Advances in Applied Mathematical Analysis.
ISSN 0973-5313 Volume 3 Number 1 (2008), pp. 75–80

http://www.ripublication.com/aama.htm

# Some Results on the Generalized Local Cohomology Modules

**Yan Gu**

*Department of Mathematics, Suzhou University*
*Suzhou 215006, P.R. China*
*E-mail: guyan@suda.edu.cn*

**Abstract**

Let $(R, m)$ be a commutative Noetherian local ring, $I$ and ideal of $R$, $M$ and $N$ finitely generated $R$-modules. Let $n_k$ = k-depth $(I + \text{Ann}(M); N)$, $J_{nk} = \cap_{j<nk} \text{Ann}(\text{Ext}^j_R(M / IM, N))$. It is shown than $n_k = \inf\{ i \mid \dim\text{Supp}(H^i_I(M, N)) \geq k \} = \{p \in \text{Ass}(\text{Ext}_R^{nk}(M / IM, N)) : \dim R/p \geq k\} = \{p \in \text{Ass}(N / (x_1, \ldots, x_{nk})N) : p \supseteq I + \text{Ann}(M), \dim R/p \geq k \}$ for any $k$-regular sequence $x_1, \ldots, x_{nk}$ of $N$ in $I + \text{Ann}(M)$. Let $t$ be a positive integer, if $\dim\text{Supp}(H^i_I(M, N)) < k$ for all $i < t$, we prove that $\{p \in \bigcup_{i\in \mathbb{N}} \text{Ass}(\text{Ext}^t_R(M / I^i M, N)) : \dim R / p \geq k\}$ is a finit set.

**AMS Subject Classification:** 13D45, 13C15.

**Keywords:** Generalized local cohomology modules, associated priames, k-depth.

## 1. Introduction

Throughout this paper, let $(R, m)$ be a commutative Noetherian local ring, $I$ a proper ideal of $R$, $M$ and $N$ finitely generated $R$-modules. The generalized local cohomology module

$$H^i_I(M, N) = \varinjlim_{n\in\mathbb{N}} \text{Ext}^i_R(M/I^nM, N).$$

was introduced by Herzog in [7] and studied further by Yassemi, Suzuki and so on. Since $H^i_I(R, N) = H^i_I(N)$, so the notion of generalized local cohomology module is an extension of the usual cohomology module.

For each integer $k \geq 0$, a sequence $x_1, \ldots, x_n \in m$ is called a $k$-regular sequence of

$M$ (cf. [5]) if $x_i \notin p$ for all $p \in$ Ass $(M / (x_1, \ldots, x_{i-1})M)$ satisfying dim $R/p \geq k$, for all $i = 1, \ldots, n$. The $k$-regular sequences of $M$ in $I$ (in [3] can be found that k-depth $(I; M)$ is well defined). It is easy to see that 0-regular sequences are exactly regular sequences; 1-regular sequences are filter regular sequences introduced in [6]; and 2-regular sequences are generalized regular sequences introduced in [11].

There are some results about $k$-regular sequence and $k$-depth, which proofs can be seen in [3, 5].

**Lemma 1.1.** (cf. [5]) Let $k \geq 0$ be an integer, and $x_1, \ldots, x_n \in m$. Then we have

(i) If $x_1, \ldots, x_n$ is a $k$-regular sequence of $M$, then $x_1/1, \ldots, x_n/1$ is a regular sequence of $M_p$ for all $p \in$ Supp$M$ containing $x_1, \ldots, x_n$ satisfying dim$R/p \geq k$, where $x_i/1$, $i = 1, \ldots, n$, is the image of $x_i$ in $R_p$.

(ii) $x_1, \ldots, x_n$ is a $k$-regular sequence of $M$ if and only if $x_1^{t1}, \ldots, x_n^{tn}$ is a $k$-regular sequence of $M$ for all positive integer $t_1, \ldots, t_n$.

(iii) Let $x \in m$. then $x$ is a $k$-regular element of $M$ if and only if dim $(0 : M^{x}) < k$. Then we have

(1) Every $k$-regular sequence of $M$ in $I$ is of finite length, all maximal $k$-regular sequences have the same length, and this common length is denoted to k-depth $(I; M)$, and k-depth $(I; M) = \inf\{ i \mid \dim\mathrm{Supp}\, (H^i_I(M)) \geq k\}$.

