

# On The Enestrom –Kakeya Theorem and Its Generalisations

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## Abstract

Many extensions of the Enestrom –Kakeya Theorem are available in the literature. In this paper we prove some results which generalize some known results .

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**Key-Words and Phrases:** Complex number, Polynomial , Zero, Enestrom – Kakeya Theorem

## 1. Introduction And Statement Of Results

The Enestrom –Kakeya Theorem (see[7]) is well known in the theory of the distribution of zeros of polynomials and is often stated as follows:

**Theorem A:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree n whose coefficients satisfy

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk  $|z| \leq 1$ .

In the literature there exist several generalizations and extensions of this result. Joyal et al [6] extended it to polynomials with general monotonic coefficients and proved the following result:

**Theorem B:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree n whose coefficients satisfy

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 .$$

Then P(z) has all its zeros in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

Aziz and zargar [1] generalized the result of Joyal et al [6] as follows:

**Theorem C:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree n such that for some  $k \geq 1$

$$ka_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 .$$

Then P(z) has all its zeros in

$$|z + k - 1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

For polynomials ,whose coefficients are not necessarily real, Govil and Rahman [2] proved the following generalization of Theorem A:

**Theorem C:** If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree n with  $\text{Re}(a_j) = \alpha_j$  and  $\text{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$ ,

such that

$$\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0 \geq 0 ,$$

where  $\alpha_n > 0$  , then P(z) has all its zeros in

$$|z| \leq 1 + \left(\frac{2}{\alpha_n}\right) \left(\sum_{j=0}^n |\beta_j|\right).$$

Govil and Mc-tume [3] proved the following generalisations of Theorems B and C:

**Theorem D:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  with  $\operatorname{Re}(a_j) = \alpha_j$  and  $\operatorname{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$ .

If for some  $k \geq 1$ ,

$$k\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

then  $P(z)$  has all its zeros in

$$|z+k-1| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

**Theorem E:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  with  $\operatorname{Re}(a_j) = \alpha_j$  and  $\operatorname{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$ .

If for some  $k \geq 1$ ,

$$k\beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \beta_0,$$

then  $P(z)$  has all its zeros in

$$|z+k-1| \leq \frac{k\beta_n - \beta_0 + |\beta_0| + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}.$$

M. H. Gulzar [4] proved the following generalizations of Theorems D and E:

**Theorem F:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  with  $\operatorname{Re}(a_j) = \alpha_j$  and  $\operatorname{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$ .

If for some real number  $\rho \geq 0$ ,

$$\rho + \alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

then  $P(z)$  has all its zeros in the disk

$$\left|z + \frac{\rho}{\alpha_n}\right| \leq \frac{\rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^n |\beta_j|}{|\alpha_n|}.$$

**Theorem G:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  with  $\operatorname{Re}(a_j) = \alpha_j$  and  $\operatorname{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$ .

If for some real number  $\rho \geq 0$ ,

$$\rho + \beta_n \geq \beta_{n-1} \geq \dots \geq \beta_1 \geq \beta_0,$$

then  $P(z)$  has all its zeros in the disk

$$\left|z + \frac{\rho}{\beta_n}\right| \leq \frac{\rho + \beta_n + |\beta_0| - \beta_0 + 2\sum_{j=0}^n |\alpha_j|}{|\beta_n|}.$$

In this paper we give generalization of Theorems F and G. In fact, we prove the following:

**Theorem 1:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  with  $\operatorname{Re}(a_j) = \alpha_j$  and  $\operatorname{Im}(a_j) = \beta_j, j=0,1,2,\dots,n$ . If

for some real numbers  $\lambda, \rho \geq 0, 1 \leq k \leq n, \alpha_{n-k} \neq 0$ ,

$$\rho + \alpha_n \geq \alpha_{n-1} \geq \dots \alpha_{n-k+1} \geq \lambda \alpha_{n-k} \geq \alpha_{n-k-1} \geq \dots \geq \alpha_1 \geq \alpha_0,$$

and  $\alpha_{n-k-1} > \alpha_{n-k}$ , then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|},$$

and if  $\alpha_{n-k} > \alpha_{n-k+1}$ , then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (1 - \lambda)\alpha_{n-k} + |1 - \lambda| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|}$$

**Remark 1:** Taking  $\lambda = 1$ , Theorem 1 reduces to Theorem F.

Taking  $k=n$ , and  $\lambda = 1$  in Theorem 1, we get a result due to M.H. Gulzar [5, Theorem 1].

