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On Asymptotically Conformal Curves

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Let f be a quasiconformal self-mapping of the complex plane. We introduce a notion of asymptotical homogeneity of the mapping f at the prescribed point and in these terms give equivalent conditions for quasiconformal curves to be asymptotically conformal. The proofs are based on a fundamental geometrical criterion of asymptotically conformal curves obtained by Ch. Pommerenke and J. Becker.

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1. INTRODUCTION

Let G be a domain in the complex plane $\mathbb{C}$ and $\mu : G \to \mathbb{C}$ be a measurable function satisfying

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_G |\mu(z)| < 1. \tag{1.1}$$

An orientation preserving homeomorphism $f : G \to \mathbb{C}$ of the Sobolev class $W^1_{2,\mathrm{loc}}$ is called *quasiconformal* with complex dilatation μ, if it satisfies the Beltrami equation

$$f_{\bar z} = \mu(z) f_z \qquad \text{a.e.} \tag{1.2}$$

A Jordan curve $\Gamma \subset \mathbb{C}$ is called a *quasiconformal curve or quasicircle* if it is the image of the unit circle under a quasiconformal mapping of $\mathbb{C}$ (see [1, p. 105], [2, p. 286]). In 1963 L. Ahlfors [3] gave a surprisingly simple geometric characterization of quasicircles. He proved that the curve Γ is a quasicircle iff the quantity

$$\gamma \equiv \gamma(w_1, w_2, w) = \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} \tag{1.3}$$

is bounded for all $w_1, w_2 \in \Gamma$ and $w \in \Gamma(w_1, w_2)$ where $\Gamma(w_1, w_2)$ denotes the subarc of Γ corresponding to $w_1, w_2 \in \Gamma$ with smaller diameter.

In 1967 L. Carleson [4] considered quasiconformal self-mappings f of the upper half-plane which are *conformal at the boundary* in the sense that

$$\operatorname*{ess\,sup}_{0<\operatorname{Im} z \le t} |f_{\bar z}/f_z| \to 0, \qquad t \to 0.$$

Let $\Gamma \subset \mathbb{C}$ be a quasicircle in the complex plane and let f denote a conformal mapping of the unit disk $D = \{z : |z| < 1\}$ onto the interior of Γ. By a result of

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L. Ahlfors [5, p. 71] f admits a quasiconformal extension over the unit circle ∂D. If there exists a quasiconformal extension with complex dilatation $\mu(z)$ such that

$$\operatorname*{ess\,sup}_{1\le|z|\le t}|\mu(z)| \to 0, \qquad t \to 1+0, \tag{1.4}$$

then the curve Γ is called *asymptotically conformal* [6, 7].

Ch. Pommerenke and J. Becker proved [7] that (1.4) is equivalent to the condition

$$\lim_{|w_1-w_2|\to 0} \frac{|w_1-w|+|w-w_2|}{|w_1-w_2|} = 1 \tag{1.5}$$

uniformly with respect to $w \in \Gamma(w_1, w_2)$.

It is well-known that quasicircles need not be rectifiable even if they are asymptotically conformal (see, for example, [13, p. 42], [8, p. 146]). Asymptotically conformal curves and related problems have been studied, for instance, by Ch. Pommerenke and S. E. Warschawski [9], J. M. Anderson, J. Becker and F. D. Lesley [10], J. M. Anderson and A. Hinkkanen [8] and by P. Mattila and M. Vuorinen [11].

We introduce a notion of *asymptotical homogeneity* of quasiconformal mappings at the prescribed point and give a description of asymptotically conformal curves in these terms.

2. ON ASYMPTOTICAL HOMOGENEITY

A mapping $f : G \to \mathbb{C}$ normalized with the condition $f(0) = 0$ is called *asymptotically homogeneous* at the origin [12] if

$$\lim_{z\to 0} \frac{f(z\zeta)}{f(z)} = \zeta \tag{2.1}$$

for any $\zeta \in \mathbb{C}$.