(2) If $x \in I$ is a $k$-regular of $M$, then k-depth $(I; M)$ = k-depth $(I; M / xM) + 1$.

**Lemma 1.3.** (cf [5] Let $k \geq 0$ be an integer. Assume that dim$M = d$, $\dim(M / I M) = d - r$. Then we have

(i) for all $k = 0, \ldots, d - r$, k-depth $(I; M) \leq r$,

(ii) for all $k > d - r$, k-depth $(I; M) = +\infty$.

**Proposition 1.4.** (cf. [5]) Let $k \geq 0$ be an integer, suppose that $\dim(M / I M) \geq k$. Then

$$\begin{aligned}\text{k-depth}(I;M) &= \min\{\mathrm{depth}(IR_p; M_p) : p \in \mathrm{Supp}(M/IM), \dim R/p \geq k\}\\ &= \min\{i : \dim \mathrm{Ext}^i_R(R/I, M) \geq k\}\end{aligned}$$

The purpose of this paper is to use $k$-regular sequences and $k$-depth to give some results on generalized local cohomology modules. Our main aim in this paper is to establish the following theorem.

**Theorem 1.5.** Let $n_k$ = k=depth $(I + \mathrm{Ann}\,(M); N)$, $J_{nk} = \cap_{j<nk} \mathrm{Ann}(\mathrm{Ext}^j_R(M / I M, N))$.

Then we have the following statements:

(1) $\dim(R / J_{nk}) \leq k - 1$, and $n_k = \inf\{ i \mid \dim\mathrm{Supp}\, (H^i_I(M, N)) \geq k\} = \inf\{ i \mid H^i_I(M, N) \ncong H^i_{Jnk}(M, N)\}$.

(2) $\{p \in \mathrm{Ass}(H^{nk}_I(M, N) : \dim R / p \geq k\} \neq \emptyset$, and $\{p \in \mathrm{Ass}(H^{nk}_I(M, N)) : \dim R / p \geq k\} = \{p \in \mathrm{Ass}(\mathrm{Ext}^{nk}_R(M / I M, N)) : \dim R / p \geq k\} = (p \in \mathrm{Ass}(N / (x_1, \ldots, x_{nk})N) : p \supseteq I + \mathrm{Ann}(M), \dim R / p \geq k\}$ is a finite set, where $x_1, \ldots, x_{nk}$ is a $k$-regular sequence of $N$ in $I + \mathrm{Ann}\,(M)$.

(3) Let $t$ be a positive integer, if $\dim\mathrm{Supp}(H^i_I(M, N) < k$ for all $i < t$, then $\{ p \in \bigcup_{i\in \mathbb{N}} \mathrm{Ass}(\mathrm{Ext}^t_R(M / I^i M, N)) : \dim R / p \geq k\}$ is a finite set.

Clearly, (1) extends a main result of [4], (2) generalize a main result of [5], (3) gives an impovement of Theorem 2.12 of [9].

## 2. Main results

Let $x_1, \ldots, x_n$ be an $N$-sequence in $I$, it has been proved that $\mathrm{Ass}(H^n_I(N)) = \mathrm{Ass}\,(H^0_I(N / (x_1, \ldots, x_n)N))$. We firstly generalize this result.

**Lemma 2.1.** Let $x_1, \ldots, x_n$ be an N-sequence in $I$ + Ann($M$). Then

$$\mathrm{Ass}(H^n_I(M, N)) = \mathrm{Ass}(H^0_I(M, N / (x_1, \ldots, x_n)N))$$

**Proof.** We use induction on n. If $n = 1$, then we have the short exact sequence $0 \to N \xrightarrow{x_1} N \to N / x_1N \to 0$, so we obtain the exact sequence $0 \to H^0_I(M, N / x_1N) \to H^1_I(M, N) \xrightarrow{x_1} H^1_I(M, N)$. By applying the functor $\mathrm{Hom}_R(R / (I + \mathrm{Ann}(M)), -)$ to the above exact sequence, we have the exact sequence $0 \to \mathrm{Hom}_R(R / (I + \mathrm{Ann}(M)), H^0_I(M, N / x_1N)) \to \mathrm{Hom}_R(R / (I + \mathrm{Ann}(M)), H^1_I(M, N) \xrightarrow{x_1} \mathrm{Hom}_R(R / (I + \mathrm{Ann}(M)), H^1_I(M, N))$. Since $x_1 \in I + \mathrm{Ann}\,(M)$, so

$$\mathrm{Hom}_R(R / (I + \mathrm{Ann}(M)), H^0_I(M, N / x_1N)) \cong \mathrm{Hom}R\,(R / (I + \mathrm{Ann}(M)), H^1_I(M, N)).$$