If  $a_j$  are real i.e.  $\beta_j = 0$  for all j, we get the following result:

**Corollary 1:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree n. If for some real numbers  $\lambda, \rho \geq 0$ ,

$$1 \leq k \leq n, a_{n-k} \neq 0,$$

$$\rho + a_n \geq a_{n-1} \geq \dots \geq a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1} \geq \dots \geq a_1 \geq a_0,$$

and  $a_{n-k-1} > a_{n-k}$ , then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + a_n + (\lambda - 1)a_{n-k} + |\lambda - 1| |a_{n-k}| + |a_0| - a_0}{|a_n|} ..$$

and if  $a_{n-k} > a_{n-k+1}$ , then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + a_n + (1 - \lambda)a_{n-k} + |1 - \lambda| |a_{n-k}| + |a_0| - a_0}{|a_n|}$$

If we apply Theorem 1 to the polynomial  $-iP(z)$ , we easily get the following result:

**Theorem 2:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree n with  $\text{Re}(a_j) = \alpha_j$  and  $\text{Im}(a_j) = \beta_j, j=0,1, 2, \dots, n$ .

If for some real numbers  $\lambda, \rho \geq 0, 1 \leq k \leq n, \beta_{n-k} \neq 0$ ,

$$\rho + \beta_n \geq \beta_{n-1} \geq \dots \geq \beta_{n-k+1} \geq \lambda \beta_{n-k} \geq \beta_{n-k-1} \geq \dots \geq \beta_1 \geq \beta_0,$$

and  $\beta_{n-k-1} > \beta_{n-k}$ , then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \beta_n + (\lambda - 1)\beta_{n-k} + |\lambda - 1| |\beta_{n-k}| + |\beta_0| - \beta_0 + 2 \sum_{j=0}^n |\alpha_j|}{|a_n|},$$

and if  $\beta_{n-k} > \beta_{n-k+1}$ , then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \beta_n + (1 - \lambda)\beta_{n-k} + |1 - \lambda| |\beta_{n-k}| + |\beta_0| - \beta_0 + 2 \sum_{j=0}^n |\alpha_j|}{|a_n|}$$

**Remark 2:** Taking  $\lambda = 1$ , Theorem 2 reduces to Theorem G.

**Theorem 3:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  such that for some real numbers  $\lambda, \rho \geq 0$ ,  $1 \leq k \leq n, a_{n-k} \neq 0$ , and  $\beta$ ,

$$|\rho + a_n| \geq |a_{n-1}| \geq \dots \geq |a_{n-k+1}| \geq \lambda |a_{n-k}| \geq |a_{n-k-1}| \geq \dots \geq |a_1| \geq |a_0|$$

and

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, \dots, n.$$

If  $|a_{n-k-1}| > |a_{n-k}|$  (i.e.  $\lambda > 1$ ), then  $P(z)$  has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos \alpha + \sin \alpha) - |a_{n-k}|(\cos \alpha - \sin \alpha - \lambda \cos \alpha - \lambda \sin \alpha - \lambda + 1) - |a_0|(\cos \alpha - \sin \alpha - 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|]}{|a_n|}$$

If  $|a_{n-k}| > |a_{n-k+1}|$  (i.e.  $\lambda < 1$ ), then  $P(z)$  has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos \alpha + \sin \alpha) + |a_{n-k}|(\cos \alpha + \sin \alpha - \lambda \cos \alpha + \lambda \sin \alpha + 1 - \lambda) + |a_0|(\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|]}{|a_n|}$$

