Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 327 (2007) 1095–1104
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Weighted norm inequalities for analytic functions

Arcadii Z. Grinshpan

Department of Mathematics, University of South Florida, Tampa, FL 33620, USA

Received 17 June 2005
Available online 5 June 2006
Submitted by Steven G. Krantz

Abstract

We present a weighted norm inequality involving convolutions of arbitrary analytic functions and certain confluent hypergeometric functions. This result implies a family of weighted norm inequalities both for entire functions of exponential type and for (generalized) hypergeometric series. The approach is based on author's general inequality for continuous functions and some hypergeometric transformations.

Keywords: Integral inequalities; Convolutions; Hadamard product; Entire functions of exponential type; Borel-associated functions; Hypergeometric series; Confluent hypergeometric functions; Hypergeometric transformations; Beta function; Gamma function; Integral operators

1. Introduction

We apply our recent general inequality for continuous functions presented in Theorem A below both to entire functions and functions $f(z)$ that are analytic in some open disk $D_{r_f} = \{z\colon |z| < r_f\}$. This leads to Theorem 1, which yields a weighted norm inequality for convolutions of arbitrary analytic functions and certain confluent hypergeometric functions, and also to its restatement in terms of the entire functions of exponential type and the generalized Borel transform. For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in D_{r_f}) \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in D_{r_g}),$$

E-mail address: azg@math.usf.edu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.013

the Hadamard product or convolution of f and g is defined by the formula (see, e.g., [18]):

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n.$$

It follows that $f*g$ is analytic in the disk D_r, where $r=r_fr_g$. Alternatively, one can use the Hadamard integral presentation

$$(f*g)(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=\rho}f(\zeta)g(z/\zeta)\frac{d\zeta}{\zeta}\quad\bigl(|z|/r_g<\rho<r_f\bigr).$$

If g is an entire function of exponential type, i.e., $\overline{\lim}_{n\to\infty}n|b_n|^{1/n}<\infty$, then $f*g$ is of exponential type as well. The function g is of exponential type $\sigma<\infty$ if the limit above equals $e\sigma$ (if g is a polynomial or if its order is less than 1, then g is of exponential type 0 according to this definition; cf. [14, Chapter 9]).

In particular, Theorem 1 implies a family of weighted norm inequalities for hypergeometric series. The (generalized) hypergeometric series ${}_pF_q(u_1,\dots,u_p;v_1,\dots,v_q;z)$, where p and q are non-negative integers and z is a complex number, is defined by

$${}_pF_q(u_1,\dots,u_p;v_1,\dots,v_q;z)=\sum_{n=0}^{\infty}\frac{(u_1)_n\cdots(u_p)_n}{(v_1)_n\cdots(v_q)_n}\cdot\frac{z^n}{n!},\tag{1}$$

provided that none of the shifted factorials

$$(v_k)_n=v_k(v_k+1)\cdots(v_k+n-1)\quad(k=1,\dots,q;\ n=1,2,\dots)$$

are equal to 0. A straightforward application of Stirling's series (see, e.g., [6, V. 1, p. 47], [8, # 8.327])

$$\Gamma(z)=e^{-z}z^{z-1/2}\sqrt{2\pi}\bigl(1+1/(12z)+O\bigl(z^{-2}\bigr)\bigr)\quad\bigl(|\arg z|<\pi,\ z\to\infty\bigr)$$

shows that any function ${}_pF_p$ presented by a non-terminating series (1) is an entire function of exponential type 1. The confluent hypergeometric functions are generated by (1) with $p=q=1$. The general functions presented in the form (1) contain most of the functions which are usually used in analysis, physics, and engineering as the special cases (see, e.g., [1,2,19]). Their applications often require the weighted norm inequalities of different kind. The standard tools used in this connection: the Cauchy–Bunyakovskii–Schwartz, Hölder, or a general inequality in functional analysis, are not sensitive enough to structure (1). One of the consequences of Theorem 1 presented in Theorem 2 gives a new tool to deal with the peculiarity of hypergeometric series.