In virture of $H^0_I(M, N / x_1N) = \mathrm{H}^0_{I+\mathrm{ANN}(M)}(M, N / x_1N)$ and $H^1_I(M, N) = H^1_{I+\mathrm{ANN}(M)}(M, N)$, thus $\mathrm{Ass}\,(H^0_I(M, N / x_1N)) = \mathrm{Ass}\,(H^1_I(M, N))$ by [1, 1.2.28].

Now let $n > 1$ and the case $n - 1$ is settled. Using the short exact sequence $0 \to N \xrightarrow{x_1} N \to N / x_1N \to 0$, we get the short exact

$$0 \to H_I^{n-1}(M, N / x_1N) \to H^n_I(M, N) \xrightarrow{x_1} H^n_I(M, N)$$

Using the similar arguments as above, we have $\mathrm{Ass}(H_I^{n-1}(M, N / x_1N) = \mathrm{Ass}\,(H_I^n(M, N))$. By the induction hypothesis, we have

$$\mathrm{Ass}(H_I^{n-1}(M, N/x_1N)) = \mathrm{Ass}(H_I^0(M, N/(x_1, \ldots, x_n)N))$$

Now the proof is complete.

Suppose $\dim(N / (I + \mathrm{Ann}(M))N) = s$, for each $k = 0, \ldots, s$, we set $n_k$ = k-depth ($I$ + Ann($M$); $N$), then Ass $(H_I^{nk}(M, N)$ is not a finite set in general. In the following theorem, we will show some cases in which it is a finite set. Furthermore, we generalize the main result in [5] and Theorem 2.7 in [8].

**Theorem 2.2.** Let $\dim(N / (I + \mathrm{Ann}(M))N) = s$. For each $k = 0, \ldots, s$, we set $n_k$ = k-depth ($I$ + Ann($M$): $N$), $H_k = \{p \in \mathrm{Ass}(H_I^{nk}(M, N)) : \dim R / p \geq k\}$. Assume that $x_1, \ldots, x_{nk} \in I + \mathrm{Ann}(M)$ is a $k$-regular sequence of $N$. Then for all $k = 0, \ldots, s$, $H_k \neq \varnothing$ and

$$\begin{aligned} H_k &= \{p \in \mathrm{Ass}(\mathrm{Ext}_R^{n_k}(M/IM, N)) : \dim R/p \geq k\} \\ &= \{p \in \mathrm{Ass}(N/(x_1, \ldots, x_{n_k})N) : p \supseteq I + \mathrm{Ann}(M), \dim R/p \geq k\} \end{aligned}$$

In particular, $H_k$ is a finite set.

*Proof.* Let $k \in \{0, \ldots, s\}$, then $n_k < +\infty$ by Lemma 1.3(i). By Proposition 1.4, we have that

$$n_k = \min\{\mathrm{depth}\,((I + \mathrm{Ann}(M))R_p, N_p) : p \in \mathrm{Supp}\,(N / (I + \mathrm{Ann}(M))N), \dim R / p \geq k\}$$

Therefore there exists a prime ideal $p \in \mathrm{Supp}\,(N / (I + \mathrm{Ann}(M))N)$ such that $\dim R / p \geq k$ and depth $(I + \mathrm{Ann}(M)\, R_p, N_p = n_k$. Then we have $(H_I^{nk}(M, N))_p = H_{IRp}{}^{nk}(M_p,$

$N_p) \neq 0$, so $p \in$ Supp $(H_I^{nk}(M, N))$. Let $p' \in$ Min (Supp$(H_I^{nk}(M, N)))$ such that $p' \subseteq p$. Then $p' \in$ Ass $(H_I^{nk}(M, N))$, $\dim R / p' \geq \dim R / p \geq k$, thus $p' \in H_k$, hence $H_k \neq \emptyset$.