**Remark 3:** Taking  $\lambda = 1$  in Theorem 3, we get the following result:

**Corollary 2:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$ . If for some real number  $\rho \geq 0$ ,

$$|\rho + a_n| \geq |a_{n-1}| \geq \dots \geq |a_1| \geq |a_0|,$$

then  $P(z)$  has all its zeros in the disk

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n|(\cos \alpha + \sin \alpha) - |a_0|(\cos \alpha - \sin \alpha - 1) + 2 \sin \alpha \sum_{j=1}^{n-1} |a_j|]}{|a_n|}.$$

**Remark 4:** Taking  $\rho = (k-1)a_n, k \geq 1$ , we get a result of Shah and Liman [8, Theorem 1].

## 2. Lemmas

For the proofs of the above results, we need the following results:

**Lemma 1:** Let  $P(z) = \sum_{j=0}^n a_j z^j$  be a polynomial of degree  $n$  with complex coefficients such that

$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, \dots, n$ , for some real  $\beta$ . Then for some  $t > 0$ ,

$$|ta_j - a_{j-1}| \leq [t|a_j| - |a_{j-1}|] \cos \alpha + [t|a_j| + |a_{j-1}|] \sin \alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [2].

**Lemma 2.** If  $p(z)$  is regular,  $p(0) \neq 0$  and  $|p(z)| \leq M$  in  $|z| \leq 1$ , then the number of zeros of  $p(z)$  in  $|z| \leq \delta, 0 < \delta < 1$ , does

not exceed  $\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|p(0)|}$  (see [9], p171).

### 3. Proofs Of Theorems

**Proof of Theorem 1:** Consider the polynomial

$$\begin{aligned}
 F(z) &= (1-z)P(z) \\
 &= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0) \\
 &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} \\
 &\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots + (a_1 - a_0)z + a_0 \\
 &= -(\alpha_n + i\beta_n)z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\
 &\quad + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots + (\alpha_1 - \alpha_0)z + \alpha_0 \\
 &\quad + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j + i\beta_0
 \end{aligned}$$

If  $\alpha_{n-k-1} > \alpha_{n-k}$ , then  $\alpha_{n-k-1} > \alpha_{n-k}$ , and we have

$$\begin{aligned}
 F(z) &= -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\
 &\quad + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (\lambda-1)\alpha_{n-k}z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots \\
 &\quad + (\alpha_1 - \alpha_0)z + \alpha_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j + i\beta_0.
 \end{aligned}$$

