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Duality in Segal–Bargmann spaces

William E. Gryc [a,*], Todd Kemp [b,1]

[a] *Muhlenberg College, Allentown, PA, United States*
[b] *UCSD, La Jolla, CA, United States*

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Abstract

For $\alpha > 0$, the *Bargmann projection* P_α is the orthogonal projection from $L^2(\gamma_\alpha)$ onto the holomorphic subspace $L^2_{hol}(\gamma_\alpha)$, where γ_α is the standard Gaussian probability measure on $\mathbb{C}^n$ with variance $(2\alpha)^{-n}$. The space $L^2_{hol}(\gamma_\alpha)$ is classically known as the *Segal–Bargmann space*. We show that P_α extends to a bounded operator on $L^p(\gamma_{\alpha p/2})$, and calculate the exact norm of this scaled L^p Bargmann projection. We use this to show that the dual space of the L^p-Segal–Bargmann space $L^p_{hol}(\gamma_{\alpha p/2})$ is an $L^{p'}$ Segal–Bargmann space, but with the Gaussian measure scaled differently: $(L^p_{hol}(\gamma_{\alpha p/2}))^* \cong L^{p'}_{hol}(\gamma_{\alpha p'/2})$ (this was shown originally by Janson, Peetre, and Rochberg). We show that the Bargmann projection controls this dual isomorphism, and gives a dimension-independent estimate on one of the two constants of equivalence of the norms.

Keywords: Segal–Bargmann spaces; Integral operators

1. Introduction and background

The Fock space is a central object in quantum mechanics, operator algebras, and probability theory. Based over the Hilbert space $\mathbb{C}^n$, it can be identified as a Hilbert space of holomorphic functions. Let $\alpha > 0$ and let $\gamma_\alpha = \gamma_\alpha^n$ denote the Gaussian measure

* Corresponding author.
E-mail addresses: wgryc@muhlenberg.edu (W.E. Gryc), tkemp@math.ucsd.edu (T. Kemp).
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$$\gamma_\alpha(dz) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2}\lambda(dz), \tag{1.1}$$

where λ is the Lebesgue measure on $\mathbb{C}^n$. Then the Fock space is $\mathscr{F}_\alpha = \mathscr{F}_\alpha(\mathbb{C}^n) \equiv L^2_{hol}(\mathbb{C}^n, \gamma_\alpha)$, the (entire) holomorphic functions in $L^2(\mathbb{C}^n, \gamma_\alpha)$. It is a reproducing kernel Hilbert space, with kernel

$$\mathcal{K}_\alpha(z, w) = e^{\alpha\langle z, w\rangle}. \tag{1.2}$$

(Note: $\langle z, w\rangle$ denotes the complex inner-product $\sum_{i=1}^n z_i \overline{w_i}$.) As usual, the existence of the reproducing kernel guarantees that $\mathscr{F}_\alpha$ is, in fact, a closed subspace of $L^2(\gamma_\alpha)$.

In this paper, we study the orthogonal projection $P_\alpha\colon L^2(\mathbb{C}^n, \gamma_\alpha) \to L^2_{hol}(\mathbb{C}^n, \gamma_\alpha) = \mathscr{F}_\alpha$. As in any reproducing kernel Hilbert subspace, this orthogonal projection has the reproducing kernel itself as its integral kernel,

$$P_\alpha f(z) = \int_{\mathbb{C}^n} e^{\alpha\langle z, w\rangle} f(w)\,\gamma_\alpha(dw). \tag{1.3}$$

The projection P_α is the $(\mathbb{C}^n, \gamma_\alpha)$ equivalent of the classical *Bergman projection* (in Bergman spaces on the unit disk in $\mathbb{C}$); in more general contexts it is sometimes called the *Riesz projection*. Since its range is the classical Segal–Bargmann space, we refer to P_α as the *Bargmann projection*. (Note, it is not the same object as the Segal–Bargmann transform cf. [1,8], though there are obvious connections.) P_α naturally controls the geometry of the imbedding of $\mathscr{F}_\alpha$ into $L^2(\gamma_\alpha)$; the interpolation scale of these holomorphic spaces can be understood well in its context. The unusual duality properties of the holomorphic L^p-spaces of γ_α (as discussed in Section 1.2 below) have the result that P_α is not bounded on $L^p(\gamma_\alpha)$ for $p \neq 2$; rather, the measure must be scaled. The main result of this paper is the following theorem.

Theorem 1.1. *Let n be a positive integer, let $1 \leqslant p < \infty$, and let $\alpha > 0$. Let p' denote the conjugate exponent to p, $\frac{1}{p} + \frac{1}{p'} = 1$. The Bargmann projection P_α is bounded on $L^p(\mathbb{C}^n, \gamma_{\alpha p/2})$, with norm*

$$\left\| P_\alpha : L^p\big(\mathbb{C}^n, \gamma_{\alpha p/2}\big) \to L^p_{hol}\big(\mathbb{C}^n, \gamma_{\alpha p/2}\big)\right\| = \left(2\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right)^n. \tag{1.4}$$

When $p = 2$, the norm in Eq. (1.4) is equal to 1 in all dimensions, as expected for an orthogonal projection; for all other p, it grows exponentially with dimension. In particular, the $L^1(\gamma_{\alpha/2})$-norm of P_α is 2^n. Note that the main theorem of [3] is the upper bound 2^n for the norm in Eq. (1.4) (a result which is actually contained in [5] in a wider context); as we show in Section 1.3, this upper bound follows simply from a reinterpretation of $\mathscr{F}_\alpha$ as a subspace of L^p functions over Lesbesgue measure.

As we discuss in Section 1.5, the norm of P_α controls the norm of the dual space $(L^p_{hol}(\gamma_{\alpha p/2}))^*$.

Theorem 1.2. *Let $1 < p < \infty$ and $\alpha > 0$. Let p' be the conjugate exponent to p. In the pairing $(f, g)_\alpha = \int_{\mathbb{C}^n} f\bar{g}\,d\gamma_\alpha$, the spaces $L^p_{hol}(\gamma_{\alpha p/2})$ and $L^{p'}_{hol}(\gamma_{\alpha p'/2})$ are dual. The norms satisfy*

$$\|h\|_{L^{p'}_{hol}(\gamma_{\alpha p'/2})} \leqslant \big\|(\cdot, h)_\alpha\big\|_{(L^p_{hol}(\gamma_{\alpha p/2}))^*} \leqslant \left(2\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right)^n \|h\|_{L^{p'}_{hol}(\gamma_{\alpha p'/2})}. \tag{1.5}$$

Remark 1.3. In fact, it is the first inequality in Eq. (1.5) that is the interesting new result; the second inequality is actually just Hölder's inequality when reinterpreted in terms of L^p spaces over Lesbesgue measure, as explained in Section 1.5 below.

Remark 1.4. The authors find it particularly worthy of note that the first inequality in Eq. (1.5) is *independent of dimension*.

In Section 1.3, we show how the problem may be simplified by viewing elements of the Fock space as elements of a subspace of L^2 over Lesbesgue measure; this transformation offers a new explanation for why P_α acts naturally on $L^p(\gamma_{\alpha p/2})$ rather than $L^p(\gamma_\alpha)$ (and hence why the holomorphic L^p-spaces of γ_α do not satisfy the usual duality relations). Since P_α, given by Eq. (1.3), has a Gaussian kernel, our approach is to use the results of [6] to calculate the norm which occurs on the subspace of Gaussian functions. Since the kernel of P_α is complex, the Gaussian maximizer may also be complex, which greatly complicates the computations.

The remainder of this paper is organized as follows. Section 1.2 explores the unusual duality relations among the holomorphic L^p spaces of Gaussian measures. In Section 1.3, we reinterpret $L^p_{hol}(\gamma_{\alpha p/2})$ as a subspace of L^p over Lesbesgue measure, which sheds light on the rescaling required for the usual L^p-duality. This allows us to reinterpret the projection P_α as a new operator Q_α in the setting of Lebesgue measure, where it is easier to analyze. In Section 1.4, we show that the norm of P_α on L^p grows exponentially with dimension, and in Section 1.5, we use the Lesbesgue perspective to prove Theorem 1.2. Section 2 is devoted to the proof of Theorem 1.1. In Section 2.1, we reduce the calculation to the $n = 1$ dimensional case with a version of Segal's lemma for tensor products of integral operators. A deep result of Lieb, cf. [6], is then used in Section 2.2 to further reduce to the case of putative Gaussian maximizers for the norm of P_α. Sections 2.3 and 2.4 then setup the appropriate calculus problem to determine the norm. The proof is completed with the lengthy calculations of Sections 2.5 and 2.6, determining critical points and identifying the global maximum to calculate the sharp norm of P_α, concluding the paper.

1.1. Gaussian integrals

Many of the calculations throughout this paper rely on the following formula for integrating Gaussian functions. Let A be a $k \times k$ complex symmetric matrix, whose real part $\Re(A)$ is positive definite. Let $v \in \mathbb{C}^k$, and let $(\cdot,\cdot)$ denote the *real* inner-product extended (bilinearly, not sesquilinearly) to $\mathbb{C}^k$. Then the (uncentered) Gaussian function $x \mapsto e^{-(x,Ax)+2(v,x)}$ is in $L^1(\mathbb{R}^k)$, and

$$\int_{\mathbb{R}^k} e^{-(x,Ax)+2(v,x)}\,dx = \frac{\pi^{k/2}}{\sqrt{\det(A)}} e^{(v,A^{-1}v)}. \tag{1.6}$$

Eq. (1.6) can be found as [7, Ch. 5, Ex. 5]. It is easy to verify for real A (by diagonalizing and completing the square); the general formula then follows by an analytic continuation argument.

1.2. Duality in $L^p_{hol}(\gamma_\alpha)$

In the holomorphic space $\mathscr{F}_\alpha(\mathbb{C}^n) = L^2_{hol}(\mathbb{C}^n, \gamma_\alpha)$, Taylor series expansions are, in fact, orthogonal sums (since the measure γ_α is rotationally-invariant). Indeed, it is easy to compute that the monomials $z^{\mathbf{j}} = z_1^{j_1} \cdots z_n^{j_n}$ are orthogonal with

$$\left(z^{\mathbf{j}}, z^{\mathbf{k}}\right)_\alpha = \delta_{\mathbf{jk}} \frac{\mathbf{j}!}{\alpha^{|\mathbf{j}|}}, \tag{1.7}$$

where $\mathbf{j}! = j_1! \cdots j_n!$ and $|\mathbf{j}| = j_1 + \cdots + j_n$. Letting $\phi_{\mathbf{j}}(z) = (\alpha^{|\mathbf{j}|/2}/\sqrt{\mathbf{j}!})z^{\mathbf{j}}$, Taylor's theorem therefore asserts that $\{\phi_{\mathbf{j}}\colon \mathbf{j} \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2_{hol}(\mathbb{C}^n, \gamma_\alpha)$. This justifies the claim that the reproducing kernel is given as in Eq. (1.2), since

$$\mathcal{K}_\alpha(z, w) \equiv \sum_{\mathbf{j}\in\mathbb{N}^n} \phi_{\mathbf{j}}(z)\overline{\phi_{\mathbf{j}}(w)} = \sum_{\mathbf{j}\in\mathbb{N}^n} \frac{\alpha^{|\mathbf{j}|}}{\mathbf{j}!} z^{\mathbf{j}} \overline{w}^{\mathbf{j}} = e^{\alpha\langle z,w\rangle}.$$

The standard estimate $|f(z)| \leqslant \mathcal{K}_\alpha(z, z)\|f\|_2$ shows that $L^2(\gamma_\alpha)$-convergence implies pointwise convergence in $L^2_{hol}(\gamma_\alpha)$; from this it is easy to see that $L^2_{hol}(\gamma_\alpha)$ is a Hilbert space.

For $p \neq 2$, the spaces $L^p_{hol}(\gamma_\alpha)$ behave somewhat differently than one would naïvely expect. Let $1 < p < \infty$, and let p' be its conjugate exponent. If $h \in L^{p'}_{hol}(\gamma_\alpha)$ then $h \in L^{p'}(\gamma_\alpha)$ and so can be viewed, via the usual pairing, as an element $(\cdot, h)_\alpha$ of the dual space to $L^p(\gamma_\alpha)$:

$$L^p(\gamma_\alpha) \ni f \mapsto (f, h)_\alpha = \int f\bar{h}\, d\gamma_\alpha.$$

Naturally this imbedding does not give all of $(L^p(\gamma_\alpha))^*$ since $L^{p'}_{hol}(\gamma_\alpha)$ is a small subspace of $L^{p'}(\gamma_\alpha)$. The same imbedding shows that $(\cdot, h)_\alpha$ is an element of the dual space to $L^p_{hol}(\gamma_\alpha)$; it is somewhat surprising that, in this context as well, the set of all $(\cdot, h)_\alpha$ with $h \in L^{p'}_{hol}(\gamma_\alpha)$ is *not* the full dual space (unless $p = 2$). This was discovered by Sjögren, cf. [9]; for completeness, we reproduce the following simpler argument, which is due to Carlen and Gross.

Proposition 1.5. *If $1 < p < \infty$ and $p \neq 2$, then the imbedding $L^{p'}_{hol}(\gamma_\alpha) \to (L^p_{hol}(\gamma_\alpha))^*$ is not surjective.*

Proof. First note that the map $h \mapsto (\cdot, h)_\alpha$ is injective. For if $(f, h)_\alpha = 0$ for all $f \in L^{p'}_{hol}(\gamma_\alpha)$, we may take f to be a monomial $f(z) = z^{\mathbf{j}}$ (which is in $L^{p'}_{hol}(\gamma_\alpha)$ for all $p > 1$); the orthogonality relations of Eq. (1.7) then yield $(f, h)_\alpha = \mathbf{j}!/\alpha^{|\mathbf{j}|} \cdot T_{\mathbf{j}}(h)$ where $T_{\mathbf{j}}(h)$ is the $\mathbf{j}$th Taylor-coefficient of the holomorphic function h. Hence, the Taylor series of h is 0, and so $h = 0$.