Since $x_1, \ldots, x_{nk} \in I$ + Ann $(M)$ is a $k$-regular sequence of $N$, then for every prime ideal $p$ with $p \in$ Supp$N$, $x_1, \ldots, x_{nk} \in p$ and $\dim R / p \geq k$, $x_1/1, \ldots, x_{nk}/1$ is a regular sequence of $N_p$ in $IR_p$ + Ann$(M_p)$ by Lemma 1.1(i). Thus

$$\mathrm{Ass}(H^{n_k}_{IR_p}(M_p, N_p)) = \mathrm{Ass}(H^0_{IR_p}(M_p, N_p/(x_1/1, \ldots, x_{n_k}/1)N_p))$$

by Lemma 2.1. Set $L = N / (x_1, \ldots, x_{nk})N$ and let a $\in IR_p$, be a $IR_p$-filter regular element of $L_p$, then $H^0_{IRp}(L_p) \cong H^0_{(a)}(L_p)$ by [12, 3.4], so we have

$$\begin{aligned} H^0_{IR_p}(M_p, L_p) \cong H^0_{IR_p}(\mathrm{Hom}_{R_p}(M_p, L_p)) &\cong \mathrm{Hom}_{R_p}(M_p, H^0_{IR_p}(L_p)) \\ &\cong \mathrm{Hom}_{R_p}(M_p, H^0_{(a)}(L_p)) \cong H^0_{(a)}(M_p, L_p) \end{aligned}$$

Therefore, there exists an exact sequence

$$0 \to H^0_{IR_p}(M_p, L_p) \to \mathrm{Hom}_{R_p}(M_p, L_p) \to (\mathrm{Hom}_{R_p}(M_p, L_p))_a$$

by [2, Remark 2.2.17], which implies the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{R_p}(R_p/IR_p, H^0_{IR_p}(M_p, L_p)) &\to \mathrm{Hom}_{R_p}(R_p/IR_p, \mathrm{Hom}_{R_p}(M_p, L_p)) \\ &\to \mathrm{Hom}_{R_p}(R_p/IR_p, (\mathrm{Hom}_{R_p}(M_p, L_p))_a) = 0 \end{aligned}$$

where for an $R$-module $P$, $P_a$ denotes the module of fraction of $P$ with respect to the multiplicatively closed subset $\{a^i : i \geq 0\}$ of $R$. So Hom$R_p$ $(R_p / I R_p, H^0_{IRp}(M_p, L_p) \cong$ Hom$R_p$ $(R_p / I R_p$, Home$R_p$ $(M_p, N_p / (x_1/1, \ldots, x_{nk}/1)N_p))$. Then

Ass$(H^0_{IRp}(M_p, L_p))$ = Ass (Hom$R_p$ $(M_p, N_p/(x_1/1, \ldots, x_{nk}/1)N_p))) \cap V(IR_p)$ by [1, 1,2,28]. Then we have

$$\mathrm{Ass}(H^{nk}_{IRp}(M_p, N_p)) = \mathrm{Supp}\,(M_p/I M_p) \cap \mathrm{Ass}(N_p/(X_1/1, \ldots, x_{nk} / 1)\, N_p)$$

Therefore $H_k = \{p \in \mathrm{Ass}(N / (x_1, \ldots, x_{nk})N) : p \supseteq I + \mathrm{Ann}(M), \dim R / p \geq k\}$ by [10, Theorem 6.2].