For  $|z| > 1$ ,

$$\begin{aligned}
 |F(z)| &\geq \left| a_n z^{n+1} + \rho z^n \right| - \left| (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \dots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \right. \\
 &\quad \left. + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + (\lambda-1)\alpha_{n-k}z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots \right. \\
 &\quad \left. + (\alpha_1 - \alpha_0)z + \alpha_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j + i\beta_0 \right| \\
 &= |z|^n \left[ \left| a_n z + \rho \right| - \left| (\rho + \alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \dots + (\alpha_{n-k+1} - \alpha_{n-k})\frac{1}{z^{k-1}} \right. \right. \\
 &\quad \left. \left. + (\lambda\alpha_{n-k} - \alpha_{n-k-1})\frac{1}{z^k} - (\lambda-1)\alpha_{n-k}\frac{1}{z^k} + (\alpha_{n-k-1} - \alpha_{n-k-2})\frac{1}{k+1} + \dots \right. \right. \\
 &\quad \left. \left. + (\alpha_1 - \alpha_0)\frac{1}{z^{n-1}} + \frac{\alpha_0}{z^n} + i \sum_{j=1}^n (\beta_j - \beta_{j-1})\frac{1}{z^{n-j}} + i\frac{\beta_0}{z^n} \right| \right] \\
 &> |z|^n \left[ \left| a_n z + \rho \right| - \left\{ \left| \rho + \alpha_n - \alpha_{n-1} \right| + \left| \alpha_{n-1} - \alpha_{n-2} \right| + \dots + \left| \alpha_{n-k+1} - \alpha_{n-k} \right| \right. \right. \\
 &\quad \left. \left. + \left| \lambda\alpha_{n-k} - \alpha_{n-k-1} \right| + \left| \lambda-1 \right| \left| \alpha_{n-k} \right| + \left| \alpha_{n-k-1} - \alpha_{n-k-2} \right| + \dots + \left| \alpha_1 - \alpha_0 \right| \right. \right. \\
 &\quad \left. \left. + \left| \alpha_0 \right| + \sum_{j=1}^n \left| \beta_j - \beta_{j-1} \right| + \left| \beta_0 \right| \right\} \right] \\
 &\geq |z|^n \left[ \left| a_n z + \rho \right| - \left\{ \rho + \alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \dots + \alpha_{n-k+1} - \alpha_{n-k} + \lambda\alpha_{n-k} - \alpha_{n-k-1} \right. \right. \\
 &\quad \left. \left. + \left| \lambda-1 \right| \left| \alpha_{n-k} \right| + \left| \alpha_{n-k-1} - \alpha_{n-k-2} \right| + \dots + \alpha_1 - \alpha_0 + \left| \alpha_0 \right| + 2 \sum_{j=0}^n \left| \beta_j \right| \right\} \right] \\
 &= |z|^n \left[ \left| a_n z + \rho \right| - \left\{ \rho + \alpha_n + (\lambda-1)\alpha_{n-k} + \left| \lambda-1 \right| \left| \alpha_{n-k} \right| - \alpha_0 + \left| \alpha_0 \right| + 2 \sum_{j=0}^n \left| \beta_j \right| \right\} \right] \\
 &> 0
 \end{aligned}$$

if

$$|a_n z + \rho| > \rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1| |\alpha_{n-k}| - \alpha_0 + |\alpha_0| + 2 \sum_{j=0}^n |\beta_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|}$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|}$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|}$$

If  $\alpha_{n-k} > \alpha_{n-k+1}$ , then  $\alpha_{n-k} > \alpha_{n-k-1}$ , and we have

$$\begin{aligned} F(z) = & -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \dots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1} \\ & + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k}z^{n-k+1} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots \\ & + (\alpha_1 - \alpha_0)z + \alpha_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j + i\beta_0. \end{aligned}$$

For  $|z| > 1$ ,

$$\begin{aligned} |F(z)| & \geq \left| a_n z^{n+1} + \rho z^n \right| - \left| (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \dots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1} \right. \\ & \quad \left. + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k}z^{n-k+1} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots \right. \\ & \quad \left. + (\alpha_1 - \alpha_0)z + \alpha_0 + i \sum_{j=1}^n (\beta_j - \beta_{j-1})z^j + i\beta_0 \right| \\ & = |z|^n \left[ |a_n z + \rho| - \left| (\rho + \alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2}) \frac{1}{z} + \dots + (\alpha_{n-k+1} - \lambda\alpha_{n-k}) \frac{1}{z^{k-1}} \right. \right. \\ & \quad \left. \left. + (\alpha_{n-k} - \alpha_{n-k-1}) \frac{1}{z^k} - (1-\lambda)\alpha_{n-k} \frac{1}{z^{k-1}} + (\alpha_{n-k-1} - \alpha_{n-k-2}) \frac{1}{z^{k+1}} + \dots \right. \right. \\ & \quad \left. \left. + (\alpha_1 - \alpha_0) \frac{1}{z^{n-1}} + \frac{\alpha_0}{z^n} + i \sum_{j=1}^n (\beta_j - \beta_{j-1}) \frac{1}{z^{n-j}} + i \frac{\beta_0}{z^n} \right| \right] \\ & > |z|^n \left[ |a_n z + \rho| - \left\{ |\rho + \alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \dots + |\alpha_{n-k+1} - \lambda\alpha_{n-k}| \right. \right. \\ & \quad \left. \left. + |\alpha_{n-k} - \alpha_{n-k-1}| + |1-\lambda| |\alpha_{n-k}| + |\alpha_{n-k-1} - \alpha_{n-k-2}| + \dots + |\alpha_1 - \alpha_0| \right. \right. \\ & \quad \left. \left. + |\alpha_0| + \sum_{j=1}^n |\beta_j - \beta_{j-1}| + |\beta_0| \right\} \right] \\ & \geq |z|^n \left[ |a_n z + \rho| - \left\{ \rho + \alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \dots + \alpha_{n-k+1} - \lambda\alpha_{n-k} + \alpha_{n-k} - \alpha_{n-k-1} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + |1 - \lambda| |\alpha_{n-k}| + \alpha_{n-k-1} - \alpha_{n-k-2} + \dots + \alpha_1 - \alpha_0 + |\alpha_0| + 2 \sum_{j=0}^n |\beta_j| \} ] \\
 = & |z|^n \left[ |a_n z + \rho| - \left\{ \rho + \alpha_n + (1 - \lambda) \alpha_{n-k} + |1 - \lambda| |\alpha_{n-k}| - \alpha_0 + |\alpha_0| + 2 \sum_{j=0}^n |\beta_j| \right\} \right] \\
 & > 0
 \end{aligned}$$