From Hölder's inequality, the map $h \mapsto (\cdot, h)_\alpha$ is, as usual, bounded. Hence, by the Open Mapping Theorem, if it is also surjective it follows that it has a bounded inverse. We will show this is not true by demonstrating there is no constant $C > 0$ such that

$$\|h\|_{p'} \leqslant C\big\|(\cdot, h)_\alpha\big\|_{(L^p_{hol}(\gamma_\alpha))^*}, \quad \text{for all } h \in L^{p'}_{hol}(\gamma_\alpha). \tag{1.8}$$

Indeed, consider the function $h(z) = K_w(z) = \mathcal{K}_\alpha(z, w) = e^{\alpha\langle z,w\rangle}$, which is of course in $L^{p'}_{hol}(\gamma_\alpha)$. A simple computation using Eq. (1.6) shows that

$$\|K_w\|_{p'} = e^{p'\alpha|w|^2/4}. \tag{1.9}$$

Now, K_w is the reproducing kernel; that is, $(f, K_w)_\alpha = f(w) \equiv \Lambda_w(f)$ for all $f \in L^2_{hol}(\gamma_\alpha)$, and so therefore also for $f \in L^p_{hol}(\gamma_\alpha)$ (true for $p > 2$ since γ_α is a finite measure so $L^p(\gamma_\alpha) \subseteq L^2(\gamma_\alpha)$; true for $p < 2$ since $L^p_{hol}(\gamma_\alpha)$ is dense in $L^2_{hol}(\gamma_\alpha)$ and $(\cdot, K_w)_\alpha$ is continuous on $L^p(\gamma_\alpha)$). Hence, the norm on the right-hand side of Eq. (1.8) is just the $(L^p_{hol}(\gamma_\alpha))^*$-norm of the evaluation functional Λ_w. This is computed elegantly by Carlen in [2, Theorem 3]; the result is

$$\|\Lambda_w\|_{(L^p_{hol}(\gamma_\alpha))^*} = e^{\alpha|w|^2/p}. \tag{1.10}$$

Therefore, Eq. (1.8) implies that

$$e^{p'\alpha|w|^2/4} \leqslant Ce^{\alpha|w|^2/p}, \quad \text{for all } w \in \mathbb{C}.$$

Rearranging and simplifying, we find

$$C \geqslant e^{\alpha|w|^2(p'/4-1/p)} = e^{\alpha|w|^2(p-2)^2/4p(p-1)}. \tag{1.11}$$

Since $(p-2)^2/4p(p-1) > 0$ as long as $p \neq 2$, the right-hand side of Eq. (1.11) tends to ∞ as $|w| \to \infty$. Hence, there can be no such constant C. □

This surprising lack of duality has material consequences for the Bargmann projection P_α.

Corollary 1.6. *The Bargmann projection P_α is not bounded on $L^p(\gamma_\alpha)$ for any $p \neq 2$.*

Remark 1.7. P_α acts, by definition, on $L^2(\gamma_\alpha)$, and so for $p > 2$ the action of P_α on $L^p(\gamma_\alpha)$ is well-defined. For $p < 2$, the corollary should be interpreted to say that P_α is not bounded on $L^2(\gamma_\alpha) \cap L^p(\gamma_\alpha)$, and hence has no extension to $L^p(\gamma_\alpha)$.

Remark 1.8. The idea of this proof is due to Brian Hall.

Proof. Suppose, to the contrary, that, for some $p \neq 2$, P_α is bounded from $L^p(\gamma_\alpha)$ to $L^p_{hol}(\gamma_\alpha)$. (If $p < 2$, the supposition is that $P_\alpha|_{L^2\cap L^p}$ extends continuously to L^p.) It then follows (by the self-adjointness of P_α on L^2 and Hölder's inequality) that P_α is also bounded from $L^{p'}(\gamma_\alpha)$ to $L^{p'}_{hol}(\gamma_\alpha)$.

Let Φ be any linear functional in $(L^p_{hol}(\gamma_\alpha))^*$. We may then define a linear functional $\hat{\Phi} \in (L^p(\gamma_\alpha))^*$ by

$$\hat{\Phi}(f) = \Phi(P_\alpha f). \tag{1.12}$$

(Note, from Eq. (1.3), P_α is related to the Fourier transform, and so it makes sense to so-name the new functional $\hat{\Phi}$.) Since P_α is a projection, $P_\alpha^2 = P_\alpha$, and so $\hat{\Phi}(P_\alpha f) = \Phi(P_\alpha^2 f) = \Phi(P_\alpha f) =$

$\hat{\Phi}(f)$. Since P_α is bounded on $L^p(\gamma_\alpha)$, $\hat{\Phi} \in (L^p(\gamma_\alpha))^* = L^{p'}(\gamma_\alpha)$. That is, there is a unique function $g \in L^{p'}(\gamma_\alpha)$ such that $\hat{\Phi}(f) = (f, g)_\alpha$. Note, then, that $\hat{\Phi}(f) = \hat{\Phi}(P_\alpha f) = (P_\alpha f, g)_\alpha = (f, P_\alpha g)_\alpha$. (The last equality holds if $f, g \in L^2(\gamma_\alpha)$, and so holds in general by the denseness of L^p in L^2.) But then $(f, g)_\alpha = (f, P_\alpha g)_\alpha$ for all $f \in L^p$. Since $g \in L^{p'}$ and so $P_\alpha g \in L^{p'}$ (by the absurd assumption of the proof), it follows that $P_\alpha g = g$, and so $g \in L^{p'}_{hol}(\gamma_\alpha)$.

Thus, the map $L^{p'}_{hol}(\gamma_\alpha) \to (L^p_{hol}(\gamma_\alpha))^*$ which sends g to the linear functional $(\cdot, g)_\alpha$ is surjective and continuous. This contradicts Proposition 1.5. □

Remark 1.9. Corollary 1.6 shows that there is a close connection between projections in L^2 and the duality relations in closed subspaces of L^p. We could have proved this result in great generality, but it is only relevent for us in this limited context.

In [5], the authors identify (up to scale) what the actual dual space of $L^p_{hol}(\gamma_\alpha)$ is, by reinterpreting the action of the Bargmann projection P_α. Their results apply to a much more general setting than the Gaussian measures γ_α. If μ is a measure on a connected region $\Omega \subseteq \mathbb{C}^n$, possessing a strictly-positive density, and if the group of gauge transformations (holomorphic bijections υ of Ω with the property that $(\upsilon^{-1})_*\mu = |\phi|^2\mu$ for some holomorphic gauge factor ϕ) is sufficiently rich, then the orthogonal projection $P : L^2(\mu) \to L^2_{hol}(\mu)$ should really be thought of as a map from $L^2[\mathcal{K}] \to L^2_{hol}[\mathcal{K}]$. Here $\mathcal{K}$ is the reproducing kernel of $L^2_{hol}(\mu)$, and $L^p[\mathcal{K}]$ is a *weighted* L^p-space, defined as the set of all functions f such that $f(z)/\mathcal{K}(z,z)^{1/2}$ is in $L^p(\mathcal{K}(z,z)\,\mu(dz))$ (with the natural norm). Of course $L^2[\mathcal{K}] = L^2(\mu)$, but for $p \neq 2$ they are distinct. Janson, Peetre, and Rochberg show that P extends to a bounded map from $L^p[\mathcal{K}] \to L^p_{hol}[\mathcal{K}]$ (with norm $\leqslant 2^n$) for such sufficiently nice μ. This leads to the correct identification of the dual space to $L^p_{hol}(\gamma_\alpha)$.

1.3. The Lesbesgue setting and the operator Q_α

Following the discussion at the end of Section 1.2, and noting that $\mathcal{K}_\alpha(z,z)^{1/2} = e^{\frac{\alpha}{2}|z|^2}$, we should consider the following spaces.

Definition 1.10. For $\alpha > 0$, let $\mathcal{S}_\alpha$ denote the space

$$\mathcal{S}_\alpha = \left\{F(z) = f(z)e^{-\frac{\alpha}{2}|z|^2}\colon\ f \text{ is holomorphic on } \mathbb{C}^n\right\}. \tag{1.13}$$

For $1 \leqslant p < \infty$, define $\mathcal{S}^p_\alpha = \mathcal{S}_\alpha \cap L^p(\mathbb{C}^n, \lambda)$.

Consider the multiplier map $\mathfrak{g}_\alpha$

$$(\mathfrak{g}_\alpha f)(z) = e^{-\frac{\alpha}{2}|z|^2} f(z), \tag{1.14}$$

determined by the density of the measure $\gamma_{\alpha/2}$. Thus $\mathcal{S}_\alpha = \mathfrak{g}_\alpha \mathrm{Hol}(\mathbb{C}^n)$. The norm on $\mathcal{S}^p_\alpha$ is given by Lebesgue measure. It is easy to see that $\mathcal{S}^p_\alpha$ is a closed subspace of $L^p(\mathbb{C}^n, \lambda)$. In particular, there is an orthogonal projection Q_α

$$Q_\alpha : L^2\big(\mathbb{C}^n, \lambda\big) \to \mathcal{S}^2_\alpha. \tag{1.15}$$

Q_α is a reinterpretation of P_α, as we now explain. We can use the map $\mathfrak{g}_\alpha$ to connect the $\mathcal{S}_\alpha^p$ spaces with the $L^p_{hol}(\gamma_\alpha)$ spaces. Indeed, $\mathfrak{g}_\alpha f \in \mathcal{S}_\alpha$ iff f is holomorphic. A simple calculation reveals that $\mathfrak{g}_\alpha$ is a dilation from $L^p(\lambda)$ to $L^p(\gamma_{\alpha p/2})$.

$$\begin{aligned}
\|\mathfrak{g}_\alpha f\|^p_{L^p(\lambda)} &= \int\limits_{\mathbb{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p \lambda(dz) \\
&= \int\limits_{\mathbb{C}^n} \left| f(z) \right|^p e^{-\frac{\alpha p}{2}|z|^2} \lambda(dz) \\
&= \left(\frac{2\pi}{p\alpha}\right)^n \|f\|^p_{L^p(\gamma_{\alpha p/2})}.
\end{aligned}$$

The multiplier function is strictly positive, and so $\mathfrak{g}_\alpha$ is a bijection. Hence, rescaling the multiplication map

$$\mathfrak{g}_{\alpha,p} = \left(\frac{p\alpha}{2\pi}\right)^{n/p} \mathfrak{g}_\alpha, \tag{1.16}$$

we have the following.

Proposition 1.11. *Let* $1 \leqslant p < \infty$ *and* $\alpha > 0$. *The map* $\mathfrak{g}_{\alpha,p}$ *of Eqs.* (1.14) *and* (1.16) *is an isometric isomorphism* $L^p(\gamma_{\alpha p/2}) \to L^p(\lambda)$. *Its restriction* $\mathfrak{g}_{\alpha,p} : L^p_{hol}(\gamma_{\alpha p/2}) \to \mathcal{S}^p_\alpha$ *is also an isometric isomorphism. Hence, the following diagram commutes.*

$$\begin{array}{ccc}
L^p(\gamma_\alpha) & \xrightarrow{\mathfrak{g}_{\alpha,p}} & L^p(\lambda) \\
{\scriptstyle P_\alpha}\downarrow & & \downarrow{\scriptstyle Q_\alpha} \\
L^p_{hol}(\gamma_\alpha) & \xrightarrow[\mathfrak{g}_{\alpha,p}]{} & \mathcal{S}^p_\alpha
\end{array}$$

Remark 1.12. One may simply use the map $\mathfrak{g}_\alpha$ in place of $\mathfrak{g}_{\alpha,p}$ in the diagram, but we find this setup more æsthetically pleasing; here, the horizontal arrows are isometric isomorphisms, and the vertical arrows are orthogonal projections.

Thus $Q_\alpha = \mathfrak{g}_{\alpha,p} P_\alpha \mathfrak{g}_{\alpha,p}^{-1} = \mathfrak{g}_\alpha P_\alpha \mathfrak{g}_\alpha^{-1}$ is the conjugated action of the Bargmann projection, from the standard L^2 space $L^2(\mathbb{C}^n, \lambda)$ onto $\mathcal{S}^2_\alpha$. From Eq. (1.3), this means Q_α has the integral representation

$$\begin{aligned}
Q_\alpha F(z) &= \mathfrak{g}_\alpha \left(\int\limits_{\mathbb{C}^n} e^{\alpha\langle z,w\rangle} \left(\mathfrak{g}_\alpha^{-1} F\right)(w)\, \gamma_\alpha(dw) \right) = \left(\frac{\alpha}{\pi}\right)^n \int\limits_{\mathbb{C}^n} e^{-\frac{\alpha}{2}|z|^2 + \alpha\langle z,w\rangle - \frac{\alpha}{2}|w|^2} F(w)\, \lambda(dw) \\
&= \int\limits_{\mathbb{C}^n} \mathcal{Q}_\alpha(z,w) F(w)\, \lambda(dw).
\end{aligned} \tag{1.17}$$

Here $\mathcal{Q}_\alpha(z,w) = (\frac{\alpha}{\pi})^n e^{-\frac{\alpha}{2}(|z|^2 - 2\langle z,w\rangle + |w|^2)}$ is the kernel of Q_α. In this form, Q_α may (a priori) act on $L^p(\lambda)$ for any p. To see that it does so boundedly, consider the operator $|Q_\alpha|$ whose integral kernel is $|\mathcal{Q}_\alpha|$:

$$
\begin{aligned}
|Q_\alpha| F(z) &= \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} \left| e^{-\frac{\alpha}{2}|z|^2 + \alpha\langle z,w\rangle - \frac{\alpha}{2}|w|^2} \right| F(w)\, \lambda(dw) \\
&= \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} e^{-\frac{\alpha}{2}|z-w|^2} F(w)\, \lambda(dw) = 2^n \cdot (\gamma_{\alpha/2} * F)(w). \qquad (1.18)
\end{aligned}
$$

That is, $|Q_\alpha|$ is convolution with $2^n \gamma_{\alpha/2}$. (Here and in the sequel, we let the symbol γ_α do double duty, representing both the measure and its density.) Young's convolution inequality therefore provides the following.