Since for every prime ideal $p$ with $p \in$ Supp$N$, $p \in V(I + \mathrm{Ann}(M))$ and $\dim R / p \geq k$, $x_1/1, \ldots, x_{nk}/1$ is a regular sequence of $N_p$ in $(I + \mathrm{Ann}(M))_p$ by Lemma 1.1(i), so we have the isomorphism:

$$\mathrm{Ext}^{n_k}_{R_p}(M_p/IM_p, N_p) \cong \mathrm{Hom}_{R_p}(M_p/IM_p, N_p/(x_1/1, \ldots, x_{n_k}/1)N_p)$$

Then Ass$(\mathrm{Ext}^{nk}_{Rp}(M_p / I M_p, N_p))$ = Supp $(M_p / I M_p) \cap$ Ass$(N_p / (x_1/1, \ldots, x_{nk}/1)N_p)$ by [1, 1.2.28]. By [10, Theorem 6.2] we get $\{p \in \mathrm{Ass}(\mathrm{Ext}^{nk}_R(M / I M, N)) : \dim R / p \geq k\} = \{p \in \mathrm{Ass}(N / (x_1, \ldots, x_{nk})N) : p \supseteq I + \mathrm{Ann}(M), \dim R / p \geq k\}$. the proof is complete.

For $k = 0, \ldots, \dim(N / (I + \mathrm{Ann}(M))N)$, we set $n_k$ = k-depth $(I + \mathrm{Ann}(M);N)$. Then we have the following two corollaries.

**Corollary 2.3.** Ass$(H_I^{no}(M, N))$ = Ass$(\mathrm{Ext}^{no}_R(M / I M, N))$. In particular, Ass$(H_I^{no}(M, N))$ is a finite set.

**Corollary 2.4.** If dim $(N / (I + \mathrm{Ann}(M))N) > 0$, then Ass$(H_I^{n1}(M, N)) \cap \{m\}$ = Ass

$(\mathrm{Ext}^{n1}_R(M/IM, N)) \cup \{m\}$. In particular, $\mathrm{Ass}(H_I^{n1}(M, N))$ is a finite set.

**Remark 2.5.** Note that $n_0$ = 0-depth $(I + \mathrm{Ann}(M); N)$ = depth $(I + \mathrm{Ann}(M); N)$ and $n1$ = 1-depth $(I + \mathrm{Ann}(M); N)$ = f-depth $(I + \mathrm{Ann}(M); N)$ whenever $\dim(N/(I + \mathrm{Ann}(M))N) > 0$, so corollary 2.3 and 2.4 generalize the known results about the finiteness of Ass $(H_I^{no}(M, N))$ and $\mathrm{Ass}(H_I^{n1}(M, N))$.

**Definition 2.6.** Let $i \geq 0$ be integer. Set dimSupp $(H^i_I(M, N)) = \max\{\dim R/p \mid p \in \mathrm{Supp}(H_{iI}(M, N))\}$.

If $H^i_I(M, N)$ is finitely generated, then $\mathrm{dimSupp}(H^i_I(M, N)) = \dim(R/\mathrm{Ann}(H^i_I(M, N)))$; if $H^i_I(M, N)$ is Artinian, then $\mathrm{Supp}(Hi_I(M, N)) \subseteq \{m\}$, thus $\mathrm{dimSupp}(H^i_I(M, N)) \leq 0$.

**Lemma 2.7.** (cf. [4] Let $\mathbb{N}$ be the set of all positive integers and $i \in \mathbb{N} \cup \{+\infty\}$. Set $J_i = \cap_{j<i} \mathrm{Ann}(\mathrm{Ext}^j_R(M/IM, N))$. Then $H^j_I(M, N) \cong H^j_{Ji}(M, N)$ for all $j < i$.

**Lemma 2.8.** (cf. [4]) Let $i \in \mathbb{N} \cup \{+\infty\}$. Then we have

$$\bigcup_{j<i} \mathrm{Supp}(H^i_I(M, N)) = \bigcup_{j<i} \mathrm{Supp}(\mathrm{Ext}^j_R(M/IM, N)).$$

**Theorem 2.9.** Set $n_k$ = k-depth $(I + \mathrm{Ann}(M), N)$ $J_{nk} = \cap_{j<nk} \mathrm{Ann}(\mathrm{Ext}^j_R(M/IM, N))$. Then $\dim(R/J_{nk}) \leq k-1$, $n_k = \inf\{i \mid \mathrm{dimSupp}(H^i_I(M, N)) \geq k\} = \inf\{i \mid H^i_I(M, N) \cong H^i_{Jnk}(M, N)\}$.