Theorem A. [10,12] *Let $\phi(t)$ and $\psi(t)$ be complex-valued continuous functions on* [0, 1]. *Then for any numbers $\alpha,\beta,\lambda>0$, the following inequality holds*:

$$\begin{aligned}&\int_0^1\tau^{\alpha+\beta-1}(1-\tau)^{\lambda-1}\left|\int_0^1t^{\alpha-1}(1-t)^{\beta-1}\phi(\tau t)\psi\bigl(\tau(1-t)\bigr)\,dt\right|^2d\tau\\&\quad\leqslant\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\alpha+\beta+\lambda)}{\Gamma(\alpha+\lambda)\Gamma(\beta+\lambda)\Gamma(\alpha+\beta)}\\&\qquad\times\int_0^1\tau^{\alpha-1}(1-\tau)^{\beta+\lambda-1}\bigl|\phi(\tau)\bigr|^2d\tau\cdot\int_0^1\tau^{\beta-1}(1-\tau)^{\alpha+\lambda-1}\bigl|\psi(\tau)\bigr|^2d\tau.\end{aligned}\tag{2}$$

The equality in (2)*, provided that* ϕ *and* ψ *are not identically* 0*, holds if and only if* $\phi(t) = \phi(0)e^{i\theta t}$ *and* $\psi(t) = \psi(0)e^{i\theta t}$ *for* $t \in [0,1]$ *and a real* θ.

A non-trivial result presented in Theorem A is generated by the multiparameter binomial inequalities [9] in conjunction with some properties of the Bernstein polynomials. Note that the Cauchy–Bunyakovskii–Schwartz inequality is just a trivial limit case of inequality (2) as $\lambda \to 0$. Applications of Theorem A and its discrete predecessor, which do not seem to be accessible through other means, include along with special functions and orthogonal polynomials, fractional integrals, bi-Hermitian forms, and univalent functions [9–12]. They emphasize the role of functions ϕ and ψ which allow us to express the weighted convolution integral on the left-hand side of (2) in terms of the basic special functions (see, e.g., [1,6–8]). It turns out that the hypergeometric transformations of certain type generate such functions ϕ and ψ in terms of the hypergeometric series. Other perspectives are connected with a presentation of the convolution integral in (2) in terms of the generalized Borel-associated functions [3,15,16].

2. Inequalities for convolutions and entire functions

Some linear operators generated by convolutions involving hypergeometric functions have been introduced and studied by several authors [4,5,13,17]. Here we apply an integral transformation related to Theorem A (see [10] and references therein) to such a convolution with the confluent hypergeometric function

$$ {}_1F_1(1;\alpha;z) = \sum_{n=0}^{\infty} \frac{z^n}{(\alpha)_n}. $$

Let

$$ f_\alpha(z) = f(z) * {}_1F_1(1;\alpha;z) = \sum_{n=0}^{\infty} \frac{a_n}{(\alpha)_n} z^n, \tag{3} $$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in a neighbourhood of the origin and $\alpha > 0$. Note that f_α is an entire function of exponential type. We define an integral operator $M(\alpha,\beta,\gamma)$ on the class of all entire functions $F(z)$:

$$ M(\alpha,\beta,\gamma)[F](z) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{\gamma t} \left|F(zt)\right|^2 dt / B(\alpha,\beta), \tag{4} $$

and an integral operator $L(\alpha,\beta,\gamma)$ on the class of all functions $f(z)$ that are analytic in a neighbourhood of the origin:

$$ L(\alpha,\beta,\gamma)[f](z) = M(\alpha,\beta,\gamma)[f_\alpha](z) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{\gamma t} \left|f_\alpha(zt)\right|^2 dt / B(\alpha,\beta), \tag{5} $$

where $B(\alpha,\beta)$ is the beta function, f_α is defined by (3), α and $\beta > 0$, γ is any real number, and z is any complex number.

Theorem 1. *Let* f *and* g *be analytic in a neighbourhood of the origin. Then for any complex* ζ*, real* γ*, and* $\alpha, \beta, \lambda > 0$*, the following inequality holds*:

$$L(\alpha+\beta,\lambda,\gamma)[fg](\zeta) \leqslant L(\alpha,\beta+\lambda,\gamma)[f](\zeta)\cdot L(\beta,\alpha+\lambda,\gamma)[g](\zeta), \tag{6}$$

where L is defined by (3)–(5).

The equality in (6), *provided that* f *and* g *are not identically* 0, *holds if and only if*

$$f(z)=f(0)\big[1-(i\theta-\gamma/2)\zeta^{-1}z\big]^{-\alpha} \quad \text{and} \quad g(z)=g(0)\big[1-(i\theta-\gamma/2)\zeta^{-1}z\big]^{-\beta}$$

in a neighbourhood of the origin ($\zeta\neq 0$, θ *is a real number*), *or* $\zeta=\gamma=0$.