Proposition 1.13. *For* $1 \leqslant p < \infty$ *and* $\alpha > 0$,

$$
\big\| |Q_\alpha| : L^p\big(\mathbb{C}^n, \lambda\big) \to L^p\big(\mathbb{C}^n, \lambda\big) \big\| = 2^n.
$$

Remark 1.14. Proposition 1.13 is actually the main theorem in [3]. Our proof is different from theirs, and is quite elementary.

Proof. By Young's convolution inequality, $\| |Q_\alpha| F\|_p = 2^n \|\gamma_{\alpha/2} * F\|_p \leqslant 2^n \|\gamma_{\alpha/2}\|_1 \|F\|_p$, and γ_α is a probability density so $\|\gamma_{\alpha/2}\|_1 = 1$. To see that the inequality is saturated, take $F = \gamma_\beta$ for any $\beta > 0$, which is in L^p for any $p > 0$; since the Gaussian probability measures γ_β form a convolution semigroup, it follows that $|Q_\alpha|(\gamma_\beta) = 2^n \gamma_{\alpha/2} * \gamma_\beta = 2^n \gamma_{\alpha/2+\beta}$. A quick calculation using Eq. (1.6) shows that

$$
\|\gamma_\beta\|_p = \left(\int_{\mathbb{R}^{2n}} \left[\left(\frac{\beta}{\pi}\right)^n e^{-\beta|z|^2} \right]^p dz \right)^{1/p} = \pi^{(1/p-1)n} p^{-n/p} \beta^{(1-1/p)n}. \qquad (1.19)
$$

Therefore

$$
\frac{\| |Q_\alpha| \gamma_\beta \|_p}{\|\gamma_\beta\|_p} = 2^n \frac{\|\gamma_{\alpha/2+\beta}\|_p}{\|\gamma_\beta\|_p} = 2^n \left(\frac{\beta}{\alpha/2 + \beta} \right)^{(1-1/p)n}.
$$

For fixed $\alpha > 0$, this tends to 2^n as $\beta \to \infty$, concluding the proof. □

1.4. Elementary bounds on the norm of P_α

Proposition 1.13 immediately yields an upper-bound of 2^n for the norm of Q_α, and therefore of P_α, as the next proposition shows. We also show that the sharp constant (of Theorem 1.1) is an easy lower-bound.

Proposition 1.15. *Let $1 \leqslant p < \infty$ and $\alpha > 0$. Then the Bargmann projection $P_\alpha : L^p(\gamma_{\alpha p/2}) \to L^p_{hol}(\gamma_{\alpha p/2})$ is bounded, with norm*

$$\left(2\frac{1}{p^{1/p}p'^{1/p'}}\right)^n \leqslant \|P_\alpha\|_{L^p(\gamma_{\alpha p/2})\to L^p_{hol}(\gamma_{\alpha p/2})} \leqslant 2^n. \tag{1.20}$$

In particular, when $p = 1$, the norm is equal to 2^n.

Proof. For any $p \geqslant 1$ and $F \in L^p(\lambda)$,

$$\begin{aligned}\|Q_\alpha F\|_p^p &= \int_{\mathbb{C}^n}\left|\int_{\mathbb{C}^n}\mathcal{Q}_\alpha(z,w)F(w)\,\lambda(dw)\right|^p\lambda(dw)\\ &\leqslant \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\big|\mathcal{Q}_\alpha(z,w)\big|\big|F(w)\big|\,\lambda(dw)\right)^p\lambda(dw) = \big\||Q_\alpha||F|\big\|_p^p.\end{aligned}$$

Thus, for $F \neq 0$, Proposition 1.13 shows that

$$\frac{\|Q_\alpha F\|_p}{\|F\|_p} \leqslant \frac{\||Q_\alpha||F|\|_p}{\|F\|_p} = \frac{\||Q_\alpha||F|\|_p}{\||F|\|_p} \leqslant \big\||Q_\alpha|\big\|_{L^p(\lambda)\to L^p(\lambda)} \leqslant 2^n.$$

From Proposition 1.11, we have $P_\alpha = \mathfrak{g}_\alpha^{-1}Q_\alpha\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha,p}^{-1}Q_\alpha\mathfrak{g}_{\alpha,p}$ (the last equality following from the fact that $\mathfrak{g}_{\alpha,p}$ is just a scalar multiple of $\mathfrak{g}_\alpha$). Since the map $\mathfrak{g}_{\alpha,p}$ is an isometric isomorphism from $L^p(\gamma_{\alpha p/2})$ onto $L^p(\lambda)$ and its inverse is an isometric isomorphism from $\mathcal{S}_\alpha^p$ onto $L^p_{hol}(\gamma_{\alpha p/2})$, it therefore follows that

$$\|P_\alpha\|_{L^p(\gamma_{\alpha p/2})\to L^p_{hol}(\gamma_{\alpha p/2})} = \|Q_\alpha\|_{L^p(\lambda)\to\mathcal{S}_\alpha^p} = \|Q_\alpha\|_{L^p(\lambda)\to L^p(\lambda)}. \tag{1.21}$$

Therefore P_α is bounded on $L^p(\gamma_{\alpha p/2})$, with norm $\leqslant 2^n$. For the lower bound, again we test the norm against functions of the form $F = \gamma_\beta$ for $\beta > 0$; so

$$\|P_\alpha\|_{L^p(\gamma_{\alpha p/2})\to L^p_{hol}(\gamma_{\alpha p/2})} \geqslant \frac{\|Q_\alpha\gamma_\beta\|_p}{\|\gamma_\beta\|_p}. \tag{1.22}$$

Set $g_\beta(w) = e^{-\beta|w|^2}$, so that $\gamma_\beta = (\beta/\pi)^n g_\beta$. Then the ratio on the right-hand side of Eq. (1.22) is equal to $\|Q_\alpha g_\beta\|_p/\|g_\beta\|_p$. This latter ratio is calculated (as a special case) in Eq. (2.11) in Section 2.3. (To match up with that formula we take $A = \beta I_{2n}$; thus $A' + I_{2n} = (\frac{2}{\alpha}\beta + 1)I_{2n}$ is a real matrix which commutes with J, and thus $\Omega((A' + I_{2n})^{-1}) = 0$.) Thus

$$\frac{\|Q_\alpha g_\beta\|_p^p}{\|g_\beta\|_p^p} = 2^{np}\sqrt{\frac{\det(A')}{|\det(A'+I_{2n})|^p}} = 2^{np}\frac{(\frac{2}{\alpha}\beta)^n}{|\frac{2}{\alpha}\beta+1|^{np}} = \left(\frac{2^p c}{(1+c)^p}\right)^n, \tag{1.23}$$

where $c = \frac{2}{\alpha}\beta > 0$. Elementary calculus shows that, when $p > 1$, this is maximized uniquely when $c = \frac{1}{p-1}$, and so (taking pth roots) Eqs. (1.22) and (1.23) yield

$$\|P_\alpha\|_{L^p(\gamma_{\alpha p/2}) \to L^p_{hol}(\gamma_{\alpha p/2})} \geqslant \left(\frac{2(\frac{1}{p-1})^{1/p}}{1+\frac{1}{p-1}}\right)^n = \left(2\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right)^n. \tag{1.24}$$

This proves the proposition for $p > 1$. Note that

$$\lim_{p\downarrow 1} \frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}} = 1,$$

and so the lower-bound and upper-bound in Eq. (1.20) converge to 2^n as $p \downarrow 1$. For a precise proof of the L^1 lower-bound, we utilize Eq. (1.23) in the case $p = 1$; thus, for any $\beta > 0$,

$$\frac{\|Q_\alpha g_\beta\|_1}{\|g_\beta\|_1} = 2^n\left(\frac{c}{1+c}\right)^n$$

where $c = \frac{2}{\alpha}\beta$. Letting $\beta \to \infty$, so $c \to \infty$, we see the ratio approaches 2^n, proving the lower-bound; thus $\|Q_\alpha : L^1(\lambda) \to L^1(\lambda)\| = 2^n$, and the L^1-result follows from Eq. (1.21). □

Remark 1.16. In the preceding proof, we showed that Q_α is bounded on $L^p(\mathbb{C}^n, \lambda)$ for any $\alpha > 0$. Through the transformations $\mathfrak{g}_{\alpha,p} : L^p(\gamma_{\alpha p/2}) \to L^p(\lambda)$, this shows why the Bargmann projection P_α indexed by α is bounded on the scaled space $L^p(\gamma_{\alpha p/2})$, rather than the space $L^p(\gamma_\alpha)$ we may have naïvely expected. As Section 1.5 demonstrates, this is the reason for the unusual scaling properties of the dual spaces of $L^p_{hol}(\gamma_\alpha)$.

Remark 1.17. The proof of the lower bound in Proposition 1.15 above actually shows that, among Gaussian test functions of the form $g(w) = e^{-\beta|w|^2}$, there is a unique maximizer for the norm of Q_α, which yields the lower bound (when $p > 1$). As we will explain in Section 2.2, to generalize this technique to determine the sharp norm of Q_α (which is given by the calculated lower bound in general), we need to expand this maximization only to the class of centered Gaussian functions, of the form $g(x) = e^{-(x,Ax)}$ (now thinking of the variable $x \in \mathbb{R}^{2n}$, with $(\cdot,\cdot)$ the real inner product) where A is a complex symmetric matrix with positive-definite real part. This may sound simple, but it is not: the real and imaginary parts of A need not commute, making the problem extremely computationally difficult. The lengthy calculations in Section 2 in the special case $n = 1$ attest to this; in fact, our approach is to first reduce to the $n = 1$ case, in Section 2.1, as the general n-dimensional optimization does not admit a simple solution.

1.5. Identifying the dual space

Since the projection Q_α is bounded on $L^p(\mathbb{C}^n, \lambda)$ for each $\alpha > 0$, and has range equal to the space $\mathcal{S}^p_\alpha$, we can use the usual duality relations in L^p-spaces to translate to duality comparisons in $\mathcal{S}^p_\alpha$. For ease of reading, denote by $\|Q_\alpha\|_{p\to p}$ the norm of $Q_\alpha : L^p(\lambda) \to \mathcal{S}^p_\alpha$.

Lemma 1.18. *Let $\alpha > 0$ and $1 < p < \infty$. Let p' denote the conjugate exponent to p. In terms of the pairing $(F,G)_\lambda = \int_{\mathbb{C}^n} F\overline{G}\,d\lambda$, the spaces $\mathcal{S}_\alpha^p$ and $\mathcal{S}_\alpha^{p'}$ are dual, with*

$$\|Q_\alpha\|_{p\to p}^{-1}\|G\|_{\mathcal{S}_\alpha^{p'}} \leqslant \big\|(\cdot,G)_\lambda\big\|_{(\mathcal{S}_\alpha^p)^*} \leqslant \|G\|_{\mathcal{S}_\alpha^{p'}} \tag{1.25}$$

for all $G \in \mathcal{S}_\alpha^{p'}$.

Proof. As in the proof of Corollary 1.6, for $\Phi \in (\mathcal{S}_\alpha^p)^*$ denote by $\hat{\Phi}$ the linear functional $\hat{\Phi} = Q_\alpha^*\Phi$; i.e. $\hat{\Phi}(F) = \Phi(Q_\alpha F)$ for $F \in L^p(\lambda)$. Since $Q_\alpha : L^p(\lambda) \to \mathcal{S}_\alpha^p$ is bounded, the linear functional $\hat{\Phi}$ is continuous on $L^p(\lambda)$; that is, $\hat{\Phi} \in (L^p(\lambda))^*$. The L^p Riesz Representation Theorem therefore shows that there is a unique function $G \in L^{p'}(\lambda)$ such that $\hat{\Phi} = (\cdot,G)_\lambda$, and moreover that

$$\big\|(\cdot,G)_\lambda\big\|_{(L^p(\lambda))^*} = \|G\|_{L^{p'}(\lambda)}. \tag{1.26}$$

Since $Q_\alpha^2 = Q_\alpha$, it follows that $\hat{\Phi}(Q_\alpha F) = \hat{\Phi}(F)$ for all $F \in L^p(\lambda)$. Thus, for any such F,

$$(F,G)_\lambda = \hat{\Phi}(F) = \hat{\Phi}(Q_\alpha F) = (Q_\alpha F, G)_\lambda = (F, Q_\alpha G)_\lambda, \tag{1.27}$$

where the last equality holds since Q_α is self adjoint on $L^2(\lambda)$ (and so by a standard density argument the self-adjointness extends to the L^p–$L^{p'}$ pairing). Since Eq. (1.27) holds for all $F \in L^p(\lambda)$, it follows that $G = Q_\alpha G$, and hence $G \in \mathcal{S}_\alpha^{p'}$. Now, for $F \in \mathcal{S}_\alpha^p \subset L^p(\lambda)$, $\hat{\Phi}(F) = \Phi(Q_\alpha F) = \Phi(F)$; that is, $\hat{\Phi}|_{\mathcal{S}_\alpha^p} = \Phi$. Thus, $\Phi(F) = (F,G)_\lambda$.

To summarize, we have shown that the map $G \mapsto (\cdot,G)_\lambda$ is surjective from $\mathcal{S}_\alpha^{p'}$ onto $(\mathcal{S}_\alpha^p)^*$; it is also continuous due to Eq. (1.26). We must now show that it is injective. Since $G \in \mathcal{S}_\alpha$, by definition there is a holomorphic g such that $G = \mathfrak{g}_\alpha g$. Consider the monomial $f(z) = z^{\mathbf{j}}$ for $\mathbf{j} \in \mathbb{N}^n$. Since f has polynomial growth, $\mathfrak{g}_\alpha f \in \mathcal{S}_\alpha^p$ (for any p). Hence, we can compute

$$(\mathfrak{g}_\alpha f, \mathfrak{g}_\alpha g)_\lambda = \int z^{\mathbf{j}}\overline{g(z)}e^{-\alpha|z|^2}\,\lambda(dz) = \left(\frac{\alpha}{\pi}\right)^n \int z^{\mathbf{j}}\overline{g(z)}\,\gamma_\alpha(dz) = \left(\frac{\alpha}{\pi}\right)^n (f,g)_\alpha.$$

Due to the orthogonality relations of Eq. (1.7), the inner product $(f,g)_\alpha$ is a scalar multiple of the $\mathbf{j}$th Taylor coefficient of f. Since g is holomorphic, these coefficients determine g uniquely, and so too G. It follows that the map $G \mapsto (\cdot,G)_\lambda$ is injective. Thus, S_α^p and $S_\alpha^{p'}$ are dual with respect to $(\cdot,\cdot)_\lambda$.