One can prove that the asymptotical homogeneity of the quasiconformal mapping f at the origin is equivalent to the conditions

$$f(z) = A(|z|)(z + o(|z|)), \tag{2.2}$$

$$\lim_{\rho\to 0} \frac{A(t\rho)}{A(\rho)} = 1, \qquad \forall \quad t > 0 \tag{2.3}$$

where $o(|z|)/|z| \to 0$ as $|z| \to 0$. Note that the differentiation of the mapping f at the prescribed point in the sense of (2.2), (2.3) is due to P. Belinskiĭ [13, p. 41].

If f is asymptotically homogeneous at the origin then

$$\lim_{r\to 0} \frac{\max_{|z|=r}|f(z)|}{\min_{|z|=r}|f(z)|} = 1$$

that is the infinitesimal circles centered at the origin are preserved and as

$$\lim_{|z|\to 0} \frac{|f(z\zeta)|}{|f(z)|} = |\zeta| \tag{2.4}$$

therefore the moduli of infinitesimal rings centered at the origin are preserved also. At last from

$$\lim_{z\to 0}[\arg f(z\zeta) - \arg f(z)] = \arg\zeta \tag{2.5}$$

for any $\zeta \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, it follows that the angles between rays outgoing from the origin in the direction of corresponding points are preserved. And vice versa, if the conditions (2.4), (2.5) hold simultaneously then the mapping f is asymptotically homogeneous at the origin by definition.

The example

$$f(z) = ze^{i(-\ln|z|)^{1/2}}, \qquad f(0) = 0$$

shows that under asymptotical homogeneity the radial lines can be transformed to the spirals.

In what follows we shall use also the notion of *uniform asymptotical homogeneity* of a function f on sets. The last means that the principal condition (2.1) is satisfied for the family of functions $f_\eta(z) = f(z+\eta) - f(\eta)$ uniformly with respect to η belonging to the given set.

3. MAIN RESULTS

We begin with the result which connects the notions of asymptotical homogeneity and asymptotical conformality and contains some different equivalent conditions for Γ to be an asymptotically conformal curve.

THEOREM 1 *A Jordan curve $\Gamma \subset \mathbb{C}$ is asymptotically conformal iff there exists a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, $\Gamma = f(\partial D)$, satisfying one of the following conditions*:

1. *The mapping f is uniformly asymptotically homogeneous with respect to $\eta \in \partial D$.*
2. *Under the assumption $|(z'-\eta)/(z-\eta)| \le \delta$*

$$\lim_{z',z\to\eta}\left\{\frac{f(z')-f(\eta)}{f(z)-f(\eta)} - \frac{z'-\eta}{z-\eta}\right\} = 0 \tag{3.1}$$

uniformly with respect to $\eta \in \partial D$ for each fixed $\delta \ge 0$.

3. *Uniformly with respect to $\eta \in \partial D$*

$$\lim_{t\to 0}\frac{1}{t^2}\iint_{|z-\eta|\le t}|\mu(z)|\,dm_z = 0. \tag{3.2}$$

Note that Theorem 1 does not require that $f(z)$ be conformal inside or outside D.

The last assertion in Theorem 1 means that the complex dilatation μ is uniformly approximately continuous on ∂D if we redefine μ on ∂D putting $\mu(\eta) = 0$, $\eta \in \partial D$ (see [14, p. 199]). In other words

$$\mu_{t,\eta}(z) \xrightarrow{\text{mes}} 0 \qquad \text{as} \quad t \to 0$$

uniformly with respect to $\eta \in \partial D$ where $\mu_{t,\eta}(z) = \mu(\eta + tz)$, $t > 0$, $\eta \in \partial D$, $|z| \le R$. Using this notion we can deduce some corollaries useful for applications.

