

On The Zeros of Polynomials

M. H. Gulzar

Department of Mathematics University of Kashmir, Srinagar, 190006, India

Abstract: In this paper we extend Enestrom -Kakeya theorem to a large class of polynomials with complex coefficients by putting less restrictions on the coefficients . Our results generalise and extend many known results in this direction.

(AMS) Mathematics Subject Classification (2010) : 30C10, 30C15

Key-Words and Phrases: Polynomials, Zeros, Bounds

I. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree n . A classical result due to Enestrom and Kakeya [9] concerning the bounds for the moduli of the zeros of polynomials having positive coefficients is often stated as in the following theorem(see [9]) :

Theorem A (Enestrom-Kakeya) : Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$0 \leq a_1 \leq a_2 \leq \dots \leq a_n.$$

Then $P(z)$ has all its zeros in the closed unit disk $|z| \leq 1$.

In the literature there exist several generalisations of this result (see [1],[3],[4],[8],[9]). Recently Aziz and Zargar [2] relaxed the hypothesis in several ways and proved

Theorem B: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,

$$ka_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0.$$

Then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq \frac{ka_n + |a_0| - a_0}{|a_n|}.$$

For polynomials ,whose coefficients are not necessarily real, Govil and Rehman [6] proved the following generalisation of Theorem A:

Theorem C: If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$,

such that

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0 \geq 0,$$

where $\alpha_n > 0$, then $P(z)$ has all its zeros in

$$|z| \leq 1 + \left(\frac{2}{\alpha_n}\right) \left(\sum_{j=0}^n |\beta_j|\right).$$

More recently, Govil and Mc-tume [5] proved the following generalisations of Theorems B and C:

Theorem D: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$.

If for some $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

then $P(z)$ has all its zeros in

$$|z + k - 1| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

Theorem E: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$.

If for some $k \geq 1$,

$$k\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \beta_0,$$

then $P(z)$ has all its zeros in

$$\left| z + k - 1 \right| \leq \frac{k\beta_n - \beta_0 + |\beta_0| + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}.$$

M.H.Gulzar [7] proved the following generalisations of Theorems D and E.

Theorem F: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$.

If for some real number $\rho \geq 0$,

$$\rho + \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

Theorem G: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$.

If for some real number $\rho \geq 0$,

$$\rho + \beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \beta_0,$$

then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{\rho}{\beta_n} \right| \leq \frac{\rho + \beta_n + |\beta_0| - \beta_0 + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}.$$

The aim of this paper is to give generalizations of Theorem F and G under less restrictive conditions on the coefficients. More precisely we prove the following :

Theorem 1: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$. If

for some real numbers $\rho \geq 0, 0 < \sigma \leq 1$,

$$\rho + \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \sigma\alpha_0,$$

then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

Remark 1: Taking $\sigma = 1$ in Theorem 1, we get Theorem F. Taking $\rho = (k-1)\alpha_n, \sigma = 1$, Theorem 1 reduces to Theorem D and taking $\rho = 0, \alpha_0 > 0$ and $\sigma = 1$, we get Theorem C.

Applying Theorem 1 to $P(tz)$, we obtain the following result:

Corollary 1 : Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$.

If for some real numbers $\rho \geq 0, 0 < \sigma \leq 1$ and $t > 0$,

$$\rho + t^n \alpha_n \geq t^{n-1} \alpha_{n-1} \geq \dots \geq t\alpha_1 \geq \sigma\alpha_0,$$

then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{\rho}{t^{n-1}\alpha_n} \right| \leq \frac{\rho + t^n \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^n |\beta_j| t^j}{t^{n-1}|\alpha_n|}.$$

In Theorem 1, if we take $\alpha_0 \geq 0$, we get the following result:

Corollary 2 : Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$.

If for some real numbers $\rho \geq 0$,

$$0 < \sigma \leq 1,$$

$$\rho + \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \sigma \alpha_0 \geq 0,$$

then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq 1 + \frac{\rho + 2(1-\sigma)\alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

If we take $\rho = \alpha_{n-1} - \alpha_n \geq 0$ in Theorem 1, we get the following result:

Corollary 3 : Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$,

such that

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \sigma \alpha_0 > 0.$$

Then P(z) has all its zeros in

$$\left| z + \frac{\alpha_{n-1}}{\alpha_n} - 1 \right| \leq \frac{\alpha_{n-1} + 2(1-\sigma)\alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

Taking $\sigma = 1$ in Cor.3, we get the following result:

Corollary 4 : Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$,

such that

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0 > 0.$$

Then P(z) has all its zeros in

$$\left| z + \frac{\alpha_{n-1}}{\alpha_n} - 1 \right| \leq \frac{\alpha_{n-1} + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

If we apply Theorem 1 to the polynomial $-iP(z)$, we easily get the following result:

Theorem 2: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$,

If for some real numbers $\rho \geq 0, 0 < \sigma \leq 1$,

$$\rho + \beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \sigma \beta_0,$$

then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{\beta_n} \right| \leq \frac{\rho + \beta_n - \sigma(|\beta_0| + \beta_0) + 2|\beta_0| + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}.$$