As for Eq. (1.25), the first inequality follows since $\|G\|_{L^{p'}(\lambda)} = \|\hat{\Phi}\|_{(L^p(\lambda))^*}$, and

$$\|\hat{\Phi}\|_{(L^p(\lambda))^*} = \sup_{F\in L^p(\lambda)} \frac{|\hat{\Phi}(F)|}{\|F\|_p} = \sup_{F\in L^p(\lambda)} \frac{|\Phi(Q_\alpha F)|}{\|F\|_p} \leqslant \|Q_\alpha\|_{p\to p} \sup_{F\in L^p(\lambda)} \frac{|\Phi(Q_\alpha F)|}{\|Q_\alpha F\|_p},$$

where the inequality is just the statement that $\|Q_\alpha F\|_p \leqslant \|Q_\alpha\|_p\|F\|_p$. Note, as shown in the first paragraph, $\|\hat{\Phi}\|_{(L^p(\lambda))^*} = \|(\cdot,G)_\lambda\|_{(L^p(\lambda))^*} = \|G\|_{L^{p'}(\lambda)} = \|G\|_{\mathcal{S}_\alpha^{p'}}$ since $G \in \mathcal{S}_\alpha^{p'}$. Since the range of Q_α on $L^p(\lambda)$ is all of $\mathcal{S}_\alpha^p$, we therefore have

$$\|G\|_{\mathcal{S}_\alpha^{p'}} = \|\hat{\Phi}\|_{(L^p(\lambda))^*} \leqslant \|Q_\alpha\|_{p\to p} \sup_{H\in\mathcal{S}_\alpha^p} \frac{|\Phi(H)|}{\|H\|_p} = \|Q_\alpha\|_{p\to p}\|\Phi\|_{(\mathcal{S}_\alpha^p)^*}$$
$$= \|Q_\alpha\|_p \big\|(\cdot, G)\big\|_{(\mathcal{S}_\alpha^p)^*}.$$

The second estimate in Eq. (1.25), that $\|(\cdot, G)_\lambda\|_{(\mathcal{S}_\alpha^p)^*} \leqslant \|G\|_{\mathcal{S}_\alpha^{p'}}$, is a straightforward consequence of Hölder's inequality. □

Due to the isometry $\mathfrak{g}_{\alpha,p} : \mathcal{S}_\alpha^p \to L^p_{hol}(\gamma_\alpha)$, Lemma 1.18 also yields the proof of Theorem 1.2.

Proof of Theorem 1.2. Dividing through by the $L^p(\gamma_{\alpha p/2}) \to L^p_{hol}(\gamma_{\alpha p/2})$ norm of P_α (following Theorem 1.1), the desired inequalities (1.5) can be written as

$$\|P_\alpha\|_{p\to p}^{-1}\|g\|_{L^{p'}_{hol}(\gamma_{\alpha p'/2})} \leqslant \left(\frac{1}{2}p^{1/p}p'^{1/p'}\right)^n \big\|(\cdot, g)_\alpha\big\|_{L^p_{hol}(\gamma_{\alpha p/2})^*} \leqslant \|g\|_{L^{p'}_{hol}(\gamma_{\alpha p'/2})}. \quad (1.28)$$

We now prove inequalities (1.28). From Eq. (1.21), $\|P_\alpha\|_{p\to p} = \|Q_\alpha\|_{p\to p}$. Any $g \in L^{p'}_{hol}(\gamma_{\alpha p'/2})$ can be written uniquely as $g = \mathfrak{g}_{\alpha,p'}^{-1}G$ for some $G \in \mathcal{S}_\alpha^p$, and since $\mathfrak{g}_{\alpha,p'}$ is an isometry, Eq. (1.25) then yields

$$\|P_\alpha\|_{p\to p}^{-1}\|g\|_{L^{p'}_{hol}(\gamma_{\alpha p'/2})} = \|Q_\alpha\|_{p\to p}^{-1}\|G\|_{\mathcal{S}_\alpha^{p'}} \leqslant \big\|(\cdot, G)_\lambda\big\|_{(\mathcal{S}_\alpha^p)^*} \leqslant \|G\|_{\mathcal{S}_\alpha^{p'}}$$
$$= \|g\|_{L^{p'}_{hol}(\gamma_{\alpha p'/2})}. \quad (1.29)$$

We are left to re-express $\|(\cdot, G)_\lambda\|_{(\mathcal{S}_\alpha^p)^*}$ in terms of g. Since $\mathfrak{g}_{\alpha,p} : \mathcal{S}_\alpha^p \to L^p_{hol}(\gamma_{\alpha p/2})$ is an isometric isomorphism, for any $F \in \mathcal{S}_\alpha^p$ there is a unique $f \in L^p_{hol}(\gamma_{\alpha p/2})$ such that $F = \mathfrak{g}_{\alpha,p}f$. Taking $G = \mathfrak{g}_{\alpha,p'}g$ as above, we can write

$$\big\|(\cdot, G)_\lambda\big\|_{(\mathcal{S}_\alpha^p)^*} = \sup_F \frac{|(F, G)_\lambda|}{\|F\|_{\mathcal{S}_\alpha^p}} = \sup_f \frac{|(\mathfrak{g}_{\alpha,p}f, \mathfrak{g}_{\alpha,p'}g)_\lambda|}{\|f\|_{L^p_{hol}(\gamma_{\alpha p/2})}}. \quad (1.30)$$

From Eqs. (1.14) and (1.16) defining the isometry $\mathfrak{g}_{\alpha,p}$, we have

$$(\mathfrak{g}_{\alpha,p}f, \mathfrak{g}_{\alpha,p'}g)_\lambda = \int_{\mathbb{C}^n} \left(\frac{p\alpha}{2\pi}\right)^{n/p} e^{-\frac{\alpha}{2}|z|^2} f(z) \left(\frac{p'\alpha}{2\pi}\right)^{n/p'} e^{-\frac{\alpha}{2}|z|^2}\overline{g(z)}\,\lambda(dz)$$
$$= \left(\frac{p^{1/p}p'^{1/p'}\alpha}{2\pi}\right)^n \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-\alpha|z|^2}\,\lambda(dz) = \left(\frac{1}{2}p^{1/p}p'^{1/p'}\right)^n (f, g)_\alpha.$$

Hence Eq. (1.30) becomes

$$\begin{aligned}\left\|(\cdot, G)_\lambda\right\|_{(\mathcal{S}_\alpha^p)^*} &= \left(\frac{1}{2} p^{1/p} p'^{1/p'}\right)^n \sup_f \frac{|(f,g)_\alpha|}{\|f\|_{L^p_{hol}(\gamma_{\alpha p/2})}} \\ &= \left(\frac{1}{2} p^{1/p} p'^{1/p'}\right)^n \left\|(\cdot, g)_\alpha\right\|_{(L^p_{hol}(\gamma_{\alpha p/2}))^*}. \qquad (1.31)\end{aligned}$$

Inequality (1.29) and Eq. (1.31) combine to prove the estimates in the inequalities (1.28). □

2. The norm of P_α

In this section, we prove Theorem 1.1. Since the sharp constant in that theorem is an nth power, we begin by showing that it is sufficient to prove the theorem in the case $n = 1$; this is a version of an idea due to Segal. The kernel $\mathcal{Q}_\alpha$ is a Gaussian function, and so to find the norm of P_α (which is the same as that of Q_α), we use the deep results of [6] to reduce to a calculation on putative Gaussian maximizers. The resulting optimization problem is difficult; the majority of this section is devoted to its resolution in the $n = 1$ case. (Note that n is the complex dimension; the maximizer thus corresponds to a 2×2 complex symmetric matrix, so the calculation involves a complicated function of 6 real variables.)

2.1. Segal's lemma

Proposition 2.1 below is a simple variant of what is colloquially known as *Segal's lemma*, based on a version appearing as Lemma 1.4 in [8] (for the case of positive kernels G_i). The proposition actually holds for kernels mapping $L^p \to L^q$ for $1 < p \leqslant q < \infty$: the following proof need only be modified by replacing the application of Tonelli's theorem in Eq. (2.3) with Minkowski's inequality for integrals. The proof we present is essentially contained in the proof of [6, Theorem 3.3].

Proposition 2.1. *Let $n, m \geqslant 1$ be integers, let $1 \leqslant p < \infty$, and let p' denote the conjugate exponent to p. Let $G_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ and $G_2 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}$ be complex functions, such that $G_1(x_1, \cdot) \in L^{p'}(\mathbb{R}^n)$ for almost every $x_1 \in \mathbb{R}^n$ and $G_2(x_2, \cdot) \in L^{p'}(\mathbb{R}^m)$ for almost every $x_2 \in \mathbb{R}^m$. Define*

$$T_1 f(x_1) = \int_{\mathbb{R}^n} G_1(x_1, y_1) f(y_1)\, dy_1, \qquad T_2 f(x_2) = \int_{\mathbb{R}^m} G_2(x_2, y_2) f(y_2)\, dy_2.$$

Let $G = G_1 \otimes G_2$: $G((x_1, x_2), (y_1, y_2)) = G_1(x_1, y_1) G_2(x_2, y_2)$; and let $T = T_1 \otimes T_2$:

$$T F(x_1, x_2) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} G\big((x_1, x_2), (y_1, y_2)\big) F(y_1, y_2)\, dy_1\, dy_2.$$

If T_1 is bounded on $L^p(\mathbb{R}^n)$ and T_2 is bounded on $L^p(\mathbb{R}^m)$, then T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, and

$$\|T\|_{p \to p} = \|T_1\|_{p \to p} \|T_2\|_{p \to p}.$$

Proof. Let $F \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$. For $1 < p < \infty$, by Tonelli's theorem, for almost every $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$ we have

$$\begin{aligned}\big\|G\big((x_1,x_2),(\cdot,\cdot)\big)\big\|_{p'}^{p'} &= \int\limits_{\mathbb{R}^n\times\mathbb{R}^m} \big|G_1(x_1,y_1)G_2(x_2,y_2)\big|^{p'}\,dy_1\,dy_2 \\ &= \int\limits_{\mathbb{R}^n}\bigg(\int\limits_{\mathbb{R}^m}\big|G_1(x_1,y_1)\big|^{p'}\big|G_2(x_2,y_2)\big|^{p'}\,dy_2\bigg)\,dy_1 \\ &= \bigg(\int\limits_{\mathbb{R}^n}\big|G_1(x_1,y_1)\big|^{p'}\,dy_1\bigg)\bigg(\int\limits_{\mathbb{R}^m}\big|G_2(x_2,y_2)\big|^{p'}\,dy_2\bigg) < \infty\end{aligned}$$

by assumption. Similarly, in the case $p = 1$ it is easy to check that $G \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Thus, using Hölder's inequality,

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} \big|G_1(x_1,y_1)G_2(x_2,y_2)F(y_1,y_2)\big|\,dy_1\,dy_2 \leqslant \big\|G\big((x_1,x_2),(\cdot,\cdot)\big)\big\|_{p'}\|F\|_p < \infty$$

for almost every $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$. Hence, calculating the L^p norm of TF, we can apply Fubini's theorem:

$$\begin{aligned}\|TF\|_p^p &= \int\limits_{\mathbb{R}^n\times\mathbb{R}^m}\bigg|\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} G_1(x_1,y_1)G_2(x_2,y_2)F(y_1,y_2)\,dy_1\,dy_2\bigg|^p\,dx_1\,dx_2 \\ &= \int\limits_{\mathbb{R}^n}\bigg(\int\limits_{\mathbb{R}^m}\bigg|\int\limits_{\mathbb{R}^m} G_2(x_2,y_2)\bigg(\int\limits_{\mathbb{R}^n} G_1(x_1,y_1)F(y_1,y_2)\,dy_1\bigg)\,dy_2\bigg|^p\,dx_2\bigg)\,dx_1 \\ &= \int\limits_{\mathbb{R}^n}\bigg(\int\limits_{\mathbb{R}^m}\bigg|\int\limits_{\mathbb{R}^m} G_2(x_2,y_2)K(x_1,y_2)\,dy_2\bigg|^p\,dx_2\bigg)\,dx_1 \qquad (2.1)\end{aligned}$$

where

$$K(x_1,y_2) = \int\limits_{\mathbb{R}^n} G_1(x_1,y_1)F(y_1,y_2)\,dy_1 = \big(T_1F(\cdot,y_2)\big)(x_1). \qquad (2.2)$$

We do not know, a priori, whether the function $K(x_1,\cdot)$ is in $L^p(\mathbb{R}^m)$; but it is nevertheless true that

$$\int\limits_{\mathbb{R}^m}\bigg|\int\limits_{\mathbb{R}^m} G_2(x_2,y_2)K(x_1,y_2)\,dy_2\bigg|^p\,dx_2 = \big\|T_2K(x_1,\cdot)\big\|_p^p \leqslant \big(\|T_2\|_{p\to p}\big)^p\big\|K(x_1,\cdot)\big\|_p^p$$

for each $x_1 \in \mathbb{R}^n$ (where the right-hand side is $+\infty$ in the case $K(x_1,\cdot) \notin L^p$). Combining with Eq. (2.1) then yields

$$\|TF\|_p^p = \int_{\mathbb{R}^n} \big\|T_2 K(x_1,\cdot)\big\|_p^p\,dx_1 \leqslant \big(\|T_2\|_{p\to p}\big)^p \int_{\mathbb{R}^n} \big\|K(x_1,\cdot)\big\|_p^p\,dx_1$$
$$= \big(\|T_2\|_{p\to p}\big)^p \int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^m} \big|K(x_1,y_2)\big|^p\,dy_2\bigg)\,dx_1. \tag{2.3}$$