COROLLARY 1 *If for some $\alpha > 0$ there exists the uniform limit with respect to $\eta \in \partial D$*

$$\lim_{t\to 0}\frac{1}{t^2}\iint_{|z-\eta|\le t}|\mu(z)|^\alpha\, dm_z = 0 \tag{3.3}$$

then Γ is asymptotically conformal.

Indeed, since $|\mu(z)| < 1$ we have $\mu_{t,\eta}(z) \xrightarrow{\text{mes}} 0$ as $t \to 0$ uniformly with respect to $\eta \in \partial D$ iff (3.3) holds for any $\alpha > 0$.

COROLLARY 2 *If for some $\alpha > 0$*

$$\iint_{|\eta-z|\le t}\frac{|\mu(z)|^\alpha}{|\eta - z|^2}\, dm_z \tag{3.4}$$

converges uniformly with respect to $\eta \in \partial D$ then Γ is asymptotically conformal.

Noting that

$$\frac{1}{t^2}\iint_{|z-\eta|\le t}|\mu(z)|^\alpha\, dm_z \le \iint_{|\eta-z|\le t}\frac{|\mu(z)|^\alpha}{|\eta - z|^2}\, dm_z$$

we deduce Corollary 2 from Corollary 1.

COROLLARY 3 *If for some $\alpha > 0$ and $t > 0$*

$$\iint_{|1-|z||\le t}\frac{|\mu(z)|^\alpha}{(1-|z|)^2}\, dm_z < \infty \tag{3.5}$$

then Γ is asymptotically conformal.

To prove Corollary 3 we have to verify that (3.5) implies the uniform convergence of the integral (3.4).

The example below shows that the condition (3.5) does not imply that $k(t) \to 0$ as $t \to 0$, where

$$k(t) = \operatorname*{ess\,sup}_{1\le|z|\le 1+t}|\mu(z)|. \tag{3.6}$$

Let $\mu(z) = \phi(|z| - 1)$, where $\phi(t) = \frac{1}{2}$ for $t \in [1/n, 1/n + 1/n^4]$, $n = 1, 2\ldots$ and $\phi(t) = 0$ for the rest t. Then $k(t) = \frac{1}{2}$ for all $t > 0$. However

$$\iint_{\mathbf{C}}\frac{|\mu(z)|^\alpha}{(1-|z|)^2}\, dm_z \le 3\cdot 2^{1-\alpha}\pi\sum_{n=1}^{\infty}1/n^2 < \infty$$

for all $\alpha > 0$.

Thus, the conditions (3.5) does not imply the well-known Dini condition

$$\int_0 k(t)^\alpha\frac{dt}{t} < \infty. \tag{3.7}$$

Note that the equivalence of (3.7) with other conditions has been discussed in [15].

In the following theorem, which applies to the conformal mappings with quasiconformal extension, we give corresponding equivalent conditions for $\Gamma = f(\partial D)$ to be asymptotically conformal.

THEOREM 2 *Let $G \subset \mathbb{C}$ be a Jordan domain and let f be a conformal mapping of D onto G. Then the following assertions are equivalent:*

1. *The boundary $\partial G \subset \mathbb{C}$ is an asymptotically conformal curve.*
2. *The mapping f is uniformly asymptotically homogeneous with respect to $\eta \in D$.*
3. *Under the assumption $|(z' - \eta)/(z - \eta)| \leq \delta$*

$$\lim_{z', z \to \eta} \left\{ \frac{f(z') - f(\eta)}{f(z) - f(\eta)} - \frac{z' - \eta}{z - \eta} \right\} = 0 \tag{3.8}$$

as z', $z \in D$ uniformly with respect to $\eta \in D$ for each fixed $\delta \geq 0$.

4. *Under the assumption $|(z' - \eta)/(z - \eta)| \leq \delta$ there exists the limit* (3.8) *as $z, z' \in \overline{D}$ uniformly with respect to $\eta \in \overline{D}$ for each fixed $\delta \geq 0$.*

4. PROOF OF THEOREMS 1 AND 2

First we need a lemma about uniformly asymptotically homogeneous families of quasiconformal mappings.