if

$$|a_n z + \rho| > \rho + \alpha_n + (1 - \lambda) \alpha_{n-k} + |1 - \lambda| |\alpha_{n-k}| - \alpha_0 + |\alpha_0| + 2 \sum_{j=0}^n |\beta_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (1 - \lambda) \alpha_{n-k} + |1 - \lambda| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|} .$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (1 - \lambda) \alpha_{n-k} + |1 - \lambda| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|} .$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{\rho + \alpha_n + (1 - \lambda) \alpha_{n-k} + |1 - \lambda| |\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2 \sum_{j=0}^n |\beta_j|}{|a_n|} .$$

That proves Theorem 1.

**Proof of Theorem 3:** Consider the polynomial

$$\begin{aligned}
 F(z) &= (1 - z)P(z) \\
 &= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0) \\
 &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} \\
 &\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots + (a_1 - a_0)z + a_0 .
 \end{aligned}$$

If  $|a_{n-k-1}| > |a_{n-k}|$ , then  $|a_{n-k+1}| > |a_{n-k}|$ ,  $\lambda > 1$  and we have, for  $|z| > 1$ , by using Lemma 1,

$$\begin{aligned}
 |F(z)| &\geq |a_n z^{n+1} + \rho z^n| - \left| (\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{n-k+1} - a_{n-k})z^{n-k+1} \right. \\
 &\quad + (\lambda a_{n-k} - a_{n-k-1})z^{n-k} + (\lambda - 1)a_{n-k} z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots \\
 &\quad \left. + (a_1 - a_0)z + a_0 \right| \\
 &= |z|^n \left[ |a_n z + \rho| - \left| (\rho + a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \frac{1}{z} + \dots + (a_{n-k+1} - a_{n-k}) \frac{1}{z^{k-1}} \right. \right. \\
 &\quad \left. + (\lambda a_{n-k} - a_{n-k-1}) \frac{1}{z^k} + (\lambda - 1)a_{n-k} \frac{1}{z^k} + (a_{n-k-1} - a_{n-k-2}) \frac{1}{z^{k+1}} + \dots \right. \\
 &\quad \left. + (a_1 - a_0) \frac{1}{z^{n-1}} + \frac{a_0}{z^n} \right]
 \end{aligned}$$