*Proof.* If $\dim(N/(I + \mathrm{Ann}(M))N) \leq k-1$, then $n_k = +\infty$ by Lemma 1.3, $\dim(R/J_{nk}) \leq k-1$, and $\mathrm{dimSupp}(H^j_I(M, N)) \leq k-1$ for all $j \geq 0$, so $\inf\{i \mid \mathrm{dimSupp}(H^j_I(M, N)) \geq k\} = +\infty$. By Lemma 2.7, we have $H^i_I(M, N) \cong H^i_{Jr}(M, N)$ for all $i \geq 0$, then $\inf\{i \mid H^i_I(M, N) \cong H^i_{Jnk}(M, N)\} = +\infty$. The result is true in this case.

If $\dim(N/(I + \mathrm{Ann}(M))N) \geq k$, then $n_k < +\infty$. Suppose that $x_1, \ldots, x_{nk}$ is a maximal $k$-regular sequence of $N$ in $I + \mathrm{Ann}(M)$. If $p \in \mathrm{Supp}(N/(I + \mathrm{Ann}(M))$ such that $\dim R/p \geq k$, then $x_1, \ldots, x_{nk}$ is a $N_p$ sequence in $(I + \mathrm{Ann}(M))_p$. Then $\mathrm{Ext}^j_R(M/IM, N)_p = 0$ for all $j < n_k$, so $\dim(\mathrm{Ext}^j_R(M/IM, N)) \leq k-1$ for all $j < n_k$, thus $\dim R/J_{nk} \leq k-1$. Since $\bigcup_{j<nk} \mathrm{Supp}(\mathrm{Ext}^j_R(M/IM, N)) = \bigcup_{j<nk} \mathrm{Supp}(\mathrm{H}^j_I(M, N))$ by Lemma 2.8, so $\mathrm{dimSupp}(H^j_I(M, N)) \leq k-1$ for all $j < n_k$.

Since $n_k = \inf\{\mathrm{depth}((I + \mathrm{Ann}(M))p, Np) \mid p \in \mathrm{Supp}(N/(I + \mathrm{Ann}(M))N), \dim R/p \geq k\}$, so there exists some $p \in \mathrm{Supp}(N/(I + \mathrm{Ann}(M))N)$, $\dim R/p \geq k$ such that $n_k$ = depth $((I + \mathrm{Ann}(M))p, Np)$. Thus $\mathrm{Ext}^{nk}_R(M/IM, N)_p \neq 0$. Hence we have $p \in \bigcup^{nk}_{j=0} \mathrm{Supp}(H^j_I(M, N))$ by Lemma 2.8. In virtue of $\mathrm{dimSupp}(H^j_I(M, N)) \leq k-1$ for all $j < n_k$, so $p \notin \bigcup_{j<nk} \mathrm{Supp}(\mathrm{H}^j_I(M, N))$, thus $p \in \mathrm{Supp}(H_I^{nk}(M, N))$, hence dimSupp $(H_I^{nk}(M, N)) \geq k$. Therefore $n_k = \inf\{i \mid \mathrm{dimSupp}(H^i_I(M, N)) \geq k\}$.

Since $\dim R/J_{nk} \leq k-1$, so dimSupp $(H^{nk}_{Jnk}(M, N) \leq k-1$. By the above equality we know that dimSupp $(H_I^{nk}(M, N)) \geq k$, so $H^{\mathrm{I}}_{nk}(M, N) \cong H^{nk}_{Jnk}(M, N)$. As $H^j_I(M, N) \cong H^j_{Jnk}(M, N)$ for all $j < n_k$ by Lemma 2.7. Then $n_k = \inf\{i \mid H^i_I(M, N) \cong H^i_{Jnk}(M, N)\}$.

**Corollary 2.10.** Let $r$ = gdepth $(I + \mathrm{Ann}(M), N)$ and $J_r = \cap_{j<r} \mathrm{Ann}(\mathrm{Ext}^j_R(M/IM, N))$, in [4] Cuong and Hoang proved that $\dim(R/J_r) \leq 1$, $r = \inf\{i \mid \mathrm{Supp}(\mathrm{H}^i_I(M, N))$ is not finite$\} = \inf\{i \mid \mathrm{H}^i_I(M, N) \cong H^i_{Jr}(M, N)\}$.