LEMMA *Let $f_j : \mathbb{C} \to \mathbb{C}$, $f_j(0) = 0$, $j \in J$ be a family of Q-quasiconformal mappings and μ_j, $j \in J$ be the corresponding family of complex dilatations. Then the following assertions are equivalent:*

1. *There exists a limit*

$$\lim_{z \to 0} \frac{f_j(z\zeta)}{f_j(z)} = \zeta, \qquad \forall \quad \zeta \in \mathbb{C} \tag{4.1}$$

uniformly with respect to $j \in J$.

2. *The limit* (4.1) *is uniform with respect to $(\zeta, j) \in K \times J$ where $K \subset \mathbb{C}$ is any compact set.*
3. *All functions of the family f_j can be represented in the form*

$$f_j(z) = A_j(|z|)(z + o_j(|z|)) \tag{4.2}$$

where $o_j(\rho)/\rho \to 0$ as $\rho \to 0$ and

$$\lim_{\rho \to 0} \frac{A_j(t\rho)}{A_j(\rho)} = 1, \qquad \forall \quad t > 0 \tag{4.3}$$

uniformly with respect to $j \in J$.

4. *Under the assumption $|z'/z| \leq \delta$ there exists a limit*

$$\lim_{z', z \to 0} \left\{ \frac{f_j(z')}{f_j(z)} - \frac{z'}{z} \right\} = 0 \tag{4.4}$$

uniformly with respect to $j \in J$ for any fixed $\delta \geq 0$.

Proof We shall prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$. Introduce the following notations

$$f_{z,j}(\zeta) \equiv f_j(\zeta z)/f_j(z), \qquad z \in \mathbb{C}^*,$$
$$f_{0,j}(\zeta) \equiv \zeta$$

for all $\zeta \in \mathbb{C}$, $j \in J$.

$1 \Rightarrow 2$. Indeed (4.1) implies that $r(f_{z,j}, f_{0,j}) \to 0$ as $z \to 0$ uniformly with respect to $j \in J$. Here

$$r(g,h) = \sum_{m=1}^{\infty} 2^{-m} \frac{|g(z_m) - h(z_m)|}{1 + |g(z_m) - h(z_m)|}$$

where $\{z_m\}_{m=1}^{\infty}$ is a countable everywhere dense subset of $\mathbb{C}$.

Now we introduce the following metric on the space of all Q-quasiconformal mappings on the plane

$$\sigma(g,h) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sigma_m(g,h)}{1 + \sigma_m(g,h)},$$

where $\sigma_m(g,h) = \max_{|z| \leq m} |g(z) - h(z)|$. We have to prove that $\sigma(f_{z,j}, f_{0,j}) \to 0$ as $z \to 0$ uniformly with respect to $j \in J$.

Supposing that it fails we can find $\varepsilon > 0$ and sequences $z_n \to 0$, $z_n \in \mathbb{C}$ and $j_n \in J$ such that $\sigma(g_n, h_n) \geq \varepsilon$ where $g_n = f_{z_n, j_n}$, $h_n = f_{0, j_n}$, $n = 1, 2, \ldots$. On the other hand the convergence $r(g_n, h_n) \to 0$ implies $\sigma(g_n, h_n) \to 0$ as $n \to \infty$ [1, p. 73]. The last contradicts to the assumption.

$2 \Rightarrow 3$. Choosing $z = \rho > 0$, $\zeta = e^{i\theta}$ and denoting $w = z\zeta = \rho e^{i\theta}$ we conclude that $f_j(w) = f_j(\rho)(\zeta + \alpha_j(\rho))$ where $\alpha_j(\rho) \to 0$ as $\rho \to 0$ uniformly with respect to $j \in J$. Thus

$$f_j(w) = \frac{f_j(\rho)}{\rho}(w + o_j(\rho))$$

where $o_j(\rho)/\rho \to 0$ as $\rho \to 0$ uniformly with respect to $j \in J$. It means that the functions f_j can be written in the form (4.2) with

$$A_j(\rho) = \frac{f_j(\rho)}{\rho}.$$

Moreover (4.1) with $z = \rho > 0$ and $\zeta = t > 0$ implies that the limit (4.3) is uniform with respect to $j \in J$.

