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INFINITE LIFETIME FOR THE STARLIKE DYNAMICS IN HELE-SHAW CELLS

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Abstract. One of the "folklore" questions in the theory of free boundary problems is the lifetime of the starlike dynamics in a Hele-Shaw cell. We prove precisely that, starting with a starlike analytic phase domain Ω_0, the Hele-Shaw chain of subordinating domains $\Omega(t)$, $\Omega_0 = \Omega(0)$, exists for an infinite time under injection at the point of starlikeness.

1. Introduction

We consider the flow of a viscous fluid in a plane Hele-Shaw cell under injection through a unique well which can be placed at the origin. Suppose that at the initial time the phase domain Ω_0 occupied by the fluid is simply connected and bounded by a smooth analytic curve Γ_0. The evolution of the phase domains $\Omega(t)$, $\Omega(0) = \Omega_0$, is described by an auxiliary conformal mapping $f(\zeta, t)$, $f(\zeta, 0) = f_0(\zeta)$, of the unit disk $U = \{\zeta, |\zeta| < 1\}$ onto $\Omega(t)$, $\Gamma(t) = \partial\Omega(t)$, normalized by $f(0,t) = 0$, $f'(0,t) > 0$. Here we denote the derivatives by $f' = \partial f/\partial\zeta$, $\dot{f} = \partial f/\partial t$, and t is a time parameter. This mapping satisfies the equation

$$\operatorname{Re}\left[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}\right] = 1, \quad \zeta = e^{i\theta}, \tag{1.1}$$

under a suitable rescaling. L. A. Galin [4] and P. Ya. Polubarinova-Kochina [11, 12] first derived the equation (1.1) and stimulated deep investigations in the complex variable approach to free boundary problems (see, e.g., [9, 20] and the references therein).

By a *classical solution* in the interval $t \in [0, T)$ to the equation (1.1) we mean a map $f(\zeta, t)$ that is conformal and univalent as a function of ζ in a neighbourhood of the closure $\bar{U}$ of the unit disk U and C^1 with respect to t in $[0, T)$ (one-sided at 0). These assumptions about the classical solution can be found, e.g., in [14].

One of the main features of the solution to the equation (1.1) is that, starting with an analytic boundary Γ_0, we obtain a one-parameter (t) chain of classical

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solutions $f(\zeta, t)$ that exists during a period $t \in [0, T)$, developing possible cusps or double points on the boundary $\Gamma(t)$, $\Gamma(0) = \Gamma_0$, at a blow-up time T. It is known [21] that the classical solution exists and is unique locally in time. Recently, in [14], it became clear that this model could be interpreted as a particular case of the abstract Cauchy problem; thus, the classical solvability (locally in time) may be proved using the nonlinear abstract Cauchy-Kovalevskaya Theorem. We also mention here that the problem is Hadamard well-posed in our case [3].

The problem of estimating T is of primary importance. One of the "folklore" questions is the lifetime of the starlike dynamics in a Hele-Shaw cell. The majority of investigators believe that this lifetime is infinite in the case of a starlike initial domain whereas some of them do not. Our aim is to settle this question. A domain $\Omega \subset \mathbb{C}$, $0 \in \Omega$, is said to be *starlike* (with respect to the origin) if each ray starting at the origin intersects Ω in a set that is either a line segment or a full ray. If a function $f(\zeta)$ maps U onto a domain that is starlike, $f(0) = 0$, then we say that $f(\zeta)$ is a *starlike function*. We denote the class of starlike functions by S^*. A criterion for a function $f(\zeta)$, $\zeta \in U$, $f(0) = 0$, $f'(0) > 0$, to be starlike is the following inequality:

$$\operatorname{Re} \frac{\zeta f'(\zeta)}{f(\zeta)} > 0, \quad \zeta \in U. \tag{1.2}$$

This standard criterion can be found, e.g., in [5, 2, 7, 13].

We prove rigorously (Theorem 3.4) that, starting with a smooth analytic starlike phase domain Ω_0, the Hele-Shaw chain of subordinating domains $\Omega(t)$ exists as a classical solution for an infinite time under injection.