We now apply Tonelli's theorem to the (non-negative) integrand in Eq. (2.3), to reverse the order of integration:

$$\int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^m} \big|K(x_1,y_2)\big|^p\,dy_2\bigg)\,dx_1 = \int_{\mathbb{R}^m}\bigg(\int_{\mathbb{R}^n} \big|K(x_1,y_2)\big|^p\,dx_1\bigg)\,dy_2. \tag{2.4}$$

For almost every $y_2 \in \mathbb{R}^m$, the function $F(\cdot, y_2)$ is in $L^p(\mathbb{R}^n)$. Referring to Eq. (2.2), it follows that for such y_2 we have

$$\int_{\mathbb{R}^n} \big|K(x_1,y_2)\big|^p\,dx_1 = \big\|T_1 F(\cdot,y_2)\big\|_p^p \leqslant \big(\|T_1\|_{p\to p}\big)^p \big\|F(\cdot,y_2)\big\|_p^p.$$

Thus, Eq. (2.4) gives

$$\int_{\mathbb{R}^m}\bigg(\int_{\mathbb{R}^n} \big|K(x_1,y_2)\big|^p\,dx_1\bigg)\,dy_2 \leqslant \big(\|T_1\|_{p\to p}\big)^p \int_{\mathbb{R}^m} \big\|F(\cdot,y_2)\big\|_p^p\,dy_2$$
$$= \big(\|T_1\|_{p\to p}\big)^p \int_{\mathbb{R}^m}\bigg(\int_{\mathbb{R}^n} \big|F(y_1,y_2)\big|^p\,dy_1\bigg)\,dy_2$$
$$= \big(\|T_1\|_{p\to p}\big)^p \|F\|_p^p. \tag{2.5}$$

Eq. (2.5) shows that $K \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$, and so in particular the bound in Eq. (2.3) is non-trivial. Combining Eqs. (2.3), (2.4), and (2.5) and taking pth roots shows that

$$\|TF\|_p \leqslant \|T_2\|_{p\to p}\|T_1\|_{p\to p}\|F\|_p$$

proving the upper bound of the proposition. The lower bound is actually the easier direction: let $f_k \in L^p(\mathbb{R}^n)$ and $g_k \in L^p(\mathbb{R}^m)$ be L^p-normalized functions saturating the L^p-norms of the operators T_1 and T_2; that is, $\|f_k\|_p = \|g_k\|_p = 1$ and

$$\lim_{k\to\infty} \|T_1 f_k\|_p = \|T_1\|_{p\to p}, \qquad \lim_{k\to\infty} \|T_2 g_k\|_p = \|T_2\|_{p\to p}.$$

Let $F_k = f_k \otimes g_k$: $F_k(y_1,y_2) = f_k(y_1)g_k(y_2)$. Then Tonelli's theorem quickly shows that $\|F_k\|_p = \|f_k\|_p\|g_k\|_p = 1$, and Fubini's theorem (as above) shows that

$$TF_k(x_1,x_2)=\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} G_1(x_1,y_1)G_2(x_2,y_2)f_k(y_1)g_k(y_2)\,dy_1\,dy_2$$
$$=\biggl(\int\limits_{\mathbb{R}^n} G_1(x_1,y_1)f_k(y_1)\,dy_1\biggr)\biggl(\int\limits_{\mathbb{R}^m} G_2(x_2,y_2)g_k(y_2)\,dy_2\biggr)=T_1f_k(x_1)\cdot T_2g_k(x_2).$$

Then Tonelli's theorem (as above) shows that

$$\|TF_k\|_p=\|T_1f_k\|_p\|T_2g_k\|_p\to\|T_1\|_{p\to p}\|T_2\|_{p\to p}\quad\text{as } k\to\infty.$$

This shows that $\|T\|_{p\to p}\geqslant\|T_1\|_{p\to p}\|T_2\|_{p\to p}$, completing the proof. □

Remark 2.2. The kernel $\mathcal{Q}_\alpha$ is, in fact, a tensor power; induction on Proposition 2.1 will therefore reduce the calculation of $\|Q_\alpha\|_{p\to p}$ to the (nth power of the) $n=1$ case. First we need to verify that $\mathcal{Q}_\alpha$ satisfies the $L^{p'}$-bound conditions of Proposition 2.1. This will follow easily from the Gaussian character of the kernel, and is the content of Corollary 2.3. Indeed, this Gaussian character gives us much more, as the next section attests to.

Corollary 2.3. *Let Q_α^1 denote the operator Q_α of Eq.* (1.17) *in the case $n=1$. Let $1<p<\infty$, and $\alpha>0$. Then*

$$\|Q_\alpha\|_{p\to p}=\bigl(\bigl\|Q_\alpha^1\bigr\|_{p\to p}\bigr)^n.$$

Proof. Note, from Eq. (1.17), that

$$Q_\alpha F(x)=\int\limits_{\mathbb{R}^{2n}}\mathcal{Q}_\alpha(z,w)F(w)\,dw$$

where

$$\mathcal{Q}_\alpha(z,w)=\left(\frac{\alpha}{\pi}\right)^n\exp\biggl\{-\frac{\alpha}{2}\bigl(|z|^2+|w|^2-2\langle z,w\rangle\bigr)\biggr\}.$$

Of course the quadratic form is a sum over independent variables,

$$|z|^2+|w|^2-2\langle z,w\rangle=\sum_{j=1}^n\bigl(|z_j|^2+|w_j|^2-2z_j\overline{w_j}\bigr)$$

and so we have

$$\mathcal{Q}_\alpha(z,w)=\prod_{j=1}^n\mathcal{Q}_\alpha^1(z_j,w_j)$$

where

$$\mathcal{Q}_\alpha^1(z_1,w_1)=\frac{\alpha}{\pi}\exp\biggl\{-\frac{\alpha}{2}\bigl(|z_1|^2+|w_1|^2-2z_1\overline{w_1}\bigr)\biggr\}.$$

Notice also that $\mathcal{Q}_\alpha^1$ is the kernel of Q_α^1; so we have

$$Q_\alpha = \bigotimes_{j=1}^{n} Q_\alpha^1.$$

Furthermore, the kernel $\mathcal{Q}_\alpha$ (in any dimension) satisfies

$$\left|\mathcal{Q}_\alpha(z,w)\right| = \left(\frac{\alpha}{\pi}\right)^n \exp\left\{-\frac{\alpha}{2}\left(|z|^2 + |w|^2 - 2\Re\langle z,w\rangle\right)\right\} = \left(\frac{\alpha}{\pi}\right)^n \exp\left\{-\frac{\alpha}{2}|z-w|^2\right\}$$

(as computed once before in Eq. (1.18)). Of course this means $\mathcal{Q}_\alpha(z,\cdot) \in L^\infty$ (with norm $(\alpha/\pi)^n$) and also in $L^{p'}$ for any $p > 1$; using Eq. (1.6),

$$\int \left|\mathcal{Q}_\alpha(z,w)\right|^{p'} dw = \left(\frac{\alpha}{\pi}\right)^{np'} \int_{\mathbb{R}^{2n}} e^{-\frac{\alpha p'}{2}|z-w|^2}\, dw = \left(\frac{\alpha}{\pi}\right)^{np'} \left(\frac{2\pi}{\alpha p'}\right)^n < \infty$$

for all z. Hence, the corollary follows by induction on Proposition 2.1. □

2.2. *Gaussian kernels*

As previously proved, the projection Q_α is a bounded map $L^p(\mathbb{C}^n,\lambda) \to \mathcal{S}_\alpha^p$, with

$$\|Q_\alpha\|_{p\to p} = \left\|Q_\alpha : L^p(\mathbb{C}^n,\lambda) \to \mathcal{S}_\alpha^p\right\| = \left\|P_\alpha : L^p(\mathbb{C}^n,\gamma_{\alpha p/2}) \to L^p_{hol}(\mathbb{C}^n,\gamma_{\alpha p/2})\right\| \leqslant 2^n$$

(cf. Theorem 1.1 and Proposition 1.11). We will investigate the bound $\|Q_\alpha\|_{p\to p}$ (and thus $\|P_\alpha\|_{p\to p}$) using a main result of [6]. Before we state the particular theorem from [6] that we will use, we will first establish some notation. For a fixed integer $k \geqslant 1$, define the set of matrices $\mathcal{A}^k$ as

$$\mathcal{A}^k = \left\{A \in \mathbb{C}^{k\times k}\colon\ A \text{ is symmetric and } \Re(A) \text{ is positive definite}\right\}. \tag{2.6}$$

In turn we define the set of *(centered) Gaussian functions* $\mathcal{G}^k$ as

$$\mathcal{G}^k = \left\{g(x) = e^{-(x,Ax)}\colon\ A \in \mathcal{A}^k\right\}. \tag{2.7}$$

In the definition above and in what follows, the inner product $(\cdot,\cdot)$ denotes the standard inner product on $\mathbb{R}^k$ extended to $\mathbb{C}^k$ such that $(\cdot,\cdot)$ is bilinear (and not sesquilinear). We can now state the theorem will we use from [6]:

Theorem 2.4 *(Lieb, 1990). Let $1 < p < \infty$. Suppose $T : L^p(\mathbb{R}^k,\lambda) \to L^p(\mathbb{R}^k,\lambda)$ is a bounded integral operator with a Gaussian kernel $G(x,y)$. Specifically, for $f \in L^p(\mathbb{R}^k,\lambda) \cap L^1(\mathbb{R}^k,\lambda)$, we can write $T(f)(x)$ as*

$$T(f)(x) = \int_{\mathbb{R}^k} f(y)G(x,y)\, dy,$$

where $G(x, y)$ has the form

$$G(x, y) = \exp\{-(x, D_{11}x) - (y, D_{22}y) - 2(x, D_{12}y)\},$$

where D_{11} and D_{22} are real symmetric matrices. If the real part of the block matrix $\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{bmatrix}$ is positive semidefinite, then the norm $\|T\|$ can be computed as

$$\|T\| = \sup_{g \in \mathcal{G}^k} \frac{\|Tg\|_p}{\|g\|_p}.$$

Theorem 2.4 is a less general version of Theorem 4.1 in [6]. To apply Theorem 2.4 to Q_α, we will need to view the space $L^p(\mathbb{C}^n, \lambda)$ as $L^p(\mathbb{R}^{2n}, \lambda)$. Recall that

$$Q_\alpha F(z) = \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} F(w) e^{-\alpha|z|^2/2 - \alpha|w|^2/2 + \alpha\langle z, w\rangle}\, dw, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the sesquilinear inner product which is linear in its first argument. Associate $z = x_1 + ix_2$ with the real vector $x = [x_1, x_2]$ and $w = y_1 + iy_2$ with the real vector $y = [y_1, y_2]$. Q_α becomes

$$Q_\alpha F(x) = \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{R}^{2n}} F(y) e^{-(x, D_{11}x) - (y, D_{22}y) - 2(x, D_{12}y)}\, dy, \tag{2.9}$$

where $D_{11} = D_{22} = (\alpha/2)I_{2n}$ and

$$D_{12} = -\alpha/2 \begin{bmatrix} I_n & -iI_n \\ iI_n & I_n \end{bmatrix}.$$

One can check that the real part of the $4n$-by-$4n$ matrix $\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{bmatrix}$ has exactly two eigenvalues 0 and α, each of multiplicity $2n$. Since α is assumed to be positive, the block matrix is positive semidefinite. Thus, we can apply Theorem 2.4 and we have

Lemma 2.5. *For any $1 < p < \infty$, the operator norm $\|Q_\alpha\|_{p\to p}$ of $Q_\alpha : L^p(\mathbb{R}^{2n}, \lambda) \to \mathcal{S}_\alpha^p \subset L^p(\mathbb{R}^{2n}, \lambda)$ is*

$$\|Q_\alpha\|_{p\to p} = \sup_{g \in \mathcal{G}^{2n}} \frac{\|Q_\alpha g\|_p}{\|g\|_p}. \tag{2.10}$$

2.3. A formula for $\frac{\|Q_\alpha g\|_p}{\|g\|_p}$

Lemma 2.6. *Let n be any positive integer, $0 < p < \infty$, and $g \in \mathcal{G}^{2n}$ so that $g(x) = e^{-(x, Ax)}$ for some $A \in \mathcal{A}^{2n}$. Let $A' = \frac{2}{\alpha}A$. Then*

$$\frac{\|Q_\alpha g\|_p^p}{\|g\|_p^p} = 2^{np} \sqrt{\frac{\det(\Re(A'))}{|\det(A' + I_{2n})|^p \det(I_{2n} + \Omega((A' + I_{2n})^{-1}))}}, \tag{2.11}$$

where for any matrix $M \in \mathbb{C}^{2n\times 2n}$

$$\Omega(M) := J^T \Re(M) J - \Re(M) - \Im(M) J - J^T \Im(M). \tag{2.12}$$

In Eq. (2.12)*, J is the $2n \times 2n$ symplectic matrix* $J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$.

Using A' instead of A above allows us to write the quotient $\frac{\|Q_\alpha g\|_p^p}{\|g\|_p^p}$ independently of α.