$3 \Rightarrow 4$. From (4.2) and (4.3) it follows that $f_{t,j}(\zeta) \to f_{0,j}(\zeta) \equiv \zeta$ as $t \to 0$, $t \in \mathbb{R}^+$ for every fixed $\zeta \in \mathbb{C}$ uniformly with respect to $j \in J$. Using the arguments of the first part we deduce that $f_{t,j}(\zeta) \to f_{0,j}(\zeta) \equiv \zeta$ as $t \to 0$, $t > 0$ uniformly with respect $(\zeta, j) \in K \times J$.

Let us note that

$$f_{z,j}(\zeta) = \frac{f_{|z|,j}(z\zeta/|z|)}{f_{|z|,j}(z/|z|)} = \frac{f_j(z')}{f_j(z)}$$

when $\zeta = z'/z$.

We shall show that $f_{z,j}(\zeta)-\zeta \to 0$ as $z \to 0$, $z \in \mathbb{C}^*$ uniformly with respect to $(\zeta, j) \in D_\delta \times J$. Here $D_\delta = \{\zeta \in \mathbb{C} : |\zeta| \leq \delta\}$.

Let us suppose that it fails. Then there exist sequences $\zeta_n \in D_\delta$, $z_n \to 0$, $z_n \in \mathbb{C}^*$, $j_n \in J$, $n = 1,2,\dots$ and $\varepsilon > 0$ such that $|g_n(\zeta_n) - \zeta_n| \geq \varepsilon$ where $g_n(\zeta) = f_{z_n,j_n}(\zeta)$, $\zeta \in \mathbb{C}$. Because the disk D_δ and the unit circle are compact one can assume additionally that $\zeta_n \to \zeta_0 \in D_\delta$ and $\eta_n = z_n/|z_n| \to \eta_0 \in \partial D$ as $n \to \infty$.

Denote by $\phi_n(\zeta)$ the mappings $f_{|z_n|,j_n}(\zeta)$, $\zeta \in \mathbb{C}$. From the above conclusions we deduce that $\phi_n(\zeta) \to \zeta$ as $n \to \infty$ uniformly in $D_\delta \cup \partial D$. Moreover

$$g_n(\zeta) = \phi_n(\eta_n \zeta)/\phi_n(\eta_n).$$

Hence $g_n(\zeta) \to \zeta$ as $n \to \infty$ uniformly in D_δ. Thus $g_n(\zeta_n) \to \zeta_0$ as $n \to \infty$. This contradicts the assumption.

$4 \Rightarrow 1$. Putting $z' = z\zeta$ and $\delta = |\zeta|$ in the expression (4.4) we obtain the relation (4.1). It completes the proof.

Proof of Theorem 1 By Lemma conditions 1 and 2 are equivalent. Now we shall prove that the condition 1 implies an asymptotical conformality of $\Gamma = f(\partial D)$. The proof is based on the fundamental geometrical criterion (1.5). Indeed let it fail for Γ. Then there exist $w_1^{(n)}$, $w_2^{(n)} \in \Gamma$, $w^{(n)} \in \Gamma(w_1^{(n)}, w_2^{(n)})$ and $\varepsilon > 0$ such that $|w_1^{(n)} - w_2^{(n)}| \to 0$ as $n \to \infty$ and

$$\frac{|w_1^{(n)} - w^{(n)}| + |w_2^{(n)} - w^{(n)}|}{|w_1^{(n)} - w_2^{(n)}|} \geq 1 + \varepsilon. \tag{4.5}$$