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{m=0}^{\infty}b_m z^m$ in some disk D_r. We apply Theorem A to functions

$$\phi(t)=e^{\gamma t/2}f_\alpha(\zeta t)=e^{\gamma t/2}\sum_{n=0}^{\infty}\frac{a_n\zeta^n}{(\alpha)_n}t^n$$

and

$$\psi(t)=e^{\gamma t/2}g_\beta(\zeta t)=e^{\gamma t/2}\sum_{m=0}^{\infty}\frac{b_m\zeta^m}{(\beta)_m}t^m.$$

In this case the convolution integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\phi(\tau t)\psi\big(\tau(1-t)\big)\,dt$$

in (2) can be presented as

$$B(\alpha,\beta)e^{\gamma\tau/2}\sum_{n,m=0}^{\infty}\frac{(\zeta\tau)^{n+m}a_n b_m}{(\alpha+\beta)_{n+m}}.$$

The double sum above is the convolution of the product $f(z)g(z)$ and ${}_1F_1(1;\alpha+\beta;z)$ for $z=\zeta\tau$. This sum equals $(fg)_{\alpha+\beta}(\zeta\tau)$ by definition (3). Inequality (2) and definition (4)–(5) allow us to obtain inequality (6). The equality statement in Theorem 1 follows from that in Theorem A. □

The same approach allows us to generalize Theorem 1 for formal power series f and g, provided that convolutions f_α and g_β are analytic in a disk D_r and $|\zeta|<r$. Moreover, Theorem 1 can be restated in terms of two entire functions of the finite exponential type:

$$F(z)=\sum_{n=0}^{\infty}A_n z^n \quad (\text{type } \sigma_1) \quad \text{and} \quad G(z)=\sum_{n=0}^{\infty}B_n z^n \quad (\text{type } \sigma_2). \tag{7}$$

Also our restatement involves two analytic functions in a neighbourhood of ∞:

$$\Phi(\alpha,z)=\sum_{n=0}^{\infty}A_n\cdot(\alpha)_n z^{-n-1} \quad \big(|z|>\sigma_1\big)$$

and

$$\Psi(\beta,z)=\sum_{n=0}^{\infty}B_n\cdot(\beta)_n z^{-n-1} \quad \big(|z|>\sigma_2\big), \tag{8}$$

which are the generalized Borel-associated functions to F and G (generalized Borel transforms) by means of the functions ${}_1F_1(1;\alpha;z)$ and ${}_1F_1(1;\beta;z)$, respectively. We have the classical case of the Borel-associated functions to F and G if $\alpha=\beta=1$ [3,16]; then

$$\Phi(z)=\Phi(1,z)=\int_0^\infty F(t)e^{-zt}\,dt \quad\text{and}\quad \Psi(z)=\Psi(1,z)=\int_0^\infty G(t)e^{-zt}\,dt$$

are the corresponding Borel transforms. The most general associated functions are introduced by Nachbin [15].

Corollary 1. *Let F and G be any entire functions of the finite exponential type. Then for any complex ζ, real γ, and $\alpha,\beta,\lambda>0$, the following inequality holds*:

$$M(\alpha+\beta,\lambda,\gamma)[H](\zeta)\leqslant M(\alpha,\beta+\lambda,\gamma)[F](\zeta)\cdot M(\beta,\alpha+\lambda,\gamma)[G](\zeta), \tag{9}$$

where M is defined by (4),

$$H(z)=\bigl(z^{-2}\Phi\bigl(\alpha,z^{-1}\bigr)\Psi\bigl(\beta,z^{-1}\bigr)\bigr)*{}_1F_1(1;\alpha+\beta;z),$$

and Φ and Ψ are defined by (7)–(8).

The equality in (9), *provided that F and G are not identically* 0, *holds if and only if*

$$F(z)=F(0)\exp\bigl[(i\theta-\gamma/2)\zeta^{-1}z\bigr]\quad\textit{and}\quad G(z)=G(0)\exp\bigl[(i\theta-\gamma/2)\zeta^{-1}z\bigr]$$

($\zeta\neq 0$, θ is a real number), or $\zeta=\gamma=0$.