$$\begin{aligned}
 &> |z|^n \left[ |a_n z + \rho| - \left\{ |\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{n-k+1} - a_{n-k}| \right. \right. \\
 &\quad \left. \left. + |\lambda a_{n-k} - a_{n-k-1}| + |\lambda - 1| |a_{n-k}| + |a_{n-k-1} - a_{n-k-2}| + \dots + |a_1 - a_0| \right. \right. \\
 &\quad \left. \left. + |a_0| \right\} \right] \\
 &\geq |z|^n \left[ |a_n z + \rho| - \left\{ (|\rho + a_n| - |a_{n-1}|) \cos \alpha + (|\rho + a_n| + |a_{n-1}|) \sin \alpha \right. \right. \\
 &\quad \left. \left. + (|a_{n-1}| - |a_{n-2}|) \cos \alpha + (|a_{n-1}| + |a_{n-2}|) \sin \alpha + \dots + (|a_{n-k+1}| - |a_{n-k}|) \cos \alpha \right. \right. \\
 &\quad \left. \left. + (|a_{n-k+1}| + |a_{n-k}|) \sin \alpha + (\lambda |a_{n-k}| - |a_{n-k-1}|) \cos \alpha + (\lambda |a_{n-k}| + |a_{n-k-1}|) \sin \alpha \right. \right. \\
 &\quad \left. \left. + |\lambda - 1| |a_{n-k}| + (|a_{n-k-1}| - |a_{n-k-2}|) \cos \alpha + (|a_{n-k-1}| + |a_{n-k-2}|) \sin \alpha \right. \right. \\
 &\quad \left. \left. + \dots + (|a_1| - |a_0|) \cos \alpha + (|a_1| + |a_0|) \sin \alpha + |a_0| \right\} \right] \\
 &= |z|^n \left[ |a_n z + \rho| - \left\{ |\rho + a_n| (\cos \alpha + \sin \alpha) - |a_{n-k}| (\cos \alpha - \sin \alpha - \lambda \cos \alpha - \lambda \sin \alpha \right. \right. \\
 &\quad \left. \left. - \lambda + 1) + |a_0| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j| \right\} \right] \\
 &> 0
 \end{aligned}$$

if

$$\begin{aligned}
 |a_n z + \rho| &> |\rho + a_n| (\cos \alpha + \sin \alpha) - |a_{n-k}| (\cos \alpha - \sin \alpha - \lambda \cos \alpha - \lambda \sin \alpha \\
 &\quad - \lambda + 1) + |a_0| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|
 \end{aligned}$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\begin{aligned}
 &[|\rho + a_n| (\cos \alpha + \sin \alpha) - |a_{n-k}| (\cos \alpha - \sin \alpha - \lambda \cos \alpha - \lambda \sin \alpha - \lambda + 1) \\
 &\quad - |a_0| (\cos \alpha - \sin \alpha - 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|] \\
 \left| z + \frac{\rho}{a_n} \right| &\leq \frac{\dots}{|a_n|} \dots
 \end{aligned}$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\begin{aligned}
 &[|\rho + a_n| (\cos \alpha + \sin \alpha) - |a_{n-k}| (\cos \alpha - \sin \alpha - \lambda \cos \alpha - \lambda \sin \alpha - \lambda + 1) \\
 &\quad - |a_0| (\cos \alpha - \sin \alpha - 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|] \\
 \left| z + \frac{\rho}{a_n} \right| &\leq \frac{\dots}{|a_n|} \dots
 \end{aligned}$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\begin{aligned}
 &[|\rho + a_n| (\cos \alpha + \sin \alpha) - |a_{n-k}| (\cos \alpha - \sin \alpha - \lambda \cos \alpha - \lambda \sin \alpha - \lambda + 1) \\
 &\quad - |a_0| (\cos \alpha - \sin \alpha - 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|] \\
 \left| z + \frac{\rho}{a_n} \right| &\leq \frac{\dots}{|a_n|} \dots
 \end{aligned}$$

If  $|a_{n-k}| > |a_{n-k+1}|$ , then  $|a_{n-k}| > |a_{n-k-1}|$ ,  $\lambda < 1$  and we have, for  $|z| > 1$ , by using Lemma 1,

$$\begin{aligned}
 |F(z)| &\geq |a_n z^{n+1} + \rho z^n| - |(\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} \\
 &\quad + (a_{n-k} - a_{n-k-1})z^{n-k} - (1 - \lambda)a_{n-k}z^{n-k+1} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots \\
 &\quad + (a_1 - a_0)z + a_0|
 \end{aligned}$$