Let η_n, $\eta_n^{(1)}$ and $\eta_n^{(2)}$ are pre-images of the points $w^{(n)}$, $w_1^{(n)}$ and $w_2^{(n)}$ on the unit circle under the mapping $f : \mathbb{C} \to \mathbb{C}$. Let us introduce the following notations: $z_n = \eta_n^{(2)} - \eta_n$, $\zeta_n = (\eta_n^{(1)} - \eta_n)/(\eta_n^{(2)} - \eta_n)$. Without loss of generality we may assume that $|\zeta_n| \leq 1$ and $z_n \neq 0$. Let

$$\Phi(\eta, \zeta, z) = \frac{f_\eta(\zeta z)}{f_\eta(z)} \tag{4.6}$$

where $f_\eta(z) = f(z + \eta) - f(\eta)$, and $z \in \mathbb{C}^*$, $\zeta \in \mathbb{C}$, $\eta \in \partial D$. Then (4.5) can be rewritten in the following form

$$\frac{1 + |\Phi_n|}{|1 - \Phi_n|} \geq 1 + \varepsilon$$

where $\Phi_n = \Phi(\eta_n, \zeta_n, z_n)$.

On the other hand by Lemma the assertion 1 about uniform asymptotical homogeneity of f implies that $\Phi(\eta, \zeta, z) \to \zeta$ as $z \to 0$ uniformly with respect to $(\eta, \zeta) \in \partial D \times \overline{D}$. Because the disk $\overline{D}$ is compact one can suppose that $\zeta_n \to \zeta_0 \in \overline{D}$ as $n \to \infty$ then $\Phi_n \to -t$ where $t \in [0,1]$. Hence $(1 + |\Phi_n|)/|1 - \Phi_n| \to 1$ as $n \to \infty$. This contradicts the assumption (4.5). Thus we proved that $\Gamma = f(\partial D)$ is asymptotically conformal.

If Γ is asymptotically conformal then by definition there is a quasiconformal mapping f conformal in D such that (1.4) holds which implies (3.2).

In order to prove that the assertion 3 implies the assertion 1 we shall consider the class $\mathfrak{F}_Q$ of all Q-quasiconformal self-mappings of extended complex plane normalized with the conditions $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Here

$$Q = \operatorname{ess\,sup}_{\mathbb{C}}(1 + |\mu(z)|)(1 - |\mu(z)|)^{-1}.$$

Note that this space of quasiconformal mappings is sequentially compact with respect to locally uniform convergence (see [1, p. 73]). Moreover all functions (4.6) with respect to variable ζ belong to the class $\mathfrak{F}_Q$. Note additionally that the assertion 3 implies that $\mu_{t,\eta}(z) \xrightarrow{\text{mes}} 0$ as $t \to 0$ uniformly with respect to $\eta \in \partial D$. Here $\mu_{t,\eta}(z) = \mu(\eta + tz)$, $t > 0$, $\eta \in \partial D$, $|z| \le R$.

Suppose that the assertion 1 fails. Then there exists $\varepsilon > 0$ such that for some $\zeta \in \mathbb{C}$

$$|g_n(\zeta) - \zeta| \ge \varepsilon \tag{4.7}$$

where $g_n(\zeta) = \Phi(\eta_n, \zeta, z_n)$ for some $z_n \in \mathbb{C}\backslash\{0\}$, $\eta_n \in \partial D$, $z_n \to 0$. We can assume that $g_n(\zeta) \to g(\zeta) \in \mathfrak{F}_Q$ and their complex dilatations $\mu_n(\zeta) \to 0$ almost everywhere (see [1, p. 73]) as $n \to 0$. By well-known Bers-Bojarski convergence theorem (see [1, p. 207], [16]) $g(\zeta) = \zeta$. The last contradicts (4.7) and we complete the proof of the theorem.