Remark 1. One can use the contour integral presentation of function H in Corollary 1 in terms of the product of the generalized Borel transforms. In particular, if F and G are the entire functions of exponential type $\leqslant\sigma$ and $\alpha=\beta=1$ in (9), we have

$$\int_0^1 t^{-5}(1-t)^{\lambda-1}e^{\gamma t}\left|\frac{1}{2\pi i}\int_{|z|=\rho} z\Phi\bigl((\zeta t)^{-1}z\bigr)\Psi\bigl((\zeta t)^{-1}z\bigr)e^{\zeta tz}\,dz-(\zeta t)^2F(0)G(0)\right|^2 dt$$
$$\leqslant\frac{\lambda+1}{\lambda}|\zeta|^6\int_0^1(1-t)^\lambda e^{\gamma t}\bigl|F(\zeta t)\bigr|^2\,dt\cdot\int_0^1(1-t)^\lambda e^{\gamma t}\bigl|G(\zeta t)\bigr|^2\,dt$$

for any complex ζ, real γ, $\lambda>0$, and $\rho>\sigma|\zeta|$. Here the equality conditions are the same as in inequality (9).

3. Inequalities for hypergeometric series

Note that for any hypergeometric function $f={}_pF_q$ in (3): $f_\alpha={}_pF_{q+1}$. We consider the hypergeometric transformations of the form

$$\begin{aligned}&{}_{p_1}F_{q_1}(a_1,\dots,a_{p_1};b_1,\dots,b_{q_1};\mu z)\,{}_{p_2}F_{q_2}(c_1,\dots,c_{p_2};d_1,\dots,d_{q_2};\nu z)\\&\quad={}_pF_q(u_1,\dots,u_p;v_1,\dots,v_q;\omega z),\end{aligned}\tag{10}$$

where $\mu,\nu\neq 0$ and $p_1,q_1,p_2,q_2,p,q\geqslant 0$. Several transformations of this type, which relate three hypergeometric series for some values of z, are the well-known results of Euler, Clausen, Kummer, and others [1,2,6,19]. Also some relations for elementary functions can be presented in the form (10):

$$ {}_0F_0(-,-;\mu z)\,{}_0F_0(-,-;\nu z) = {}_0F_0\big[-,-;(\mu+\nu)z\big], \tag{11}$$

$$ {}_1F_0(a,-;z)\,{}_1F_0(b,-;z) = {}_1F_0(a+b,-;z). \tag{12}$$

Here are some classical examples: Euler's transformation [6, V. 1, p. 64 (23)]

$$ {}_1F_0(a+b-c,-;z)\,{}_2F_1(c-a,c-b;c;z) = {}_2F_1(a,b;c;z); \tag{13}$$

Kummer's transformation [6, V. 1, p. 253 (7)]

$$ {}_0F_0(-,-;z)\,{}_1F_1(b;a+b;-z) = {}_1F_1(a;a+b;z); \tag{14}$$

Clausen's formula [6, V. 1, p. 185 (1)]

$$ \big[{}_2F_1(a,b;a+b+1/2;z)\big]^2 = {}_3F_2(2a,a+b,2b;a+b+1/2,2a+2b;z); \tag{15}$$

and the transformations given by Orr and Bailey [6, V. 1, pp. 185 (2), 186 (8, 9), 190 (3)] who reworked some earlier results of Goursat, Orr, Preece, and Ramanujan

$$ {}_0F_1(-,a;z)\,{}_0F_1(-,b;z) = {}_2F_3\big[(a+b)/2,(a+b-1)/2;a,b,a+b-1;4z\big], \tag{16}$$

$$ \begin{aligned} &{}_2F_1(a,b;a+b-1/2;z)\,{}_2F_1(a,b;a+b+1/2;z) \\ &\quad = {}_3F_2(2a,2b,a+b;2a+2b-1,a+b+1/2;z), \end{aligned} \tag{17}$$

$$ \begin{aligned} &{}_2F_1(a,b;a+b-1/2;z)\,{}_2F_1(a-1,b;a+b-1/2;z) \\ &\quad = {}_3F_2(2a-1,2b,a+b-1;2a+2b-2,a+b-1/2;z), \end{aligned} \tag{18}$$

$$ \begin{aligned} &{}_1F_0(2a-2b,-;z)\,{}_3F_2(2b-1,b+1/2,b-a-1/2;b-1/2,a+b+1/2;z) \\ &\quad = {}_3F_2(2a-1,a+1/2,a-b-1/2;a-1/2,a+b+1/2;z). \end{aligned} \tag{19}$$

An explicit description of all situations in which (10) is valid is an old and fundamentally difficult open problem. However what is crucial to us is the fact that the known solutions of Eq. (10) can immediately be used for some important applications. This is because each transformation (10) leads to a new weighted norm inequality involving three hypergeometric series ${}_{p_1}F_{q_1+1}$, ${}_{p_2}F_{q_2+1}$, and ${}_pF_{q+1}$, and because the known solutions of (10) allow us to obtain such inequalities for the most applicable hypergeometric series.