On applying Theorem 2 to the polynomial P(tz), one gets the following result:

Corollary 5 : Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j,$

$j=0,1,2,\dots,n$. If for some real numbers $\rho \geq 0, 0 < \sigma \leq 1$ and $t > 0$,

$$\rho + t^n \beta_n \geq t^{n-1} \beta_{n-1} \geq \dots \geq t \beta_1 \geq \beta_0,$$

then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{\rho}{t^{n-1} \beta_n} \right| \leq \frac{\rho + t^n \beta_n - \sigma(|\beta_0| + \beta_0) + 2|\beta_0| + 2 \sum_{j=0}^n |\alpha_j| t^j}{t^{n-1} |\beta_n|}.$$

II. Proofs of the Theorems

Proof of Theorem 1.

Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i(\beta_n - \beta_{n-1})z^n \\ &\quad + \dots + i(\beta_1 - \beta_0)z + i\beta_0 \\ &= -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \dots + (\alpha_1 - \alpha_0)z \\ &\quad + (\alpha_0 - \alpha_0)z + \alpha_0 + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \dots + (\beta_1 - \beta_0)z + \beta_0\}. \end{aligned}$$

Then

$$\begin{aligned} |F(z)| &= \left| -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \dots + (\alpha_1 - \alpha_0)z \right. \\ &\quad \left. + (\alpha_0 - \alpha_0)z + \alpha_0 + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \dots + (\beta_1 - \beta_0)z + \beta_0\} \right| \\ &\geq |z|^n \left\{ \begin{aligned} &|\alpha_n z + \rho| - |\rho + \alpha_n - \alpha_{n-1}| - |\alpha_0| \frac{1}{|z|^n} - |\alpha_1 - \alpha_0| \frac{1}{|z|^{n-1}} \\ &-(1-\sigma)|\alpha_0| - \sum_{j=2}^{n-1} |\alpha_j - \alpha_{j-1}| \frac{1}{|z|^{n-j}} \end{aligned} \right\} \\ &\quad - \left| -\beta_n z^{n+1} + \dots + (\beta_1 - \beta_0)z + \beta_0 \right|. \end{aligned}$$

Thus, for $|z| > 1$,

$$\begin{aligned} |F(z)| &> |z|^n \left\{ \begin{aligned} &|\alpha_n z + \rho| - (\rho + \alpha_n - \alpha_{n-1}) - |\alpha_0| - (\alpha_{n-1} - \alpha_{n-2}) - \dots - (\alpha_2 - \alpha_1) \\ &-(\alpha_1 - \alpha_0) - (1-\sigma)|\alpha_0| \end{aligned} \right\} \\ &\quad - (|\beta_n| + |\beta_0|) - \sum_{j=1}^n (|\beta_j| + |\beta_{j-1}|) \\ &= |z|^n \left[|\alpha_n z + \rho| - \left\{ \rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2 \sum_{j=0}^n |\beta_j| \right\} \right] \\ &> 0 \end{aligned}$$

if

$$|\alpha_n z + \rho| > \rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2 \sum_{j=1}^n |\beta_j|.$$

Hence all the zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2 \sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Hence it follows that all the zeros of $F(z)$ lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

Since all the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

This completes the proof of Theorem 1.

REFERENCES

- [1] N. Anderson , E. B. Saff , R. S.Verga , An extension of the Enestrom-Kakeya Theorem and its sharpness, SIAM. Math. Anal. , 12(1981), 10-22.
- [2] A.Aziz and B.A.Zargar, Some extensions of the Enestrom-Kakeya Theorem , Glasnik Matematiki, 31(1996) , 239-244.
- [3] K.K.Dewan, N.K.Govil, On the Enestrom-Kakeya Theorem, J.Approx. Theory, 42(1984) , 239-244.
- [4] R.B.Gardner, N.K. Govil, Some Generalisations of the Enestrom-Kakeya Theorem , Acta Math. Hungar Vol.74(1997), 125-134.
- [5] N.K.Govil and G.N.Mc-tume, Some extensions of the Enestrom-Kakeya Theorem , Int.J.Appl.Math. Vol.11,No.3,2002, 246-253.
- [6] N.K.Govil and Q.I.Rehman, On the Enestrom-Kakeya Theorem, Tohoku Math. J.,20(1968) , 126-136.
- [7] M.H.Gulzar, On the Location of Zeros of a Polynomial, Anal. Theory. Appl., vol 28, No.3(2012)
- [8] A. Joyal, G. Labelle, Q.I. Rahman, On the location of zeros of polynomials, Canadian Math. Bull.,10(1967) , 55-63.
- [9] M. Marden , Geometry of Polynomials, IInd Ed.Math. Surveys, No. 3, Amer. Math. Soc. Providence,R. I,1996.
- [10] G.V. Milovanoic , D.S. Mitrinovic and Th. M. Rassias, Topics in Polynomials, Extremal Problems, Inequalities, Zeros, World Scientific Publishing Co. Singapore,New York, London, Hong-Kong, 1994.