2. Monotone change of strong α-starlikeness

The class S^* is a union of classes S^*_α of so-called *strongly starlike functions of order* α, $0 < \alpha \le 1$, defined by D. A. Brannan and W. E. Kirwan [1] and J. Stankiewicz [19]. A function $f : U \to \mathbb{C}$, $f(0) = 0$, $f'(0) > 0$, is said to be from S^*_α if for all $\zeta \in U$,

$$\left| \arg \frac{\zeta f'(\zeta)}{f(\zeta)} \right| < \alpha \frac{\pi}{2}. \tag{2.1}$$

This class of functions is characterized as follows. Every level line $f(re^{i\theta})$, $\theta \in [0, 2\pi)$, $f \in S^*_\alpha$, is reachable from outside by the radial angle $\pi(1-\alpha)$. We will call the images of the unit disk under functions from S^*_α strongly starlike of order α as well. Clearly, $S^* = S^*_1$.

We prove that, starting with a phase domain Ω_0 that is strongly starlike of order α and bounded by an analytic curve, we obtain a subordination chain of domains $\Omega(t)$ (and functions $f(\zeta, t)$) strongly starlike of order $\alpha(t)$ with a decreasing order $\alpha(t)$.

Theorem 2.1. *Let $f_0 \in S^*_\alpha$, $\alpha \in (0, 1]$, be analytic and univalent in a neighbourhood of $\bar{U}$. Then the classical solution $f(\zeta, t)$ to the Polubarinova-Galin equation* (1.1) *forms a subordination chain of strongly starlike functions of order $\alpha(t)$ with a strictly decreasing $\alpha(t)$ during the time of existence.*

Proof. Let T be such that the classical solution $f(\zeta, t)$ exists during the time $t \in [0, T)$, $T > 0$. Since all functions $f(\zeta, t)$ have analytic univalent extension into a neighbourhood of $\bar{U}$ during the time of the existence of the classical solution to

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(1.1), their derivatives $f'(\zeta,t)$ are continuous and do not vanish in $\bar{U}$. Moreover, $f(\zeta,t)$ are starlike in U (see [8, 20]). Therefore, there exists $\alpha(t)$, $0<\alpha(t)\le 1$, such that $f(\zeta,t)\in S^*_{\alpha(t)}$ and $f(\zeta,t)\notin S^*_{\alpha(t)-\varepsilon}$ for any $\varepsilon>0$.

Let us fix $t_0\in[0,T)$ and consider the set A of all points ζ, $|\zeta|=1$, for which $|\arg\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}|=\alpha\pi/2$. First, we deal with the subset A^+ of A where

$$\arg\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}=\frac{\alpha\pi}{2}. \tag{2.2}$$

The sets A^+ and $A^-=A\setminus A^+$ are closed and do not intersect. One of the sets A^+ and A^- is allowed to be empty. Without loss of generality, we suppose that $A^+\neq\emptyset$. For any point $\zeta\in A^+$, we have

$$\operatorname{Im}\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}>0. \tag{2.3}$$

The argument $\arg\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}$ attains its maximum on $\zeta\in\partial U$ at the points of A^+. Therefore,

$$\frac{\partial}{\partial\theta}\arg\frac{e^{i\theta}f'(e^{i\theta},t_0)}{f(e^{i\theta},t_0)}=0,\quad \zeta=e^{i\theta}\in A^+.$$

The argument $\arg\frac{re^{i\theta}f'(re^{i\theta},t_0)}{f(re^{i\theta},t_0)}$, $e^{i\theta}\in A^+$, attains its maximum on $r\in[0,1]$ at $r=1$. Hence,

$$\frac{\partial}{\partial r}\arg\frac{re^{i\theta}f'(re^{i\theta},t_0)}{f(re^{i\theta},t_0)}\bigg|_{r=1}\ge 0.$$

We calculate

$$\operatorname{Re}\left[1+\frac{\zeta f''(\zeta,t_0)}{f'(\zeta,t_0)}-\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}\right]=0, \tag{2.4}$$

$$\operatorname{Im}\left[1+\frac{\zeta f''(\zeta,t_0)}{f'(\zeta,t_0)}-\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}\right]\ge 0, \tag{2.5}$$

where $\zeta\in A^+$.