Theorem 2. *Let the hypergeometric series* ${}_{p_1}F_{q_1}(a_1,\dots,a_{p_1};b_1,\dots,b_{q_1};z)$, ${}_{p_2}F_{q_2}(c_1,\dots,c_{p_2};d_1,\dots,d_{q_2};z)$, *and* ${}_pF_q(u_1,\dots,u_p;v_1,\dots,v_q;z)$ *converge and satisfy relation* (10) *for all z in a neighbourhood of the origin.*

Then for any complex z, real γ, and $\alpha,\beta,\lambda>0$, the following inequality holds:

$$ \begin{aligned} &\int_0^1 t^{\alpha+\beta-1}(1-t)^{\lambda-1}e^{\gamma t}\big|{}_pF_{q+1}(u_1,\dots,u_p;v_1,\dots,v_q,\alpha+\beta;\omega zt)\big|^2\,dt \\ &\quad\leqslant \frac{B(\alpha+\beta,\lambda)}{B(\alpha,\beta+\lambda)B(\beta,\alpha+\lambda)} \\ &\qquad\times\int_0^1 t^{\alpha-1}(1-t)^{\beta+\lambda-1}e^{\gamma t}\big|{}_{p_1}F_{q_1+1}(a_1,\dots,a_{p_1};b_1,\dots,b_{q_1},\alpha;\mu zt)\big|^2\,dt \\ &\qquad\times\int_0^1 t^{\beta-1}(1-t)^{\alpha+\lambda-1}e^{\gamma t}\big|{}_{p_2}F_{q_2+1}(c_1,\dots,c_{p_2};d_1,\dots,d_{q_2},\beta;\nu zt)\big|^2\,dt. \end{aligned} \tag{20}$$

The equality in (20) *holds if and only if* ${}_{p_1}F_{q_1}$ *and* ${}_{p_2}F_{q_2}$ *are identically* 1 *and* $\gamma = 0$; *or* $z = \gamma = 0$; *or* $\mu z = \nu z = -\gamma/2 + i\theta$ (θ *is real*) *and* ${}_{p_1}F_{q_1}(\ldots;\ldots;z) = (1-z)^{-\alpha}$, ${}_{p_2}F_{q_2}(\ldots;\ldots;z) = (1-z)^{-\beta}$.

Proof. To prove Theorem 2 we apply Theorem 1 with

$$f(z) = {}_{p_1}F_{q_1}(a_1,\ldots,a_{p_1}; b_1,\ldots,b_{q_1}; \mu z)$$

and

$$g(z) = {}_{p_2}F_{q_2}(c_1,\ldots,c_{p_2}; d_1,\ldots,d_{q_2}; \nu z),$$

and then use definitions (1), (3)–(5) and relation (10). □

Alternatively, one can prove Theorem 2 directly via Theorem A. Then the integral transformation for convolutions used in the proof of Theorem 1 should be presented in terms of the hypergeometric functions. It seems that the corresponding integro-hypergeometric transformation associated with relation (10) is of interest. The following lemma gives this result. The integral convolution formula (21) given in it is a generalization of the integral addition theorem for ${}_1F_1$ and the ${}_0F_1$-version of the Sonin(e)'s integral for Bessel functions, which are associated with the simplest cases of (10), (12) and (11), respectively [6, V. 1, p. 271 (15); V. 2, p. 46], [11].