$$\begin{aligned}
 &= |z|^n \left[ |a_n z + \rho| - \left| (\rho + a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \frac{1}{z} + \dots + (a_{n-k+1} - \lambda a_{n-k}) \frac{1}{z^{k-1}} \right. \right. \\
 &\quad \left. \left. + (a_{n-k} - a_{n-k-1}) \frac{1}{z^k} - (1-\lambda) a_{n-k} \frac{1}{z^{k-1}} + (a_{n-k-1} - a_{n-k-2}) \frac{1}{z^{k+1}} + \dots \right. \right. \\
 &\quad \left. \left. + (a_1 - a_0) \frac{1}{z^{n-1}} + \frac{a_0}{z^n} \right| \right] \\
 &> |z|^n \left[ |a_n z + \rho| - \left\{ |\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{n-k+1} - \lambda a_{n-k}| \right. \right. \\
 &\quad \left. \left. + |a_{n-k} - a_{n-k-1}| + |1 - \lambda| |a_{n-k}| + |a_{n-k-1} - a_{n-k-2}| + \dots + |a_1 - a_0| \right. \right. \\
 &\quad \left. \left. + |a_0| \right\} \right] \\
 &\geq |z|^n \left[ |a_n z + \rho| - \left\{ (|\rho + a_n| - |a_{n-1}|) \cos \alpha + (|\rho + a_n| + |a_{n-1}|) \sin \alpha \right. \right. \\
 &\quad \left. \left. + (|a_{n-1}| - |a_{n-2}|) \cos \alpha + (|a_{n-1}| + |a_{n-2}|) \sin \alpha + \dots + (|a_{n-k+1}| - |\lambda a_{n-k}|) \cos \alpha \right. \right. \\
 &\quad \left. \left. + (|a_{n-k+1}| + |\lambda a_{n-k}|) \sin \alpha + (|a_{n-k}| - |a_{n-k-1}|) \cos \alpha + (|a_{n-k}| + |a_{n-k-1}|) \sin \alpha \right. \right. \\
 &\quad \left. \left. + |1 - \lambda| |a_{n-k}| + (|a_{n-k-1}| - |a_{n-k-2}|) \cos \alpha + (|a_{n-k-1}| + |a_{n-k-2}|) \sin \alpha \right. \right. \\
 &\quad \left. \left. + \dots + (|a_1| - |a_0|) \cos \alpha + (|a_1| + |a_0|) \sin \alpha + |a_0| \right\} \right] \\
 &= |z|^n \left[ |a_n z + \rho| - \left\{ |\rho + a_n| (\cos \alpha + \sin \alpha) + |a_{n-k}| (\cos \alpha + \sin \alpha - \lambda \cos \alpha + \lambda \sin \alpha \right. \right. \\
 &\quad \left. \left. + 1 - \lambda) + |a_0| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j| \right\} \right] \\
 &> 0
 \end{aligned}$$

if

$$\begin{aligned}
 &|a_n z + \rho| > |\rho + a_n| (\cos \alpha + \sin \alpha) + |a_{n-k}| (\cos \alpha + \sin \alpha - \lambda \cos \alpha + \lambda \sin \alpha \\
 &\quad + 1 - \lambda) + |a_0| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|
 \end{aligned}$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\begin{aligned}
 &|z + \frac{\rho}{a_n}| \leq \frac{|\rho + a_n| (\cos \alpha + \sin \alpha) + |a_{n-k}| (\cos \alpha + \sin \alpha - \lambda \cos \alpha + \lambda \sin \alpha + 1 - \lambda) \\
 &\quad + |a_0| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|}{|a_n|}
 \end{aligned}$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) and therefore P(z) lie in

$$\begin{aligned}
 &|z + \frac{\rho}{a_n}| \leq \frac{|\rho + a_n| (\cos \alpha + \sin \alpha) + |a_{n-k}| (\cos \alpha + \sin \alpha - \lambda \cos \alpha + \lambda \sin \alpha + 1 - \lambda) \\
 &\quad + |a_0| (\sin \alpha - \cos \alpha + 1) + 2 \sin \alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|}{|a_n|}
 \end{aligned}$$

That proves Theorem 3.

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