Let us represent the derivative

$$\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}=\operatorname{Im}\frac{\partial}{\partial t}\log\frac{f'(\zeta,t)}{f(\zeta,t)}=\operatorname{Im}\left(\frac{\frac{\partial}{\partial t}f'(\zeta,t)}{f'(\zeta,t)}-\frac{\frac{\partial}{\partial t}f(\zeta,t)}{f(\zeta,t)}\right). \tag{2.6}$$

Now we differentiate the Polubarinova-Galin equation (1.1) with respect to θ as

$$\operatorname{Im}\left(\overline{f'(\zeta,t)}\frac{\partial}{\partial t}f'(\zeta,t)-\overline{\zeta f'(\zeta,t)}\dot f(\zeta,t)-\overline{\zeta^2 f''(\zeta,t)}\dot f(\zeta,t)\right)=0,\quad \zeta=e^{i\theta}. \tag{2.7}$$

This equality is equivalent to

$$\begin{aligned}|f'(\zeta,t)|^2\operatorname{Im}&\left(\frac{\frac{\partial}{\partial t}f'(\zeta,t)}{f'(\zeta,t)}-\frac{\frac{\partial}{\partial t}f(\zeta,t)}{f(\zeta,t)}\right)\\ &=\operatorname{Im}\left[\overline{\zeta f'(\zeta,t)}\dot f(\zeta,t)\left(\overline{\left(\frac{\zeta f''(\zeta,t)}{f'(\zeta,t)}\right)}-\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}+1\right)\right].\end{aligned}$$

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Substituting (1.1) and (2.4) in the latter expression, we have

$$\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\Big|_{\zeta\in A^+,t=t_0}=\frac{-1}{|f'(\zeta,t_0)|^2}\operatorname{Im}\left(\frac{\zeta f'(\zeta,t_0)}{f(\zeta,t_0)}+\frac{\zeta f''(\zeta,t_0)}{f'(\zeta,t_0)}\right).$$

The right-hand side of this equality is continuous on A^+ and strictly negative because of (2.3) and (2.5). Therefore,

$$\max_{\zeta\in A^+}\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\Big|_{t=t_0}=-\delta<0.$$

There exists a neighborhood $A^+(\delta)$ on the unit circle of A^+ such that $A^+(\delta)$ and A^- do not intersect and

$$\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\Big|_{\zeta\in A^+(\delta),t=t_0}<-\frac{\delta}{2}.$$

There is a positive number σ such that

$$\max_{\zeta\in\partial U\setminus A^+(\delta)}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\Big|_{t=t_0}=\frac{\alpha\pi}{2}-\sigma.$$

We choose such $s>0$ that

(i) $t_0+s<T$;

(ii) $\frac{\partial}{\partial t}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\Big|_{\zeta\in A^+(\delta)}<0, \quad t\in[t_0,t_0+s]$;

(iii) $\max_{\zeta\in\partial U\setminus A^+(\delta)}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\le\frac{\alpha\pi}{2}-\frac{\sigma}{2}, \quad t\in[t_0,t_0+s]$.

The condition (ii) implies that

$$\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}<\frac{\alpha\pi}{2}, \quad t\in(t_0,t_0+s],\ \zeta\in A^+(\delta).$$

Thus, the condition (iii) yields

$$\alpha^+(t):=\max_{\zeta\in\partial U}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}<\frac{\alpha\pi}{2}=\alpha(t_0), \quad \text{for all } t\in(t_0,t_0+s].$$

This means that $\alpha^+(t)$ is strictly decreasing in $[0,T)$.

If the set $A^-\neq\emptyset$, then we can define the function

$$\alpha^-(t):=-\min_{\zeta\in\partial U}\arg\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}.$$

Similar argumentation shows that $\alpha^-(t)$ is strictly decreasing.

If $A^-=\emptyset$ (or $A^+=\emptyset$), then $\alpha(t)=\alpha^+(t)$ (or $=\alpha^-(t)$) for $t\in[t_0,t_0+s]$, s sufficiently small.

We set the function $\alpha(t)=\max\{\alpha^+(t),\alpha^-(t)\}$ in the case $A^+\neq\emptyset$ and $A^-\neq\emptyset$. This function $\alpha(t)$ is strictly decreasing, and the proof is complete. □

We remark that a similar result, with estimates of $\alpha(t)$, has been independently obtained by O. Kuznetsova [10].

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3. Infinite lifetime

In this section first we will prove that if the classical solution to (1.1) exists during the time interval $[0,T)$, then the limiting function $\lim_{t\to T-0} f(\zeta,t) \equiv f(\zeta,T)$ is analytic in some neighbourhood of the unit disk U. Here the limit is taken with respect to the uniform convergence on compacts of the unit disk U. It exists because $f(\zeta,t)$ is a subordination chain, and due to the Carathéodory Kernel Theorem. Then we will obtain the main result about the infinite lifetime.