Proof. Note that $g(x) = e^{-(x, \frac{\alpha}{2} A' x)}$. Using Eq. (1.6) above we have

$$\|g\|_p^p = \left(\frac{2\pi}{p\alpha}\right)^n \frac{1}{\sqrt{\det(\Re(A'))}}. \tag{2.13}$$

To calculate $\|Q_\alpha g\|_p^p$, note that $D_{12} = -\frac{\alpha}{2}(I_{2n} + iJ)$ and $J^T = -J$; using this together with Eq. (1.6), one can calculate

$$\|Q_\alpha g\|_p^p = \left(\frac{2^{np}}{\sqrt{|\det(A' + I_{2n})|^p}}\right) \cdot \int_{\mathbb{R}^{2n}} e^{-\frac{p\alpha}{2}(x,x)} \left|e^{(x, \frac{\alpha}{2}(I_{2n}+iJ)(A'+I_{2n})^{-1}(I_{2n}+iJ^T)x)}\right|^p dx. \tag{2.14}$$

One can verify that

$$\Re\big((I_{2n} + iJ)\big(A' + I_{2n}\big)^{-1}(I_{2n} - iJ)\big) = -\Omega\big(\big(A' + I_{2n}\big)^{-1}\big). \tag{2.15}$$

Thus, plugging in Eq. (2.15) into Eq. (2.14) and again applying Eq. (1.6) yields

$$\begin{aligned}\|Q_\alpha g\|_p^p &= \left(\frac{2^{np}}{\sqrt{|\det(A' + I_{2n})|^p}}\right) \int_{\mathbb{R}^{2n}} e^{-\frac{p\alpha}{2}(x,x)} e^{(x, -\frac{p\alpha}{2}\Omega((A'+I_{2n})^{-1})x)}\, dx \\ &= \left(\frac{(\frac{2\pi}{p\alpha})^n 2^{np}}{\sqrt{|\det(A' + I_{2n})|^p \det(I_{2n} + \Omega((A' + I_{2n})^{-1}))}}\right).\end{aligned} \tag{2.16}$$

Dividing (2.16) by (2.13) gives the lemma. □

Now we have a new characterization of the norm $\|Q_\alpha\|_{p\to p}$.

Lemma 2.7. *Let $1 < p < \infty$. Then*

$$\big(\|Q_\alpha\|_{p\to p}\big)^p = 2^{np} \sup_{A\in\mathcal{A}^{2n}} \sqrt{\frac{\det(\Re(A))}{|\det(A + I_{2n})|^p \det(I_{2n} + \Omega((I_{2n} + A)^{-1}))}}.$$

Proof. Note that the mapping $A \to \frac{2}{\alpha}A$ is a bijection of $\mathcal{A}^{2n}$ to itself. Thus, combining Lemmas 2.5 and 2.6 gives the result. □

2.4. The optimization problem

Lemma 2.7 reduces the determination of the sharp norm of Q_α to an optimization problem over the space $\mathcal{A}^{2n}$ of $2n \times 2n$ complex symmetric matrices with positive definite real part. While more tractable than the general optimization over L^p, the domain of this function is an open subset of a $2n(2n+1)$ (real) dimensional space, and the function is quite complicated. The task of identifying all critical points of this function in general is quite difficult. Instead, we use Proposition 2.1 to reduce to the case $n = 1$, where we devote the remainder of this paper to an analysis of the optimization. In particular, Corollary 2.3 and Lemma 2.7 show that for general n

$$\left(\|Q_\alpha\|_{p\to p}\right)^p = \left(2^p \sup_{A\in\mathcal{A}^2} \sqrt{\frac{\det(\Re(A))}{|\det(A+I_2)|^p \det(I_2 + \Omega((I_2+A)^{-1}))}}\right)^n, \tag{2.17}$$

where

$$\Omega(M) = J^T \Re(M) J - \Re(M) - J^T \Im(M) - \Im(M) J,$$

I_2 is the 2×2 identity matrix, and J is the counter-clockwise rotation by 90°

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Accordingly, we define the function $h_p : \mathcal{A}^2 \to \mathbb{R}$ by

$$h_p(A) := \frac{\det(\Re(A))}{|\det(A+I_2)|^p \det(I_2 + \Omega((I_2+A)^{-1}))}. \tag{2.18}$$

From Eq. (2.17) and the lower bound of Proposition 1.15, and since $\|P_\alpha\|_{p\to p} = \|Q_\alpha\|_{p\to p}$ by Eq. (1.21), it therefore follows that to prove Theorem 1.1, it suffices to show that

$$h_p(A) \leqslant \left(\frac{1}{p^{1/p} p'^{1/p'}}\right)^{2p}, \quad \forall A \in \mathcal{A}^2. \tag{2.19}$$

What's more, we already showed (in the proof of Proposition 1.15) that the inequality (2.19) holds for matrices of the form $A = \beta I_2$; indeed, when $p > 1$, $h_p(\beta I_2)$ is maximized at its unique critical point $\beta = \frac{1}{p-1}$, where h_p indeed takes the desired value. This suggests the outline of the remainder of this section: we will show that, on the 6-dimensional open set $\mathcal{A}^2$, the function h_p has the unique critical point $A_p = \frac{1}{p-1} I_2$, and achieves its maximum there. Note that since Q_α is an orthogonal projection on L^2, the norm is already known to be 1 in that case, so in the sequel we consider only $p \neq 2$ in $(1, \infty)$.

2.5. The critical point of h_p

The first step in looking for critical points is to write h_p in terms of coordinates. So for $A \in \mathcal{A}^2$, we write

$$A = \begin{bmatrix} a+ie & b+if \\ b+if & d+ig \end{bmatrix} \to (a, d, b, e, g, f), \tag{2.20}$$

and with this coordinate mapping consider h_p as a function on an open subset of $(0,\infty)^2 \times (-\infty,\infty)^4$. The next lemma and the technical lemma that follows are long but straightforward calculations in coordinates. We omit the proofs, but the results can be checked by hand or by a computer algebra system.

Lemma 2.8. *The function* $h_p : \mathcal{A}^2 \to (0,\infty)$ *can be written in coordinates as*

$$h_p(a,d,b,e,g,f) = \frac{ad-b^2}{(\Psi^2+\Phi^2)^{\frac{p-2}{2}}\tau},$$

where

$$\tilde{a} := a+1, \qquad \tilde{d} =: d+1, \tag{2.21}$$

$$\Psi := \tilde{a}\tilde{d} - b^2 - eg + f^2, \qquad \Phi := \tilde{d}e + \tilde{a}g - 2bf, \tag{2.22}$$

$$\psi := a-d+2f, \qquad \phi := e-g-2b, \tag{2.23}$$

$$\tau := \Psi^2+\Phi^2-\psi^2-\phi^2. \tag{2.24}$$

It will be useful to have the expression τ in Lemma 2.8 written in the following forms form in the case that $b=0$; again, the elementary calculations are omitted.

Lemma 2.9. *Consider the expression* $\tau = \tau(a,d,b,e,g,f)$*, defined as* $\tau = \Psi^2+\Phi^2-\psi^2-\phi^2$*. When* $b=0$*,* $\tau(a,d,0,e,f,g)$ *can be written as*

$$\begin{aligned}
&\tau(a,d,0,e,g,f)\\
&\quad= \big(ad-eg+f^2-1\big)^2 + 8ad + 2ad(a+d) + 2a(f-1)^2\\
&\qquad + 2d(f+1)^2 + (ag+de)^2 + 2\big(de^2+ag^2\big) &(2.25)\\
&\quad= \big(eg-f^2+1\big)^2 + a^2d^2 + 2ad(a+d) + 6ad + 2a(f-1)^2\\
&\qquad + 2adf^2 + 2d(f+1)^2 + e^2\big(d^2+2d\big) + g^2\big(a^2+2a\big). &(2.26)
\end{aligned}$$

We are now ready to compute partial derivatives of h_p to look for a critical point. First, we will consider critical points of a certain type. More specifically, define a set of matrices $\mathcal{A}'^2 \subset \mathcal{A}^2$ by

$$\mathcal{A}'^2 = \big\{A \in \mathcal{A}^2\colon\ \Re(A) \text{ is diagonal}\big\}. \tag{2.27}$$

The fact that elements of $\mathcal{A}'^2$ have diagonal real parts will be quite useful in proving the following proposition, and it will turn out that proving statements on the set $\mathcal{A}'^2$ will lead to general results on $\mathcal{A}^2$. As we mentioned in the introduction of this section, we will impose the condition that $p \neq 2$.

Proposition 2.10. *Let* $1<p<\infty$ *and* $p\neq 2$*. Suppose* $A \in \mathcal{A}'^2$ *and is a critical point of* h_p*. Then* $A = \frac{1}{p-1}I_2$.

Proof. We first derive a formula for $\frac{\partial h_p}{\partial x}$ where x is any variable. For an expression ω, we let ω_x denote the partial derivative of ω with respect to x. One can easily calculate

$$\frac{\partial h_p}{\partial x} = \frac{(ad-b^2)[-(\Psi\Psi_x+\Phi\Phi_x)(p\tau+2(\psi^2+\phi^2))]}{(\Psi^2+\Phi^2)^{\frac{p}{2}}\tau^2} + \frac{2(ad-b^2)(\psi\psi_x+\phi\phi_x)(\Psi^2+\Phi^2)+(\Psi^2+\Phi^2)(ad-b^2)_x\tau}{(\Psi^2+\Phi^2)^{\frac{p}{2}}\tau^2}.$$

Let $A=\left[\begin{smallmatrix} a+ie & if \\ if & d+ig \end{smallmatrix}\right]\in\mathcal{A}'^2$ be a critical point of h_p. Then all six partial derivatives are 0, which imply the following six equations

$$\begin{aligned}
&0=ad\bigl[-\alpha\bigl(p\tau+2\bigl(\psi^2+\phi^2\bigr)\bigr)+2\psi\bigr]+d\tau,\\
&0=ad\bigl[-\delta\bigl(p\tau+2\bigl(\psi^2+\phi^2\bigr)\bigr)-2\psi\bigr]+a\tau,\\
&0=-\beta\bigl(p\tau+2\bigl(\psi^2+\phi^2\bigr)\bigr)-2\phi, \qquad 0=\sigma\bigl(p\tau+2\bigl(\psi^2+\phi^2\bigr)\bigr)+2\psi,\\
&0=\epsilon\bigl(p\tau+2\bigl(\psi^2+\phi^2\bigr)\bigr)+2\phi, \qquad 0=\gamma\bigl(p\tau+2\bigl(\psi^2+\phi^2\bigr)\bigr)-2\phi,
\end{aligned}$$

where we define $\alpha,\delta,\beta,\sigma,\epsilon,\gamma$ as

$$(A+I_2)^{-1}=\begin{bmatrix} \alpha+i\epsilon & \beta+i\sigma \\ \beta+i\sigma & \delta+i\gamma \end{bmatrix}.$$

Since $p\tau+2(\psi^2+\phi^2)>0$, we can solve each equation for the corresponding value of $(A+I_2)^{-1}$. Define C_p as

$$C_p := p\tau+2\bigl(\psi^2+\phi^2\bigr).$$

Then the six equations above become

$$\begin{aligned}
&\alpha=\frac{\tau}{aC_p}+\frac{2\psi}{C_p}, \qquad \delta=\frac{\tau}{dC_p}-\frac{2\psi}{C_p}, \qquad \beta=\frac{-2\phi}{C_p},\\
&\sigma=\frac{-2\psi}{C_p}, \qquad \epsilon=\frac{-2\phi}{C_p}, \qquad \gamma=\frac{2\phi}{C_p}.
\end{aligned} \tag{2.28}$$

Note that

$$\beta=\epsilon=-\gamma.$$

Thus we can write $(A+I_2)^{-1}=\left[\begin{smallmatrix} \alpha+i\beta & \beta+i\sigma \\ \beta+i\sigma & \delta-i\beta \end{smallmatrix}\right]$ and thus $I_2=(A+I_2)(A+I_2)^{-1}$ gives us the following eight equations:

$$1 = \tilde{a}\alpha - e\beta - f\sigma, \qquad 1 = \tilde{d}\delta - f\sigma + g\beta, \tag{2.29}$$

$$0 = e\alpha + \tilde{a}\beta + f\beta, \qquad 0 = \tilde{a}\beta - e\sigma + f\beta, \tag{2.30}$$

$$0 = \tilde{d}\beta - f\beta - g\sigma, \qquad 0 = -\tilde{d}\beta + f\beta + g\delta, \tag{2.31}$$

$$0 = \tilde{a}\sigma + e\beta + f\delta, \qquad 0 = \tilde{d}\sigma + f\alpha + g\beta. \tag{2.32}$$

Subtracting the equations in (2.30) yields $e(\alpha+\sigma) = 0$. So either $e = 0$ or $\alpha + \sigma = 0$. If $\alpha + \sigma = 0$, then

$$0 < \frac{\tau}{aC_p} = \left(\frac{\tau}{aC_p} + \frac{2\psi}{C_p}\right) - \frac{2\psi}{C_p} = \alpha + \sigma = 0,$$

a contradiction. Thus, we must have $e = 0$. Similarly, adding the equations of (2.31) gives $g(\delta - \sigma) = 0$. If $\delta - \sigma = 0$, then

$$0 < \frac{\tau}{dC_p} = \left(\frac{\tau}{dC_p} - \frac{2\psi}{C_p}\right) - \frac{-2\psi}{C_p} = \delta - \sigma = 0,$$

a contradiction. Thus, we also have $g = 0$. Thus

$$A = \begin{bmatrix} a & if \\ if & d \end{bmatrix}, \qquad (A + I_2)^{-1} = \begin{bmatrix} \alpha & i\sigma \\ i\sigma & \delta \end{bmatrix}.$$

Using this new information, the eight equations above reduce to

$$1 = \tilde{a}\alpha - f\sigma, \qquad 1 = \tilde{d}\delta - f\sigma, \tag{2.33}$$

$$0 = \tilde{a}\sigma + f\delta, \qquad 0 = \tilde{d}\sigma + f\alpha. \tag{2.34}$$

Using (2.28) in (2.34) and clearing the C_p in denominator gives

$$0 = \frac{f}{d}\tau + 2(-\tilde{a} - f)\psi, \qquad 0 = \frac{f}{a}\tau + 2(-\tilde{d} + f)\psi. \tag{2.35}$$

Subtracting the two above equations (in reverse order of their appearance) of Eq. (2.35) yields

$$0 = -\left(\frac{a-d}{ad}\right) f\tau + 2\psi^2. \tag{2.36}$$

We would like to know that $\psi = 0$, for then Eq. (2.36) would show that $a = d$ and $f = 0$. Showing that $\psi = 0$ is an involved argument, and thus we shall prove it as a separate lemma.