**Theorem 2.11.** Let $t$ be an in non-negative integer such that $\mathrm{dimSupp}(\mathrm{H}^i_I(M, N)) < k$ for all $i < t$, then $\{p, \in \bigcup_{i \in \mathbb{N}} \mathrm{Ass}(\mathrm{Ext}^t_R(M/I^iM, N)) : \dim R/p \geq k\}$ is a finite set.

*Proof*. Suppose that $i \in \mathbb{N}$, $p \in$ Ass($\mathrm{Ext}^t_R(M / I^i M, N)$) such that dim$R / p \geq k$, then $pR_p \in$ Ass($\mathrm{Ext}^t_R(M / I^i M, N))_p$. Since $n_k$ = k-depth ($I$ + Ann($M$)) $\geq t$, so there exists an $k$-regular sequence $x_1, \ldots, x_t$ in $I$ + Ann ($M$) of $N$. Thus there exist $n_1, \ldots, n_t \in \mathbb{N}$ such that $x_1^{n1}, \ldots, x_t^{nt}$ is a $k$-regular sequence of $N$ in $I^i$ + Ann ($M$) for all $i \in \mathbb{N}$, so $x_1^{n1}/1, \ldots, x_t^{nt}/1$ is a $N_p$-regular sequence, thus $\mathrm{Ext}^t_R(M / I^i M, N)_p \cong \mathrm{Hom}(M / I^i M, N / (x_1^{n1}, \ldots, x_t^{nt})N)_p$, thus $pR_p \in$ Ass (Hom($M / I^i M, N / (x_1^{n1}, \ldots, x_t^{nt})N))_p$, $pR_p \in$ V ($IR_p$) $\cap$ Ass($N / (x_1^{n1}, \ldots, x_t^{nt})N)p$. As $x_1^{n1}/1, \ldots, x_t^{nt}/1$ is a $N_p$-regular sequence, Ass($N / (x_1, \ldots, x^t)N)p$, therefore $p \in$ Ass ($N / (x_1, \ldots, x_t)N$), the proof is complete.

**Corollary 2.12.** If Supp($H^i_I(N)$) is a finite set for all $i < t$, then the set $\{p \in \bigcup_{i\in\mathbb{N}}$ Ass($\mathrm{Ext}^t_R(M / I^i M, N)$) : dim$R / p\}$ is finite.

## References


[1] Bruns, W. and Herzog, J., Cohen-Macaulay Rings. Cambridge Studies in Advanced Mathe-matics, 39. Cambridge University Press. 1993.

[2] Brodmann, M, P. and Sharp, R.Y., Local cohomology-An algebraic introduction with geometric applications. Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press. 1998.

[3] Brodmann, M.P. and Nhan, L.T., A finite nesresult for associated primes of certain EXT-modules, to appear in Comm. Algebra.

[4] Coung, N.T. and Hoang, N.V., On the vanishing and the finiteness of supports of generalized local cohomology modules, to appear in Manuscript a Math.

[5] Chinh, N.Q. and Nhan, L.T., On the associated primes and the support of local cohomology modules, to appear in Algebra. Colloq.

[6] Cuong, N.T., Schenzel, P. and Trung, N.V., Verallgemeinerte Cohen-Macaulay Moduln, Math. Nachr. 85, pp. 57-73, 1978.

[7] Herzog, J., Komplex Auflösungen und Dualitä in der lokalen Algebra. Habilitationsschrift, Universität Regensburg, 1974.

[8] Khashyarmanesh, K., On the finiteness properties of $\cap_i \mathrm{AssExt}^n_R(R / a^i, M)$, Comm. Algebra 34, pp. 779-784, 2006.

[9] Mafi, A., On the finiteness results of the generalized local cohomology modules, to appear in Algebra. Colloq.

[10] Matsumura, H., Commutative Ring Theory. Cambridge Studies in Advanced Mathematics, 8. Cambridge : Cambridge University Press, 1986.

[11] Nhan, L.T., On generalized regular sequence and the finiteness for associated primes of local cohomology modules, Comm. Algebra 33, pp. 793-806, 2005.

[12] Nagel, U. and Schenzel, P., Cohomological annihilators and Castelnuovo Mumford regularity. Commutative Algebra : Syzygies, Multiplicities, and Birational Algebra (Southe Hadley, Ma, 1992). Providence, RI, pp. 307-328, 1994.