Lemma 1. *Let* ${}_{p_1}F_{q_1}$, ${}_{p_2}F_{q_2}$, *and* ${}_pF_q$ *be the hypergeometric series considered in Theorem* 2. *Then the following integral formula holds for any complex* z *and* α, β *with* $\Re\alpha, \Re\beta > 0$:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}U(t)V(1-t)\,dt = B(\alpha,\beta)\,{}_pF_{q+1}(u_1,\ldots,u_p; v_1,\ldots,v_q,\alpha+\beta;\omega z), \tag{21}$$

where

$$U(t) = {}_{p_1}F_{q_1+1}(a_1,\ldots,a_{p_1}; b_1,\ldots,b_{q_1},\alpha; \mu z t)$$

and

$$V(t) = {}_{p_2}F_{q_2+1}(c_1,\ldots,c_{p_2}; d_1,\ldots,d_{q_2},\beta; \nu z t) \tag{22}$$

for $t \in [0,1]$.

Proof. For a fixed z, functions $U(t)$ and $V(t)$ on $[0,1]$ can be presented in the form

$$U(t) = \sum_{n=0}^{\infty} \frac{A_n}{(\alpha)_n}(zt)^n \quad \text{and} \quad V(t) = \sum_{n=0}^{\infty} \frac{B_n}{(\beta)_n}(zt)^n, \tag{23}$$

where coefficients A_n and B_n are defined by the expansions

$$\begin{aligned} {}_{p_1}F_{q_1}(a_1,\ldots,a_{p_1}; b_1,\ldots,b_{q_1}; \mu z) &= \sum_{n=0}^{\infty} A_n z^n, \\ {}_{p_2}F_{q_2}(c_1,\ldots,c_{p_2}; d_1,\ldots,d_{q_2}; \nu z) &= \sum_{n=0}^{\infty} B_n z^n. \end{aligned} \tag{24}$$

Note that both series in (23) converge absolutely for any t and z since both series in (24) converge in a neighbourhood of the origin. From (23), (24), and (10) we have that

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}U(t)V(1-t)\,dt$$
$$= B(\alpha,\beta)\sum_{n,k=0}^{\infty}\frac{z^{n+k}}{(\alpha+\beta)_{n+k}}A_nB_k$$
$$= B(\alpha,\beta)\,{}_pF_{q+1}(u_1,\ldots,u_p;v_1,\ldots,v_q,\alpha+\beta;\omega z). \qquad \square$$

Proof of Theorem 2 via Theorem A. We apply Theorem A to functions

$$\phi(t)=e^{yt/2}U(t) \quad \text{and} \quad \psi(t)=e^{yt/2}V(t),$$

where U and V are defined by (22). Then we use formula (21), where z is replaced by $z\tau$. $\square$

4. Remarks on applications of Theorem 2

Via Theorem 2 even the trivial cases of transformation (10) lead to some non-trivial inequalities. Namely, relations (12) and (11), i.e., formulas

$$(1-z)^{-a}(1-z)^{-b}=(1-z)^{-a-b} \quad \text{and} \quad e^{\mu}e^{\nu}=e^{\mu+\nu}$$

lead to the general integral inequalities for the confluent hypergeometric and confluent hypergeometric limit functions, respectively. A detailed description of these inequalities as well as their applications to Bessel and Whittaker functions and Laguerre and Hermite polynomials are given in [11] (see also [10,12]).

We mention some applications of Theorem 2 implied by the classical transformations (13)–(19). For example, Euler's transformation (13) leads to the integral inequality

$$\int_0^1 t^{\alpha+\beta-1}(1-t)^{\lambda-1}\left|{}_2F_2(a,b;c,\alpha+\beta;zt)\right|^2dt$$
$$\leqslant \frac{B(\alpha+\beta,\lambda)}{B(\alpha,\beta+\lambda)B(\beta,\alpha+\lambda)}\int_0^1 t^{\alpha-1}(1-t)^{\beta+\lambda-1}\left|{}_1F_1(a+b-c;\alpha;zt)\right|^2dt$$
$$\times\int_0^1 t^{\beta-1}(1-t)^{\alpha+\lambda-1}\left|{}_2F_2(c-a,c-b;c,\beta;zt)\right|^2dt, \tag{25}$$

where z is any complex number and $\alpha,\beta,\lambda>0$. The equality in (25) holds if and only if $z=0$; or $a=c, b=0$; or $b=c, a=0$.