Lemma 3.1. *Let the classical solution to* (1.1) *exist during the time interval* $[0,T)$, $0 < T < \infty$, $\Omega(t) = f(U,t)$, *and let the initial function* $f(\zeta,0)$ *be analytic and univalent in a neighbourhood of the closure of the unit disk* U. *Then, there is* $\eta > 0$ *such that the function* $f(\zeta,t)$ *is analytic in* $U_{1+\eta} = \{\zeta : |\zeta| < 1+\eta\}$ *for all* $t \in [0,T]$, *univalent in* U, *and possibly* $f(\zeta,T)$ *has a vanishing derivative at some points of the unit circle* ∂U *or is not univalent on* ∂U. *It follows that* $\Omega(T) \equiv f(U,T)$ *is a simply connected domain with an analytic boundary* $\Gamma(T) = \partial\Omega(T)$ *with possible analytic singularities in the form of finitely many cusps and double points.*

Proof. By the Carathéodory Kernel Theorem the domain

$$\Omega(T) = \bigcup_{t\in[0,T)} \Omega(t)$$

is just the same as in the formulation of the lemma, and $\Omega(T)$ is a simply connected domain.

It is well known (see, e.g., [6, 15, 16]) and easily verified directly that for any analytic and integrable function Φ in $\Omega(T)$ the identity

$$\iint_{\Omega(t)} \Phi(z)d\sigma_z = \iint_{\Omega(0)} \Phi(z)d\sigma_z + 2\pi t\Phi(0) \tag{3.1}$$

holds for $t \in [0,T)$. We note also that since the normal velocity on the boundary never vanishes, we have the strict monotonicity of the subordination chain of domains:

$$\overline{\Omega(s)} \subset \Omega(t) \quad \text{for } s < t \text{ and } s,t \in (0,T). \tag{3.2}$$

Letting $t \to T$, we see that (3.1) and (3.2) hold for $t = T$, i.e., for $\Omega(T)$ as well.

In order to give a rigorous proof of the statement about the properties of $f(\zeta,T)$ we analytically extend the mapping $f(\zeta,T)$ following the scheme proposed in [6, Theorem 9]; see also [17, Theorem 1.7]. Let us choose the function $\Phi(z) = (z-w)^{-1}$ for $w \notin \Omega(t)$, $t \in (0,T]$, in (3.1). This gives

$$\hat{\chi}_{\Omega(t)} = \hat{\chi}_{\Omega(0)} + 2\pi t\hat{\delta}, \tag{3.3}$$

where

$$\hat{\mu}(w) = -\frac{1}{\pi}\iint_{\Omega(t)} \frac{d\mu(z)}{z-w}$$

denotes the Cauchy transformation of a measure μ, normalized so that $\partial\hat{\mu}/\partial\bar{w} = \mu(w)$, χ stands for the characteristic function of the corresponding domain, and δ is the Dirac distribution.

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We define a function $S(z,t)$ by

$$S(z,t) = \bar{z} - \hat{\chi}_{\Omega(t)}(z) + \hat{\chi}_{\Omega(0)}(z) + \frac{2t}{z}$$

for $z \in \overline{\Omega(t)} \setminus \overline{\Omega(0)}$, for each $t \in (0,T]$. Then $S(z,t)$ as a function of z is continuous in $\overline{\Omega(t)} \setminus \overline{\Omega(0)}$, analytic in $\Omega(t) \setminus \overline{\Omega(0)}$ (one easily calculates that $\partial S/\partial \bar{z} = 0$), and $S(z,t) = \bar{z}$ on $\Gamma(t) = \partial\Omega(t)$ by (3.3). Thus, $S(z,t)$ is a one-sided Schwarz function of $\Gamma(t)$ [16, 18]. The transformation $z \mapsto \overline{S(z,t)}$ is a one-sided reflection over $\Gamma(t)$, which we will use to extend $f(\zeta,t)$, $t \in (0,T]$. Let us fix $t \in (0,T]$. We set $r(t) = \max_{z\in\partial\Omega(0)} |f^{-1}(z,t)|$ and consider a point ζ, $r(t) < |\zeta| < 1$. The point $1/\bar{\zeta}$ is its reflection through ∂U.

