \frac{|d(f(v), y_0) - d(f(w), y_0)|}{d(f(v), f(w))} \leq 1. \quad \square$$

Corollary 3.7. *Let $f : X \to Y$ be a local homeomorphism, X complete and Y strongly contractible at a point in $f(X)$. If*

$$D_x^- f \geq \beta > 0, \quad \forall x \in X, \tag{3.4}$$

then f is a global homeomorphism.

Proof. First, observe that condition (3.4) implies condition (2.1). Now, let $x_0 \in X$ be fixed and define $k(x) = d(x, x_0)$. Obviously, k is coercive. Let v and w in X; then for all $w \in X$,

$$\limsup_{v \to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} \leq \limsup_{v \to w} \frac{d(v, w)}{d(f(v), f(w))} = (D_w^- f)^{-1} = \frac{1}{\beta}. \quad \square$$

By means of a mountain-pass theorem for metric spaces and in terms of the Ioffe–Katriel surjection constant (see Example 2.2), Katriel [11] proved results similar to Corollaries 3.6 and 3.7 with more complicate metric and topological hypotheses over X and Y.

In more generality, we have the following corollary (a similar result, in the Banach space context, appears in a paper by John [10]).

Corollary 3.8. *Let $f : X \to Y$ be a local homeomorphism, X complete and Y strongly contractible at a point in $f(X)$. If there exists a continuous map $\omega : [0, \infty) \to (0, \infty)$ such that*

$$\int_0^\infty \frac{1}{\omega(t)} \mathrm{d}t = \infty, \tag{3.5}$$

and for some $x_0 \in X$, for every $w \in X$,

$$\frac{1}{\omega(d(w, x_0))} \leq D_w^- f.$$

then f is a global homeomorphism.

Proof. Note that, by continuity of ω, condition (2.1) is satisfied. On the other hand, for every $x \in X$, let

$$k(x) = \int_0^{d(x, x_0)} \frac{1}{\omega(t)} \mathrm{d}t.$$

Note that k is by hypothesis coercive. We are going to prove that condition (3.1) is satisfied. First, note that for every $w, v \in X$, $w \neq v$:

$$|k(v) - k(w)| = \left| \int_{d(v, x_0)}^{d(w, x_0)} \frac{1}{\omega(t)} \, \mathrm{d}t \right| \leq \frac{|d(v, x_0) - d(w, x_0)|}{y_{w,v}}$$

where $y_{w,v} = \min\{\omega(x) : x \in [d(w, x_0), d(v, x_0)]\}$, then,

$$\frac{|k(v) - k(w)|}{d(f(v), f(w))} \leq \frac{d(v, w)}{d(f(v), f(w))} \cdot \frac{1}{y_{w,v}}.$$

Therefore,

$$\limsup_{v \to w} \frac{|k(v) - k(w)|}{d(f(v), f(w))} \leq (D_w^- f)^{-1} \cdot \frac{1}{\omega(d(w, x_0))} \leq 1. \quad \square$$

Note that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map such that $f'(x)$ is invertible, for every $x \in \mathbb{R}^n$, the function ω,

$$\omega(t)^{-1} = \min_{|x|=t} \|f'(x)^{-1}\|^{-1}$$

is continuous, and we get from Corollary 3.8 the celebrated Hadamard's Global Inversion Theorem, [8]. More generally, consider the following condition for a map $f : X \to Y$ between metric spaces:

$$\mu(t)^{-1} = \inf_{d(x,x_0)\le t} D_x^- f, \quad \text{with} \int_0^\infty \mu(t)^{-1}\mathrm{d}t = \infty. \tag{3.6}$$

In [7, Lemma 4.5] we proved that the condition (3.6) is satisfied if and only if there exists a nondecreasing map $\omega : [0,\infty] \to (0,\infty)$ (not necessarily continuous) satisfying (3.5). But, since ω is nondecreasing, it has just a countable number of jump discontinuities, so there exists a continuous map $\bar{\omega} : [0,\infty] \to (0,\infty)$ "above" ω satisfying (3.5). So, if X is complete, Y is strongly contractible, and f is a local homeomorphism, then condition (3.6) implies that f is a global homeomorphism.

Example 3.9. Consider the function $f : (x, y) \mapsto (x^3 + y, -x^3 + y^3 - 1)$. It is not difficult to see that $f'(x, y)$ is invertible, since $\det f'(x, y) = 9x^2y^2 + 3x^2 + 1 > 1$; but

$$\int_0^\infty \min_{\sqrt{x^2+y^2}=t} \|f'(x,y)^{-1}\|^{-1}\mathrm{d}t = \int_0^\infty g(t)\mathrm{d}t < 1,$$

where $g(t) = \sqrt{\frac{9}{2}t^4 + 1 - \frac{3}{2}\sqrt{9t^8 + 4t^4}}$. Nevertheless, if we consider the coercive functional $k(x, y) = \log(x^2 + y^2 + 1)$, we can check that

$$\|k'(x,y) \circ f'(x,y)^{-1}\| \le \frac{2|3xy^2 + 3yx^2 + y| + |3yx^2 - x|}{(9x^2y^2 + 3x^2 + 1)(x^2 + y^2 + 1)} \le \frac{3}{x^2 + y^2 + 1} + 2 \le 5.$$

Therefore, f is a global homeomorphism, but does not satisfy the Hadamard condition.

Remark 3.10 (*Relation with the Palais–Smale Condition*). Let E be a Banach space and $k : E \to \mathbb{R}$ a C^1 map. We say that k satisfies the Palais–Smale condition if every sequence $\{x_n\}$ with $\{k(x_n)\}$ bounded, and such that $\lim_{n\to\infty} k'(x_n) = 0$, has a converging subsequence. It is well known that if k is bounded from below and satisfies the Palais–Smale condition, then k is coercive. Katriel proved a generalization of this result to metric spaces (see [11, Lemma 6.5]). So, in Theorem 3.1 we can replace the coerciveness condition with k being bounded from below and satisfying the Palais–Smale condition.

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