Lemma 2.11. *Let $1 < p < \infty$ and $p \neq 2$. Suppose $A \in \mathcal{A}'^2$ and is a critical point of h_p. Then $\psi = 0$.*

Proof. We will proceed by contradiction, assuming $\psi \neq 0$ and showing that this implies $p = 2$. So suppose $\psi \neq 0$. We can clear the denominators of the two equations in (2.35) and subtract the resulting equations to get

$$0 = 2\big(a - d - (a+d)f\big)\psi.$$

Since we assumed $\psi \neq 0$, we can divide the above equation by ψ and solve for f, yielding

$$f = \frac{a-d}{a+d}. \tag{2.37}$$

One can use the above expression of f in the definition of ψ in (2.23) to compute

$$\psi = \frac{a-d}{a+d}(\tilde{a} + \tilde{d}). \tag{2.38}$$

We can use (2.37) and (2.38) to rewrite (2.36) as

$$0 = -\frac{(a-d)^2}{ad(a+d)}\tau + 2\frac{(a-d)^2}{(a+d)^2}(\tilde{a} + \tilde{d})^2. \tag{2.39}$$

Since we assume $\psi \neq 0$, by (2.38) we have $a - d \neq 0$. Thus, we can multiply each side of (2.39) by $\frac{ad(a+d)^2}{(a-d)^2}$

$$0 = -(a+d)\tau + 2ad(\tilde{a} + \tilde{d})^2. \tag{2.40}$$

Now by (2.25) we have

$$\tau = \big(ad + f^2 - 1\big)^2 + 8ad + 2ad(a+d) + 2a(f-1)^2 + 2d(f+1)^2. \tag{2.41}$$

Using (2.37) in (2.41), one (or a computer algebra system) can show that

$$\tau = \left(ad - 1 + \frac{(a-d)^2}{(a+d)^2}\right)^2 + \frac{2ad}{a+d}(\tilde{a} + \tilde{d})^2. \tag{2.42}$$

Plugging (2.42) into (2.40), one (or a computer algebra system) can show that

$$0 = -(a+d)\left(ad - 1 + \frac{(a-d)^2}{(a+d)^2}\right)^2.$$

Since $a + d > 0$, we can divide by $a + d$ above, take the square root, clear the denominator, and rearrange the equality to get

$$ad(a+d)^2 = (a+d)^2 - (a-d)^2 = 4ad.$$

Dividing each side by ad (which, note, is not 0) and taking the square root yields

$$a + d = 2. \tag{2.43}$$

The above equation (2.43) allows us to simplify things more. In fact, we can change (2.37), (2.38), (2.40) into

$$f = \frac{a-d}{2}, \qquad \psi = 2(a-d), \qquad \tau = 16ad. \tag{2.44}$$

One can use (2.44) above to give an simple expression for C_p:

$$C_p = 32 + 16(p-2)ad. \tag{2.45}$$

However, we also have the first equation in (2.33) which, combined with (2.28) and (2.44) says

$$C_p = C_p(\tilde{a}\alpha - f\sigma) = \tilde{a}\left(\frac{\tau}{a} + 2\psi\right) + 2f\psi = 6a^2 + 8ad + 2d^2 + 12d + 4a. \tag{2.46}$$

Similarly, one can rewrite the second equation in (2.33) as

$$C_p = C_p(\tilde{d}\delta - f\sigma) = 2a^2 + 8ad + 6d^2 + 12a + 4d. \tag{2.47}$$

We now combine Eqs. (2.45), (2.46), and (2.47) to get

$$\begin{aligned} 32 + 16(p-2)ad = C_p &= \frac{1}{2}C_p(\tilde{a}\alpha - f\sigma + \tilde{d}\delta - f\sigma) \\ &= \frac{1}{2}\left(8a^2 + 16ad + 8d^2 + 16a + 16d\right) = 32. \end{aligned}$$

Thus, we must have

$$16(p-2)ad = 0. \tag{2.48}$$

But $a > 0$ and $d > 0$, so the only way the above equation can hold is if $p = 2$. This contradicts our assumption that $p \neq 2$, proving the lemma. □

Thus $\psi = 0$. By (2.36), we know that either $a - d = 0$ or $f = 0$. Since $0 = \psi = (a-d) + 2f$, this implies that *both* $a - d = 0$ and $f = 0$. Thus the critical point $A = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$. We can rewrite the equation the first equation in (2.33) as $1 = \tilde{a}\alpha$. Plugging in (2.28) into $1 = \tilde{a}\alpha$ and solving for a yields $a = \frac{1}{p-1}$, proving $A = \frac{1}{p-1}I_2$. It is elementary to verify that the matrix $\frac{1}{p-1}I_2$ is a critical point, proving that is the unique critical point in $\mathcal{A}'^2$. □

We can turn this special case into a general theorem. However, we will need an intermediate lemma to prove the theorem.

Lemma 2.12. *Let $A \in \mathcal{A}^2$. If $U \in SO(2)$ (i.e. U is a real orthogonal matrix with* $\det(U) = 1$), *then $h_p(A) = h_p(U^T A U)$.*

Proof. Let $A \in \mathcal{A}^2$ and $U \in SO(2)$. Since U is real, $SO(2)$ is commutative, and $U, J \in SO(2)$, we have

$$h_p(U^T AU) = \frac{\det(\Re(U^T AU))}{|\det(U^T(A+I_2)U)|^p \det(I_2 + \Omega((U^T AU + I_2)^{-1}))}$$
$$= \frac{\det(\Re(A))}{|\det(A+I_2)|^p \det(I_{2n} + \Omega(B))} = h_p(A),$$

as claimed. □

With Lemma 2.12, we can prove that there is only one critical point in $\mathcal{A}^2$.

Theorem 2.13. *Let $1 < p < \infty$ and $p \neq 2$. The function $h_p : \mathcal{A}^2 \to (0, \infty)$ defined by*

$$h_p(A) := \frac{\det(\Re(A))}{|\det(A+I_2)|^p \det(I_{2n} + \Omega((I_2+A)^{-1}))} \tag{2.49}$$

has exactly one critical point, namely $A = \frac{1}{p-1} I_2$.

Proof. Let $A \in \mathcal{A}^2$ be a critical point of h_p (as stated at the end of the proof of Proposition 2.10, the reader may readily verify that $A = \frac{1}{p-1} I_2$ is a critical point, so at least one critical point does exist). Since $\Re(A)$ is symmetric, there exists a $U \in SO(2)$ such that $U^T AU \in \mathcal{A}'^2$. Since the mapping $B \to UBU^T$ is a diffeomorphism of $\mathcal{A}^2$ to itself and $h_p(B) = h_p(UBU^T)$ by Lemma 2.12, $U^T AU$ must also be a critical point of h_p. By Lemma 2.10, we must have $U^T AU = \frac{1}{p-1} I_2$, which forces $A = \frac{1}{p-1} I_2$. □

Here we note the value of h_p at the critical point $A = \frac{1}{p-1} I_2$. It is straightforward to calculate that

$$h_p\left(\frac{1}{p-1} I_2\right) = \left(\frac{(p-1)^{p-1}}{p^p}\right)^2 = \left(\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right)^{2p}.$$

We will need to know the behavior of this prospective maximum value in what follows.

Lemma 2.14. *The function $j : (1, \infty) \to \mathbb{R}$ defined as $j(p) \equiv \frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}$ takes a minimum value at $p = 2$ and $j(2) = \frac{1}{2}$.*

Proof. This can be shown using elementary calculus. The details are omitted. □

We will use the above lemma when proving that our critical point gives us a unique maximum when $p \neq 2$.

2.6. Proving the maximum occurs at the critical point

We have a unique critical point $\frac{1}{p-1} I_2$ for our function h_p, and next we want to show that this critical point gives us our maximum. Our plan is to define a compact set $\mathcal{K} \subset \mathcal{A}^2$ such that h_p takes on values strictly less than $h_p(\frac{1}{p-1} I_2) = (\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}})^{2p}$ outside of and on the boundary of $\mathcal{K}$. Thus our first job is going to be finding appropriate bounds with which we will construct $\mathcal{K}$.

To find these bounds, we are going to define two common operator norms on matrices. For a vector $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{R}^2$, denote the 2-norm by $|v|_2$ ($|v|_2 = (|v_1|^2 + |v_2|^2)^{1/2}$). For $B \in \mathbb{R}^{2\times 2}$, denote the operator 2-norm as $|B|_2$ ($|B|_2 = \max\{|Bv|_2\colon |v|_2 = 1\}$). We will use the fact that

$$\left|U^T BU\right|_2 = |B|_2 \quad \text{for any } U \in SO(2) \text{ and any } B \in \mathbb{R}^{2\times 2}. \tag{2.50}$$

We also denote the maximum norm as $|B|_{\max}$ ($|B|_{\max} = \max_{i=1,2;\, j=1,2} |b_{ij}|$). The norms $|\cdot|_2$ and $|\cdot|_{\max}$ are equivalent. In fact,

$$|B|_{\max} \leqslant |B|_2 \leqslant 2|B|_{\max} \quad \text{for any } B \in \mathbb{R}^{2\times 2}. \tag{2.51}$$

Eqs. (2.50) and (2.51) are well-known facts. See, for example, [4].

First, we start with a bound for the operator $|Q_\alpha|$ (whose definition can be found in Eq. (1.18)) that will prove useful. Since it is as easy to prove in general dimension as in 1 dimension, we state it for general n.

Lemma 2.15. *Let A_r be a real positive definite $2n$-by-$2n$ matrix with eigenvalues λ_j for $j = 1, \ldots, 2n$. Let $g_r(x) = e^{-(x, \frac{\alpha}{2} A_r x)}$. Then we have*

$$\frac{\| |Q_\alpha| g_r \|_p^p}{\|g_r\|_p^p} = 2^{np} \prod_{i=1}^{2n} \sqrt{\frac{1}{(1+\lambda_i)^{p-1}}}.$$

Proof. Using (1.6), one can calculate

$$\begin{aligned}\frac{\| |Q_\alpha| g_r \|_p^p}{\|g_r\|_p^p} &= 2^{np} \sqrt{\frac{\det(A_r)}{\det(A_r + I_{2n})^p \det(I_{2n} - (A_r + I_{2n})^{-1})}} \\ &= 2^{np} \prod_{j=1}^{2n} \sqrt{\frac{1}{(\lambda_j + 1)^{p-1}}}. \qquad \square\end{aligned}$$

In the next lemma, we define our first bound $M_p^{a,d}$.

Lemma 2.16. *Let $1 < p < \infty$. There exists a positive real number $M_p^{a,d}$ such that for any $A \in \mathcal{A}^2$ if $|\Re(A)|_2 \geqslant M_p^{a,d}$, then $h_p(A) < (\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}})^{2p}$.*

Proof. Let $A \in \mathcal{A}^2$, and define $A_r = \Re(A)$. Let λ_1 and λ_2 be the eigenvalues of A_r. Define $g(x) = e^{-(x, \frac{\alpha}{2} Ax)}$ and $g_r(x) = e^{-(x, \frac{\alpha}{2} A_r x)}$. Note that $\|g\|_p = \|g_r\|_p$. Then using Lemma 2.15 we have

$$\begin{aligned} h_p(A) &\leqslant \left(\frac{1}{2^{np}} \frac{\| |Q_\alpha| g_r \|_p^p}{\|g_r\|_p^p}\right)^2 = \frac{1}{(\lambda_1 + 1)^{p-1}} \frac{1}{(\lambda_2 + 1)^{p-1}} \\ &< 1 \cdot \frac{1}{(\max(\lambda_1, \lambda_2) + 1)^{p-1}} = \frac{1}{(|\Re(A_r)|_2 + 1)^{p-1}}. \end{aligned}$$

Since $p-1>0$, we have

$$\lim_{x\to\infty}\frac{1}{(x+1)^{p-1}}=0,$$

and thus there exists a $M_p^{a,d}$ such that for $x\geqslant M_p^{a,d}$ we have $\frac{1}{(x+1)^{p-1}}<[\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}]^{2p}$. Thus, if $|\Re(A)|_2\geqslant M_p^{a,d}$, we have

$$h_p(A)<\frac{1}{(|\Re(A')|_{\max}+1)^{p-1}}<\left[\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right]^{2p},$$

as desired. □

For our next bound $M_p^{e,f,g}$ we will just consider matrices in $\mathcal{A}'^2$.