We define the function $f(\zeta,t)$ in a neighbourhood of the unit circle $1 < |\zeta| < 1/r(t)$ outside U by $f(1/\bar{\zeta},t) \equiv \overline{S(f(\zeta,t),t)}$. This defines f analytically in the annulus $1 < |\zeta| < 1/r(t)$. Across ∂U we have a certain form of continuity because of the continuity of $S(z,t)$. Indeed, as $|\zeta| \to 1$ with $\zeta \in U$ we have

$$|f(\zeta,t) - f(1/\bar{\zeta},t)| = |f(\zeta,t) - \overline{S(f(\zeta,t),t)}| \to 0, \tag{3.4}$$

where $z = \overline{S(z,t)}$ on $\Gamma(t)$, and therefore, given $\varepsilon > 0$, we have $|z - \overline{S(z,t)}| < \varepsilon$ for $z \in \Omega(t)$ in some neighbourhood of $\Gamma(t)$. By now the function $f(\zeta,t)$ is defined in U as well as in the annulus $1 < |\zeta| < 1/r(t)$, and hence, almost everywhere in the disk $|\zeta| < 1/r(t)$. Let us prove that the distributional derivative $\partial f(\zeta,t)/\partial\bar{\zeta}$ vanishes in $|\zeta| < 1/r(t)$, using (3.4). Obviously, we must verify this across the circle $|\zeta| = 1$. Given a test function φ with compact support in $|\zeta| < 1/r(t)$, we have

$$\begin{aligned}
\langle \frac{\partial f}{\partial\bar{\zeta}}, \varphi\rangle &= -\iint\limits_{\mathbb{C}} f(\zeta,t)\frac{\partial\varphi}{\partial\bar{\zeta}} d\sigma_\zeta \\
&= -\frac{1}{2i}\iint\limits_{U} f(\zeta,t)\frac{\partial\varphi}{\partial\bar{\zeta}} d\bar{\zeta}d\zeta - \frac{1}{2i}\iint\limits_{|\zeta|>1} f(\zeta,t)\frac{\partial\varphi}{\partial\bar{\zeta}} d\bar{\zeta}d\zeta \\
&= -\frac{1}{2i}\lim_{\varepsilon\downarrow 0}\left(\int\limits_{|\zeta|=1-\varepsilon} f(\zeta,t)\varphi(\zeta)\,d\zeta - \int\limits_{|\zeta|=1+\varepsilon} f(\zeta,t)\varphi(\zeta)\,d\zeta\right) \\
&= -\frac{1}{2i}\lim_{\varepsilon\downarrow 0}\int\limits_{|\zeta|=1-\varepsilon} (f(\zeta,t) - f(1/\bar{\zeta},t))\varphi(\zeta)\,d\zeta = 0.
\end{aligned}$$

In the above curve integrals we take the counterclockwise direction on the circles. Thus, the function $f(\zeta,t)$ is analytic in the disk $|\zeta| < 1/r(t)$.

For any pair of numbers s,t such that $0 < s < t \le T$, we have that the function $h(\zeta,s,t) \equiv f^{-1}(f(\zeta,s),t)$ maps the unit disk into itself and $h(0,s,t) \equiv 0$. A simple application of the Schwarz Lemma to the function h shows that

$$f^{-1}(\Omega(0),t) \subset U_{r(s)}.$$

Therefore, $r(t) \le r(s)$. For a sufficiently small $\varepsilon > 0$ we choose δ such that the solution $f(\zeta,t)$ is analytic in the disk $U_{1+\varepsilon}$ for any $t \in [0,\delta)$. Then we define

$$\eta = \min\left(\varepsilon, \frac{1}{r(\delta)} - 1\right).$$

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Thus, we have defined the function $f(\zeta,t)$ in the disk $U_{1+\eta}$ for all $t\in[0,T]$. To get a pointwise definition of f on the circle $|\zeta|=1$ one may use the Cauchy integral formula for the disk $|\zeta|<1+\eta/2$. This finishes the proof of the lemma. □

Lemma 3.2. *Let $f_0\in S^*$ be analytic and univalent in a neighbourhood of the closure $\bar{U}$ of the unit disk U. If the solution $f(\zeta,t)$ to the Polubarinova-Galin equation* (1.1) *exists during the time interval $[0,T)$, then it forms a subordination chain of starlike functions such that the limiting domain $\Omega(T)$ has a smooth analytic boundary.*