Proof of Theorem 2 $1 \Rightarrow 2$. Let ∂G be an asymptotically conformal curve. It means that for f there exists a quasiconformal extension over the unit disk D with complex dilatation $\mu(z)$ such that

$$\operatorname*{ess\,sup}_{1<|z|\le 1+t} |\mu(z)| \to 0, \qquad t \to 0.$$

Hence $\mu_{t,\eta}(z) = \mu(\eta + tz) \to 0$ as $t \to 0$ in the sense of Lebesque measure convergence uniformly with respect to $\eta \in D$. Arguing as in the proof of Theorem 1 we can deduce that the last implies $\Phi(\eta, \zeta, z) \to \zeta$ as $z \to 0$ uniformly with respect to $\eta \in D$. Here the function $\Phi(\eta, \zeta, z)$ is defined by (4.6).

$2 \Rightarrow 3$. By Lemma it is equivalent that (4.4) holds for the family of functions

$$f_\eta(z) = f(\eta + z) - f(\eta)$$

uniformly with respect to $\eta \in D$.

Hence under the assumption $|(z' - \eta)/(z - \eta)| \le \delta$

$$\Psi(\eta, z, z') = \frac{f(z') - f(\eta)}{f(z) - f(\eta)} - \frac{z' - \eta}{z - \eta} \to 0$$

as $z, z' \to \eta$ uniformly with respect to $\eta \in D$ for each $\delta \ge 0$.

$3 \Rightarrow 4$. Note that the function f realizes a homeomorphism of $\overline{D}$ onto $\overline{G}$ by a well-known theorem on boundary correspondence under conformal mappings.

Let $\lambda_n \in (0, 1)$, $n = 1, 2, \ldots$, and $\lambda_n \to 1$ as $n \to \infty$. For any η, z and $z' \in \overline{D}$, $z \ne \eta$ we put

$$\Psi_n(\eta, z, z') = \Psi(\eta\lambda_n, z\lambda_n, z'\lambda_n).$$

Note that

$$\begin{aligned}|z'\lambda_n - \eta\lambda_n| &\le |z' - \eta|, |z\lambda_n - \eta\lambda_n| \\ &\le |z - \eta|, |(z'\lambda_n - \eta\lambda_n)/(z\lambda_n - \eta\lambda_n)| \\ &= |(z' - \eta)/(z - \eta)|\end{aligned}$$

and $\eta\lambda_n$, $z\lambda_n$, $z'\lambda_n \in D$. By assertion 3 under the condition $|(z' - \eta)/(z - \eta)| \le \delta$

$$\lim_{z,z' \to \eta} \Psi_n(\eta, z, z') = 0 \tag{4.8}$$

as $z, z' \in \overline{D}$ uniformly with respect to n and $\eta \in \overline{D}$ for each fixed $\delta \ge 0$.

Suppose that the limit (3.8) is not uniform with respect to $\eta \in \overline{D}$. Then there exist $\delta \ge 0$, $\varepsilon > 0$, η_k, z_k and $z'_k \in \overline{D}$, $|(z'_k - \eta_k)/(z_k - \eta_k)| \le \delta$, $k = 1, 2, \ldots, z'_k - \eta_k \to 0$ and $z_k - \eta_k \to 0$ such that

$$|\Psi(\eta_k, z_k, z'_k)| \ge \varepsilon. \tag{4.9}$$

On the other hand

$$|\Psi(\eta_k, z_k, z'_k) - \Psi_{n_k}(\eta_k, z_k, z'_k)| < \varepsilon/2 \tag{4.10}$$

for any fixed k and some n_k. From inequality (4.9) and (4.10) we deduce that

$$|\Psi_{n_k}(\eta_k, z_k, z'_k)| > \varepsilon/2, \qquad \forall \quad k = 1, 2, \ldots .$$

The last contradicts (4.8) and hence the limit (3.8) is uniform with respect to $\eta \in \overline{D}$.

$4 \Rightarrow 1$. To prove it we have to repeat the corresponding arguments based on the geometrical criterion (1.5). It completes the proof of Theorem 2.

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