Lemma 2.17. *Let $1<p<\infty$. There exists a positive real number $M_p^{e,f,g}$ such that for any $A'\in\mathcal{A}'^2$, if $|\Re(A')|_2<M_p^{a,d}$ and $|\Im(A')|_{\max}\geqslant M_p^{e,f,g}$, then $h_p(A')<(\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}})^{2p}$.*

Proof. First we define the bound $M_p^{e,f,g}$ as

$$M_p^{e,f,g}:=\left[\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right]^{-2}\cdot\big(2\big(M_p^{a,d}\big)^2+2M_p^{a,d}+3\big)^{1/p}. \tag{2.52}$$

Let $A'\in\mathcal{A}'^2$ with $|\Re(A')|_2<M_p^{a,d}$ and $|\Im(A')|_{\max}\geqslant M_p^{e,f,g}$. Note that since $\Re(A')$ is diagonal, we have $|\Re(A')|_{\max}=|\Re(A')|_2<M_p^{a,d}$ and we can write

$$h_p\big(A'\big)=\frac{ad(\Psi^2+\Phi^2)}{(\Psi^2+\Phi^2)^{\frac{p}{2}}\tau}=\frac{ad}{(\Psi^2+\Phi^2)^{\frac{p}{2}}}\left(1+\frac{\psi^2+\phi^2}{\tau}\right).$$

First we bound $ad\frac{\psi^2+\phi^2}{\tau}$. Repeatedly using Cauchy's inequality ($2|xy|\leqslant(x^2+y^2)$) yields

$$\begin{aligned}\psi^2+\phi^2&\leqslant 3a^2-6ad+3d^2+6f^2+2e^2+2g^2\\&<3a^2+3d^2+6f^2+2e^2+2g^2.\end{aligned} \tag{2.53}$$

Thus, using the above equation, the expression of τ as (2.26), and the fact $a,d<M_p^{a,d}$ we have

$$\begin{aligned}ad\frac{\psi^2+\phi^2}{\tau}&<\frac{ad(3a^2+3d^2+6f^2+2e^2+2g^2)}{\tau}\\&\leqslant\frac{3a^3d}{6ad}+\frac{3ad^3}{6ad}+\frac{6adf^2}{2adf^2}+\frac{2ade^2}{2de^2}+\frac{2adg^2}{2ag^2}\\&\leqslant\big(M_p^{a,d}\big)^2+2M_p^{a,d}+3.\end{aligned} \tag{2.54}$$

Now, using (2.26) one (or a computer algebra system) can show

$$\begin{aligned}\Psi^2+\Phi^2 &= \big(eg-f^2\big)^2+a^2d^2+2ad(a+d)+6ad+2adf^2+2af^2 \\ &\quad +2(a+d)f^2+2(a+d)+e^2\big(d^2+2d\big)+g^2\big(a^2+2a\big) \\ &\quad +(a-d)^2+1+2f^2+e^2+g^2. \end{aligned} \tag{2.55}$$

We have four useful inequalities that we can deduce from (2.55) that we will summarize as one inequality

$$\Psi^2+\Phi^2 \geqslant \max\big(1,f^2,e^2,g^2\big). \tag{2.56}$$

Using (2.54) and (2.56) we have

$$\begin{aligned} h_p\big(A'\big) &< \frac{1}{(\Psi^2+\Phi^2)^{\frac{p}{2}}}\big(2\big(M_p^{a,d}\big)^2+2M_p^{a,d}+3\big) \\ &\leqslant \frac{1}{(\max(f^2,e^2,g^2))^{\frac{p}{2}}}\big(2\big(M_p^{a,d}\big)^2+2M_p^{a,d}+3\big) \\ &\leqslant \frac{1}{(M_p^{e,f,g})^p}\big(2\big(M_p^{a,d}\big)^2+2M_p^{a,d}+3\big) = \left[\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right]^{2p}, \end{aligned}$$

proving the lemma. □

We have one last bound to define, a lower bound that we will call $m_p^{a,d}$. We need a lower bound for the real parts of matrices in $\mathcal{A}^2$ since h_p does not extend continuously over the set of symmetric matrices whose real part is positive semidefinite. The issue is that for a matrix in the closure of $\mathcal{A}^2$, it is possible for the τ in the denominator of h_p to vanish when either $a=0$ or $d=0$. In fact, if we just consider matrices in the closure of $\mathcal{A}'^2$, one can check that τ vanishes in exactly two cases:

(1) $a=0, e=0, f=-1$,
(2) $d=0, g=0, f=1$.

We will have to consider how $h_p(A')$ behaves when $A'\in\mathcal{A}'^2$ has entries close to one of the two cases above. As we will see, these two cases on the boundary will require us to again impose the condition that $p\neq 2$, a condition that was not needed in the previous two lemmas.

Lemma 2.18. *Let $1<p<\infty$ and $p\neq 2$. There exists a positive real number $m_p^{a,d}$ such that for any $A'=\left[\begin{smallmatrix} a+ie & if \\ if & d+ig\end{smallmatrix}\right]\in\mathcal{A}'^2$ if $\min(a,d)\leqslant m_p^{a,d}$, $|\Re(A')|_2<M_p^{a,d}$ and $|\Im(A')|_{\max}<M_p^{e,f,g}$, then $h_p(A')<(\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}})^{2p}$.*

Proof. Let $A'\in\mathcal{A}'^2$ satisfy $|\Re(A')|_2<M_p^{a,d}$ and $|\Im(A')|_{\max}<M_p^{e,f,g}$. We will first concern ourselves with $h_p(A')$ when A' has entries close to the two cases enumerated just before the

lemma where τ vanishes. The two cases are related due to symmetry in h_p. In fact, one can check that

$$h_p\left(\begin{bmatrix} a+ie & if \\ if & d+ig \end{bmatrix}\right) = h_p\left(\begin{bmatrix} d+ig & -if \\ -if & a+ie \end{bmatrix}\right). \tag{2.57}$$

We will use (2.57) to concentrate on the $a = 0, e = 0, f = -1$ case. Define a new function $\tilde{h}_p : [0, M_p^{a,d}]^2 \times [-M_p^{e,f,g}, M_p^{e,f,g}]^2 \times [-M_p^{e,f,g}, 0] \to [0, \infty)$ by

$$\tilde{h}_p(a, d, e, g, f) := \frac{d(\Psi^2 + \Phi^2)^{1-p/2}}{ad^2 + 2d(a+d) + 6d + 2df^2 + 2(f-1)^2 + g^2(a+2)}. \tag{2.58}$$

First note that by (2.26) we have

$$\tau \geqslant a^2 d^2 + 2ad(a+d) + 6ad + 2adf^2 + 2a(f-1)^2 + g^2\left(a^2 + 2a\right),$$

an inequality that is close to equality when e and $f + 1$ are close to 0. Using the above one can show that $h_p(A') \leqslant \tilde{h}_p(a, d, e, g, f)$. Note that $\tilde{h}_p$ is uniformly continuous on its domain. Let ϵ_p be defined as

$$\epsilon_p := \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p} - \left(\frac{1}{2}\right)^p \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^p. \tag{2.59}$$

By Lemma 2.14, we have $\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}} > \frac{1}{2}$ for $p \neq 2$. Thus, $\epsilon_p > 0$ for $p \neq 2$. By the uniform continuity of $\tilde{h}_p$, there exists a $\delta_p > 0$ not dependent on the choice of A' such that

$$\begin{aligned} \max\left(a, |e|, |f+1|\right) < \delta_p \quad &\Longrightarrow \quad \left|\tilde{h}_p(a, d, e, g, f) - \tilde{h}_p(0, d, 0, g, -1)\right| < \epsilon_p \\ \Longrightarrow \quad h_p\left(A'\right) &\leqslant \tilde{h}_p(a, d, e, g, f) < \tilde{h}_p(0, d, 0, g, -1) + \epsilon_p. \end{aligned} \tag{2.60}$$

We want to maximize $\tilde{h}_p(0, d, 0, g, -1)$ (and justify our choice of ϵ_p). To that end,

$$\tilde{h}_p(0, d, 0, g, -1) = \frac{d}{2((d+2)^2 + g^2)^{p/2}} \leqslant \frac{d}{2(d+2)^p}.$$

Maximizing this last expression over $[0, \infty)$, we see that the maximum occurs at $x = \frac{2}{p-1}$. So we have

$$\tilde{h}_p(0, d, 0, g, -1) \leqslant \frac{\frac{2}{p-1}}{2(\frac{2}{p-1} + 2)^p} = \left(\frac{1}{2}\right)^p \left(\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right)^p.$$

Thus, using (2.59) we have

$$\tilde{h}_p(0, d, 0, g, -1) + \epsilon_p \leqslant \left(\frac{1}{2}\right)^p \left(\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right)^p + \epsilon_p = \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p}. \tag{2.61}$$

Putting (2.60) and (2.61) together we have

$$\max(a, |e|, |f+1|) < \delta_p \quad \Longrightarrow \quad h_p(A') < \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p}. \tag{2.62}$$

By the symmetry (2.57), Eq. (2.62) also gives us

$$\max(d, |g|, |f-1|) < \delta_p \quad \Longrightarrow \quad h_p(A') < \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p}. \tag{2.63}$$

We also need to consider the case that either $|e|$ or $|f+1|$ are greater than δ_p. Note that by (2.53) and (2.26) and since $|\Re(A')|_{\max} < M_p^{a,d}$ and $|\Im(A')|_{\max} < M_p^{e,f,g}$ with $\max(|e|, |f+1|) \geqslant \delta_p$, we have

$$\begin{aligned} h_p(a,d,e,f,g) &\leqslant \frac{ad(\Phi^2+\Psi^2)}{\tau} \leqslant a\left(d + \frac{d(\phi^2+\psi^2)}{\tau}\right) \\ &\leqslant a\left(d + \frac{d(\phi^2+\psi^2)}{2d(f+1)^2+2de^2}\right) \leqslant a\left(d + \frac{(\phi^2+\psi^2)}{2\delta_p^2}\right) \\ &\leqslant a\left(d + \frac{3a^2+3d^2+6f^2+2e^2+2g^2}{2\delta_p^2}\right) \\ &\leqslant a\left(\frac{M_p^{a,d}\delta_p^2 + 3(M_p^{a,d})^2 + 5(M_p^{e,f,g})^2}{\delta_p^2}\right). \end{aligned} \tag{2.64}$$

Thus, if we set

$$\delta'_p := \frac{\delta_p^2}{M_p^{a,d}\delta_p^2 + 3(M_p^{a,d})^2 + 5(M_p^{e,f,g})^2} \frac{1}{2}\left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p},$$

then (2.64) proves the implication

$$a \leqslant \delta'_p,\ \max(|e|, |f+1|) \geqslant \delta_p \quad \Longrightarrow \quad h_p(A') < \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p}. \tag{2.65}$$

Again, by symmetry (2.57), we also have

$$d \leqslant \delta'_p, \quad \max(|g|, |f-1|) \geqslant \delta_p \quad \Longrightarrow \quad h_p(A') < \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p}. \tag{2.66}$$

Let $m_p^{a,d} = \min(\delta_p, \delta'_p, M_p^{a,d})$. Note that since δ_p and δ'_p do not depend on our choice of A', $m_p^{a,d}$ also does not depend on our choice of A'. Then combining (2.62), (2.63), (2.65) and (2.66) we have

$$\min(a,d) \leqslant m_p^{a,d} \quad \Longrightarrow \quad h_p(A') < \left[\frac{1}{p^{1/p}} \frac{1}{p'^{1/p'}}\right]^{2p},$$

proving the lemma. □

The combination of Lemmas 2.16, 2.17, and 2.18 give

Proposition 2.19. *Let $1 < p < \infty$ and $p \neq 2$. There exists positive real numbers $m_p^{a,d}$, $M_p^{a,d}$, $M_p^{e,f,g}$ such that for any $A' = \left[\begin{smallmatrix} a+ie & if \\ if & d+ig \end{smallmatrix}\right] \in \mathcal{A}'^2$ if either $|\Re(A')|_2 \geqslant M_p^{a,d}$, or $|\Im(A')| \geqslant M_p^{e,f,g}$, or $\min(a,d) \leqslant m_p^{a,d}$, then $h_p(A') < (\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}})^{2p}$.*

We now have all the bounds we need to prove our final theorem.

Theorem 2.20. *Let $1 < p < \infty$ with $p \neq 2$. Then the function $h_p(A)$ takes a unique maximum value at $A = \frac{1}{p-1}I_2$ of $[\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}]^{2p}$.*

Proof. First we will use the bounds from Proposition 2.19 to create a compact set. Define the set $\mathcal{K} \subset \mathcal{A}^2$ as

$$\mathcal{K} := \big\{A \in \mathcal{A}^2\colon \big|\Im(A)\big|_2 \leqslant 2M_p^{e,f,g}, \text{ the eigenvalues } \lambda_1, \lambda_2 \\ \text{of } \Re(A) \text{ satisfy } m_p^{a,d} \leqslant \lambda_j \leqslant M_p^{a,d} \text{ for } j = 1, 2\big\}. \tag{2.67}$$

Note that $\mathcal{K}$ is compact. Thus, h_p takes a maximum value on $\mathcal{K}$ that occurs either at the critical point $\frac{1}{p-1}I_2$ or the boundary of $\mathcal{K}$. We will show that if $A \in \mathcal{A}^2 - \text{int}(\mathcal{K})$ (here $\text{int}(\mathcal{K})$ is the interior of $\mathcal{K}$), then $h_p(A) < h_p(\frac{1}{p-1}I_2) = [\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}]^{2p}$. This will prove that $h_p(\frac{1}{p-1}I_2) = [\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}]^{2p}$ is indeed the maximum over all of $\mathcal{A}^2$, proving the result of the theorem. So let $A \in \mathcal{A}^2 - \text{int}(\mathcal{K})$ with eigenvalues λ_1 and λ_2. Since $A \notin \text{int}(\mathcal{K})$, either

$$\big|\Im(A)\big|_2 \geqslant 2M_p^{e,f,g}, \quad \min(\lambda_1,\lambda_2) \leqslant m_p^{a,d}, \quad \text{or} \quad \max(\lambda_1,\lambda_2) = \big|\Re(A)\big|_2 \geqslant M_p^{a,d}. \tag{2.68}$$

Choose $U \in SO(2)$ such that $U^T\Re(A)U$ is diagonal and let $A' = U^TAU$. By Lemma 2.12 $h_p(A) = h_p(A')$. Write A' as $A' = \left[\begin{smallmatrix} a+ie & if \\ if & d+ig \end{smallmatrix}\right]$. Without loss of generality, $a = \lambda_1$, $d = \lambda_2$. Also, since U is real, note that $\Re(A') = U^T\Re(A)U$ and $\Im(A') = U^T\Im(A)U$. So Eqs. (2.50), (2.51) and (2.68) imply

$$\big|\Im\big(A'\big)\big|_{\max} \geqslant M_p^{e,f,g}, \quad \min(a,d) \leqslant m_p^{a,d}, \quad \text{or} \quad \big|\Re\big(A'\big)\big|_2 \geqslant M_p^{a,d}.$$

Thus, by Proposition 2.19 and Lemma 2.12, we must have

$$h_p(A) = h_p\big(A'\big) < \left[\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}\right]^{2p},$$

proving that $h_p(\frac{1}{p-1}I_2) = [\frac{1}{p^{1/p}}\frac{1}{p'^{1/p'}}]^{2p}$ is the maximum of h_p. □

Following Section 2.4, this concludes the proof that $\|P_\alpha\|_{p\to p} = (2\frac{1}{p^{1/p}p'^{1/p'}})^n$ in general; i.e. we have proved Theorem 1.1.

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