Proof. By the assumptions on f_0 the initial domain $\Omega(0)$ has a smooth analytic boundary and the solution $f(\zeta,t)$ to (1.1) exists locally in time $t\in[0,s)$. The function $f_0\in S^*_\alpha$ for some $\alpha\in(0,1]$. The function $f(\zeta,t)$ belongs to the class $S^*_{\alpha(t)}$ with $\alpha(t)<1$ for any $t\in(0,s)$, due to Theorem 2.1. Therefore, we can prove Lemma 3.2 choosing $f(\zeta,t)$ with some $t\in(0,s)$ as an initial mapping, $f(\zeta,t)\in S^*_{\alpha(t)}$ with $\alpha(t)<1$. We prove that $\partial\Omega(T)$ does not contain a cusp or a double point on its boundary.

Define the limiting function $f(\zeta,T)=\lim\limits_{t\to T-0}f(\zeta,t)$, where the limit is taken locally uniformly in U. The function $f(\zeta,T)$ is univalent, strongly starlike of order $\alpha(T)=\lim\limits_{t\to T-0}\alpha(t)<1$, and has a continuous extension on $\bar{U}$. According to the geometric characterization of the class $S^*_{\alpha(T)}$, the boundary of the domain $\Omega(T)=f(U,T)$ is reachable by the radial external angles $\pi(1-\alpha)$, which implies that there is no cusp or double point on the boundary of $\Omega(T)$. This completes the proof. □

Lemma 3.3. *Let $f(\zeta,t)$ be the classical solution to the equation* (1.1) *that exists during the time interval $[0,T)$ with a starlike initial mapping f_0 as in Lemma 3.2. Then, there exists $\varepsilon>0$ such that the classical solution exists during the time interval $[0,T+\varepsilon)$.*

Proof. The limiting domain $\Omega(T)$ is simply connected and has an analytic boundary. The limiting mapping $f(\zeta,T)$ is analytic in a neighbourhood of $\bar{U}$ by Lemma 3.1 and strongly starlike of order $\alpha(T)<1$ by Lemma 3.2. Therefore, there exists an $\eta>0$ such that $f(\zeta,T)$ is starlike and univalent in the disk $|\zeta|<1+\eta$. Let us construct the subordination chain of mappings $f_2(\zeta,t)$ satisfying the Polubarinova-Galin equation (1.1) with the initial data $f_2(\zeta,0)\equiv f(\zeta,T)$. The classical solution exists and is unique locally in time, say $t\in[0,\varepsilon)$. Moreover, we have

$$\lim_{t\to T-0}f(\zeta,t)=\lim_{t\to 0+0}f_2(\zeta,t)=f(\zeta,T)$$

and

$$\lim_{t\to T-0}f'(\zeta,t)=\lim_{t\to 0+0}f_2'(\zeta,t)=f'(\zeta,T)$$

locally uniformly in $U_{1+\eta}$. We rewrite the equation (1.1) in U using the Schwarz kernel as

$$\dot{f}(\zeta,t)=\zeta f'(\zeta,t)\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|f'(e^{i\theta},t)|^2}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\theta,\quad t\in[0,T),\ \ |\zeta|<1.$$

A similar equation is valid for the chain $f_2(\zeta,t)$ in the time interval $[0,\varepsilon)$. Taking the limit in the above equation as $t\to T-0$ we observe that there exists the one-sided limit $\dot{f}(\zeta,T-0)$. Similarly, the one-sided limit $\dot{f}_2(\zeta,0+0)$ exists, and

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they are equal. Let us define $f(\zeta, t) \equiv f_2(\zeta, t - T)$ in the interval $t \in [T, T + \varepsilon)$. The above observations yield that the function thus extended is continuous in the interval $t \in [0, T + \varepsilon)$, analytic, univalent and starlike in some neighbourhood of $\bar{U}$. Moreover, it is differentiable at the point $t = T$, and being extended onto the unit circle, satisfies the equation (1.1). Thus, it is a unique classical solution in the interval $t \in [0, T + \varepsilon)$, and the lemma is proved. □

Theorem 3.4. *Starting with a starlike phase domain Ω_0 with an analytic boundary, the lifetime of the classical Hele-Shaw starlike dynamics $\Omega(t)$ is infinite.*

Proof. Indeed, if the classical solution exists during the finite interval $t \in [0, T)$ and does not exist in $t \in [T, T + \varepsilon)$ for any $\varepsilon > 0$, then this contradicts Lemma 3.3. □

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