Kummer's transformation (14) generates an integral inequality for ${}_0F_1$ and ${}_1F_2$. Clausen's formula (15) leads, e.g., to the following inequality

$$\int_0^1 t^{2\alpha-1}(1-t)^{\lambda-1}e^{yt}\left|{}_3F_3(2a,a+b,2b;a+b+1/2,2a+2b,2\alpha;zt)\right|^2dt$$
$$\leqslant \frac{B(2\alpha,\lambda)}{B^2(\alpha,\alpha+\lambda)}\left[\int_0^1 t^{\alpha-1}(1-t)^{\alpha+\lambda-1}e^{yt}\left|{}_2F_2(a,b;a+b+1/2,\alpha;zt)\right|^2dt\right]^2 \tag{26}$$

for any complex z, real y, and $\alpha, \lambda > 0$. The equality in (26) holds if and only if $abz = 0$ and $y = 0$.

Finally we note that Orr's and Bailey's transformations (16), (17) and (18), and (19) lead to the integral inequalities for ${}_0F_2$ and ${}_2F_4$, ${}_2F_2$ and ${}_3F_3$, and ${}_1F_1$ and ${}_3F_3$, respectively.

5. The limit case

By induction, inequality (6) of Theorem 1 with $g = f^m$, $\beta = m\alpha$, and $\gamma = 0$ ($m = 1, 2, \ldots$) leads to the following inequality:

$$L(n\alpha, \lambda, 0)\big[f^n\big](\zeta) \leqslant L^n\big(\alpha, (n-1)\alpha + \lambda, 0\big)[f](\zeta), \tag{27}$$

where $n = 2, 3, \ldots$. Theorem 3 gives the limit inequality as $n \to \infty$. A particular case of it, when $f(z) = e^z$, is discussed in [11].

Theorem 3. *Let f, $f(0) = 1$, be analytic in a neighbourhood of the origin. Then for any $\alpha, \lambda > 0$, and complex ζ, the following inequality holds*:

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1}\big|f_\alpha(\zeta t)\big|^2\,dt \leqslant B(\alpha,\lambda)\exp\left\{\alpha\int_0^1 t^{-1}(1-t)^{\alpha+\lambda-1}\big[\big|f_0(\zeta t)\big|^2 - 1\big]\,dt\right\}, \tag{28}$$

where f_α is defined by (3) *and*

$$f_0(z) = 1 + \big(\alpha^{-1}\log f(z)\big) * \big(ze^z\big). \tag{29}$$

Proof. We replace f by $f^{1/n}$ ($f^{1/n}(0) = 1$) and α by α/n in (27). Then according to definition (4)–(5), we obtain

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1}\big|f_\alpha(\zeta t)\big|^2\,dt/B(\alpha,\lambda) \leqslant \left[\int_0^1 t^{\alpha/n-1}(1-t)^{(n-1)\alpha/n+\lambda-1}\big|f_{\alpha/n}^{1/n}(\zeta t)\big|^2\,dt/B\big(\alpha/n, (n-1)\alpha/n+\lambda\big)\right]^n \tag{30}$$

for $n = 2, 3, \ldots$.

Let $\log f(z) = \sum_{k=1}^{\infty} c_k z^k$ in some disk D_r. It follows that the expression in the brackets on the right-hand side of (30) can be presented in the form

$$1 + n^{-1}\int_0^1 t^{-1}(1-t)^{\alpha+\lambda-1}\left\{2\Re\left[\sum_{k=1}^{\infty}\frac{c_k\zeta^k}{(k-1)!}t^k\right] + \frac{1}{\alpha}\left|\sum_{k=1}^{\infty}\frac{c_k\zeta^k}{(k-1)!}t^k\right|^2\right\}dt + O\big(n^{-2}\big), \quad \text{as } n \to \infty. \tag{31}$$

We use (30), (31) and (29) to obtain inequality (28). □

Corollary 2. *Let* F, $F(0) = 1$, *be an entire function of the finite exponential type. Then for any* $\alpha, \lambda > 0$, *and complex* ζ, *the following inequality holds:*

$$\int_0^1 t^{\alpha-1}(1-t)^{\lambda-1}\left|F(\zeta t)\right|^2 dt \leqslant B(\alpha,\lambda)\exp\left\{\alpha\int_0^1 t^{-1}(1-t)^{\alpha+\lambda-1}\left[\left|\omega(\zeta t)\right|^2-1\right]dt\right\}, \tag{32}$$

where $\omega(z) = 1 + (\alpha^{-1}\log[z^{-1}\Phi(\alpha, z^{-1})]) * (ze^z)$ *and* Φ *is the generalized Borel-associated function to* F *defined by* (7)–(8).

Remark 2. The contour integral presentation of function ω in Corollary 2 is worth mentioning.

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